Solution. Assignment 5

We consider the linear homogeneous differential equation

$$x'' + x = 0. \tag{1}$$

1. From

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

we obtain by termwise differentiation and substitution into (1)

$$\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

We shift the index in the first sum and obtain

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) t^n + \sum_{n=0}^{\infty} a_n t^n = 0,$$

hence

$$\sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) + a_n\right) t^n = 0,$$

and by the identity principle

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

For $n = 2k$ even it follows by induction that

$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0$$

and for $n = 2k+1$ odd

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1.$$

The initial value problem $x(0) = 1$, $x'(0) = 0$ is solved by taking $a_0 = 1$ and $a_1 = 0$, and hence

$$\mathrm{cs}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}.$$

The initial value problem $x(0) = 0$, $x'(0) = 1$ is solved by taking $a_0 = 0$ and $a_1 = 1$, and hence

$$\mathrm{sn}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}.$$

These series have infinite radius of convergence by the ratio test, since

$$|a_n / a_{n+2}| = (n+2)(n+1) \to \infty$$

as $n \to \infty$.

2. sn is odd because the defining power series has only odd powers of t. Likewise cs is even because its series has only even powers. The derivatives of the functions are determined by termwise differentiation:

$$\begin{aligned}\operatorname{sn}'(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2k+1)t^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}t^{2k} = \operatorname{cs}(t).\end{aligned}$$

The equation

$$\operatorname{cs}'(t) = -\operatorname{sn}(t)$$

is seen in the same way.

3. Let s be fixed. The function $x_1(t) = \operatorname{sn}(s+t)$ satisfies (1) because the equation is autonomous and $t \mapsto \operatorname{sn}(t)$ solves it. Furthermore $x_1(0) = \operatorname{sn}(s)$ and $x_1'(0) = \operatorname{cs}(s)$ (by the previous item). The function $x_2(t) = \operatorname{sn}(s)\operatorname{cs}(t) + \operatorname{cs}(s)\operatorname{sn}(t)$ solves (1) since it is a linear combination of the two solutions $\operatorname{cs}(t)$ and $\operatorname{sn}(t)$. Furthermore, $x_2(0) = \operatorname{sn}(s)$ and $x_2'(0) = \operatorname{cs}(s)$ (again by the previous item). Hence $x_1 = x_2$ by the uniqueness theorem for linear equations, that is,

$$\operatorname{sn}(s+t) = \operatorname{sn}(s)\operatorname{cs}(t) + \operatorname{cs}(s)\operatorname{sn}(t).$$

The formula

$$\operatorname{cs}(s+t) = \operatorname{cs}(s)\operatorname{cs}(t) - \operatorname{sn}(s)\operatorname{sn}(t)$$

is proved in the same fashion.

4. If we take $s = -t$ in the last formula above, we obtain

$$\operatorname{cs}^2(t) + \operatorname{sn}^2(t) = 1$$

(by use of item 2).

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a C^2-function for which $f'(t) > 0$ and $f''(t) \geq 0$ for all $t > 0$. By Taylor's formula with remainder

$$f(t) = f(1) + (t-1)f'(1) + \frac{1}{2}f''(t_1)$$

for $t > 1$, where t_1 is some number between 1 and t. Since $f'' \geq 0$ it follows that

$$f(t) \geq f(1) + (t-1)f'(1)$$

and since $f'(1) > 0$ it follows that $f(t) \to \infty$ for $t \to \infty$.

6. Claim: There exists $b > 0$ such that $\operatorname{cs}(b) = 0$. Otherwise, since cs is continuous and $c(0) = 1$, it follows that $\operatorname{cs}(t) > 0$ for all t. Since $\operatorname{sn}' = \operatorname{cs}$ it then follows that $\operatorname{sn}(t)$ is strictly increasing for $t \geq 0$, hence $\operatorname{sn}(t) > \operatorname{sn}(0) = 0$ for all $t > 0$. Applying item 5 with $f = -\operatorname{cs}$ we infer that $\operatorname{cs}(t) \to -\infty$ for $t \to \infty$, and this contradicts item 4.

7. Let ϖ be defined by $\frac{\varpi}{2} := \inf\{b > 0 \mid \mathrm{cs}(b) = 0\}$. It follows from item 6 that the infimum is taken over a non-empty set, and hence ϖ is well-defined. We have

$$\mathrm{cs}(\frac{\varpi}{2}) = 0$$

since cs is continuous. Moreover, $\mathrm{cs}(t) > 0$ for $0 \leq t < \frac{\varpi}{2}$, and this implies that sn is increasing in this interval, and hence $\mathrm{sn}(\frac{\varpi}{2}) > 0$. By item 4 it then follows that

$$\mathrm{sn}(\frac{\varpi}{2}) = 1.$$

Now

$$\mathrm{cs}(\varpi) = -1, \quad \mathrm{sn}(\varpi) = 0$$

follows by taking $s = t = \frac{\varpi}{2}$ in item 2. By taking $s = \varpi$ in item 3 we conclude $\mathrm{sn}(t + \varpi) = -\mathrm{sn}(t)$ and $\mathrm{cs}(t + \varpi) = -\mathrm{cs}(t)$, and it follows that cs and sn are periodic with period 2ϖ.