Solution. Assignment 3

(a) The function $x \mapsto |x|^\alpha$, $(x \in \mathbf{R})$, is locally Lipschitz when $\alpha \geq 1$. Hence so is X_α by Lemma 7.16 (for $\alpha > 1$ one could also argue by the fact that it is C^1). If $0 < \alpha < 1$, then $X_\alpha(t, y)$ is not locally Lipschitz at $(t_0, 1)$ for any $t_0 \in \mathbf{R}$, since that would imply the existence of $R, L > 0$ such that for all t with $|t - t_0| < R$ and y with $|y - 1| < R$

$$|X_\alpha(t, y) - X_\alpha(t, 1)| = \frac{\cosh(t)|y + 1|^\alpha |y - 1|^\alpha}{y} \leq L|y - 1|.$$

Hence for $y \neq 1$

$$\cosh(t)|y + 1|^\alpha \leq L|y - 1|^{1-\alpha} y.$$

The right hand side of this inequality tends to 0 as $y \to 1$, whereas the left side is ≥ 1, a contradiction.

(b) X_α is not Lipschitz for any $\alpha > 0$. To prove this its suffices to regard $X_\alpha(0, y)$ for $0 < y < \frac{1}{2}$. If it were Lipschitz, there would exist $R, L > 0$ such that for all $y, y' > 0$ with $|y' - y| < R$ we would have

$$|X_\alpha(0, y) - X_\alpha(0, y')| \leq L|y - y'|. \qquad (1)$$

Take $y' = 2y$. Then

$$X_\alpha(0, y) - X_\alpha(0, 2y) = \frac{(1 - y^2)^\alpha}{y} - \frac{(1 - 4y^2)^\alpha}{2y} = y^{-1} \left((1 - y^2)^\alpha - \frac{1}{2}(1 - 4y^2)^\alpha \right)$$

and

$$(1 - y^2)^\alpha - \frac{1}{2}(1 - 4y^2)^\alpha \geq (1 - y^2)^\alpha - \frac{1}{2}.$$

If y is sufficiently small then $(1 - y^2)^\alpha \geq \frac{3}{4}$ and hence

$$X_\alpha(0, y) - X_\alpha(0, 2y) \geq \frac{1}{4} y^{-1}$$

Since we also have

$$|y - y'| = |y - 2y| = y < R$$

for y sufficiently small, we then obtain from (1)

$$\frac{1}{4} y^{-1} \leq Ly,$$

for all such y, which is a contradiction.

(c) X_α is C^1 on $\mathbf{R} \times \big((0, 1) \cup (1, \infty)\big)$, since $y^2 - 1 \neq 0$ on this domain. Hence it is locally Lipschitz.

(d) It follows from (c) and Corollary 8.5 that there exists a unique maximal solution (I, y) in $\mathbf{R}\times\big((0,1)\cup(1,\infty)\big)$ with $y(t_0) = y_0$. Note that (I, y) is also a solution in $\mathbf{R}\times(0,\infty)$ (but as such it is not necessarily maximal).

(e) Let $(I, \tilde{y})$ be a solution in $\mathbf{R} \times (0, \infty)$ with $\tilde{y}(t_0) = y_0$ and with the same interval I as in (d). Assume $\tilde{y} \neq y$. Then $\tilde{y}$ must attain the value 1 somewhere, since (I, y) is the unique $(0,1)\cup(1,\infty)$-valued solution to the initial value problem. Say for example that 1 is attained in $I \cap (t_0, \infty)$ (the other case is similar), and let $b = \inf\{t > t_0 | \tilde{y}(t) = 1\}$. By continuity $\tilde{y}(b) = 1$. On the other hand, since $\tilde{y}(t) \neq 1$ for all $t \in [t_0, b)$, we have $\tilde{y}(t) = y(t)$ for $t \in [t_0, b)$ by uniqueness of the $(0,1) \cup (1,\infty)$-valued solutions. By continuity of $\tilde{y}$ and y we conclude $\tilde{y}(b) = y(b) \neq 1$, a contradiction. Hence $\tilde{y} = y$.

To answer the question we can therefore take $U = I$.

(f) Exercise 1 corresponds to the present exercise with $\alpha = \frac{1}{2}$. The solution denoted y_1 in Exercise 1(a) corresponds to the solution y of (d) above. The last two of the three solutions in Exercise 1(b) both solve the initial value problem $y(1) = \cosh(1)$ and they both agree with y_1 on $I = (0, \infty)$. This is in accordance with what we proved in (e) above.