Assignment 5

Due March 13

In this exercise we take the viewpoint that the sine and cosine functions have not yet been defined. We consider the linear homogeneous differential equation

$$x'' + x = 0. \tag{1}$$

1. Let $\mathrm{cs}(t)$, $t \in \mathbf{R}$ denote the solution with $\mathrm{cs}(0) = 1$, $\mathrm{cs}'(0) = 0$ and $\mathrm{sn}(t)$, $t \in \mathbf{R}$, the solution with $\mathrm{sn}(0) = 0$, $\mathrm{sn}'(0) = 1$. Determine the power series of these functions and their radius of convergence.

2. Prove that sn is an odd function, cs is even, and that $\mathrm{sn}' = \mathrm{cs}$, $\mathrm{cs}' = -\mathrm{sn}$.

3. Prove the addition formulas

$$\mathrm{sn}(s+t) = \mathrm{sn}(s)\,\mathrm{cs}(t) + \mathrm{cs}(s)\,\mathrm{sn}(t), \quad \mathrm{cs}(s+t) = \mathrm{cs}(s)\,\mathrm{cs}(t) - \mathrm{sn}(s)\,\mathrm{sn}(t)$$

   by showing that for fixed s both sides solve (1) as functions of t with the same initial condition at $t = 0$.

4. Prove that $\mathrm{cs}^2(t) + \mathrm{sn}^2(t) = 1$.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a C^2-function for which $f'(t) > 0$ and $f''(t) \geq 0$ for all $t > 0$. Show that $f(t) \to \infty$ for $t \to \infty$.

6. Prove there exists $b > 0$ such that $\mathrm{cs}(b) = 0$.

   (Hint: Otherwise a contradiction can be reached with $f = -\mathrm{cs}$ in the previous item).

7. Let ϖ be defined by $\frac{\varpi}{2} := \inf\{b > 0 \mid \mathrm{cs}(b) = 0\}$. Find

$$\mathrm{cs}(\frac{\varpi}{2}),\ \mathrm{sn}(\frac{\varpi}{2}),\ \mathrm{cs}(\varpi),\ \mathrm{sn}(\varpi),\ \mathrm{cs}(2\varpi),\ \mathrm{sn}(2\varpi)$$

   and show that cs and sn are periodic with period 2ϖ.