Solution. Assignment 2

We consider

$$(1+s^2)\frac{d^2u}{ds^2}+s\frac{du}{ds}-u=\frac{1}{\sqrt{1+s^2}} \tag{1}$$

on $\mathbf{R}\times\mathbf{R}$.

(a) Let $s=\sinh t$ then

$$\frac{ds}{dt}=\cosh t$$

and

$$\frac{d}{ds}=\frac{1}{\cosh t}\frac{d}{dt}.$$

Hence with

$$u(s)=u(\sinh t)=x(t)$$

we find

$$\frac{du}{ds}=\frac{1}{\cosh t}\frac{dx}{dt}$$

and

$$\begin{aligned}\frac{d^2u}{ds^2}&==\frac{1}{\cosh t}\frac{d}{dt}\left(\frac{1}{\cosh t}\frac{dx}{dt}\right)\\&=\frac{1}{\cosh t}\left(\frac{1}{\cosh t}\frac{d^2x}{dt^2}-\frac{\sinh t}{\cosh^2 t}\frac{dx}{dt}\right)\\&=\frac{1}{\cosh^2 t}\frac{d^2x}{dt^2}-\frac{\sinh t}{\cosh^3 t}\frac{dx}{dt}\end{aligned}$$

By insertion in (1) we obtain

$$(1+\sinh^2 t)\left(\frac{1}{\cosh^2 t}\frac{d^2x}{dt^2}-\frac{\sinh t}{\cosh^3 t}\frac{dx}{dt}\right)+\sinh t\frac{1}{\cosh t}\frac{dx}{dt}-x=\frac{1}{\sqrt{1+\sinh^2 t}}.$$

Simplifying with $\cosh^2 t-\sinh^2 t=1$ we finally obtain

$$\frac{d^2x}{dt^2}-x=\frac{1}{\cosh t} \tag{2}$$

as stated.

(b) Let $y(t)=(y_1(t),y_2(t))=(x(t),\dot{x}(t))$, then

$$\dot{y}=Ay+b \tag{3}$$

with

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad b(t)=\begin{pmatrix}0\\\frac{1}{\cosh t}\end{pmatrix}.$$

(c) Eigenvectors for A are $(1,1)$ with eigenvalue 1 and $(1,-1)$ with eigenvalue -1. Hence $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This implies $e^{tA} = Ce^{tD}C^{-1}$ from which it follows that

$$e^{tA} = C \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} C^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

(d) It follows from Theorem 6.28 that

$$\begin{aligned} y_0(t) &= e^{tA} \int_0^t e^{-sA} b(s)\, ds \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\cosh t} \end{pmatrix} ds \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} -\tanh s \\ 1 \end{pmatrix} ds \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} -\ln(\cosh t) \\ t \end{pmatrix} \\ &= \begin{pmatrix} -\ln(\cosh t)\cosh t + t\sinh t \\ -\ln(\cosh t)\sinh t + t\cosh t \end{pmatrix} \end{aligned}$$

(e) The set of maximal solutions of (3) is $\{y_0 + e^{tA}\eta \mid \eta \in \mathbf{R}^2\}$, that is,

$$\left\{ y = \begin{pmatrix} -\ln(\cosh t)\cosh t + t\sinh t + \eta_1 \cosh t + \eta_2 \sinh t \\ -\ln(\cosh t)\sinh t + t\cosh t + \eta_1 \sinh t + \eta_2 \cosh t \end{pmatrix} \,\middle|\, \eta_1, \eta_2 \in \mathbf{R} \right\}.$$

(f) The maximal solutions $x(t)$ of (2) are obtained as the first coordinates of $y(t)$, and the set of maximal solutions of (1) is then

$$\{x = -\ln(\sqrt{1+s^2})\sqrt{1+s^2} + ts + \eta_1\sqrt{1+s^2} + \eta_2 s \mid \eta_1, \eta_2 \in \mathbf{R}\}.$$