Assignment 6

Due Wednesday March 20.

Consider Bessel's equation with $p = 0$,

$$t^2x'' + tx' + t^2x = 0 \qquad (t > 0). \tag{1}$$

We have seen that the equation has a power series solution of the form

$$J_0(t) = \sum_{k=0}^{\infty} a_k t^k$$

given by the recursion formula

$$k^2 a_k + a_{k-2} = 0 \qquad (k \geq 2). \tag{2}$$

and with $a_k = 0$ for odd k.

Since it is a second order linear equation, the solution space is two dimensional. The purpose of this exercise is to find a solution independent of J_0.

(a) Show that with $a_0 = 1$ we have

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

(b) Let B denote the differential operator

$$\mathrm{B} = t^2\frac{d^2}{dt^2} + t\frac{d}{dt} + t^2.$$

Show that

$$\mathrm{B}(\ln(t)x) = \ln(t)\,\mathrm{B}\,x + 2tx'$$

for every two times differentiable function $x = x(t)$. Conclude that if x solves the Bessel equation $Bx = 0$ and y is a solution to the inhomogeneous equation $\mathrm{B}\,y = -2tx'$, then $\ln(t)x + y$ also solves the Bessel equation.

(c) Let $x(t) = J_0(t) = \sum_{k=0}^{\infty} a_k t^k$ and consider the Ansatz

$$y(t) = \sum_{k=1}^{\infty} b_k t^k, \qquad \mathrm{B}\,y = -2tx'.$$

Show that if termwise differentiation is allowed in the series for y, then

$$k^2 b_k + b_{k-2} = -2ka_k \qquad (k \geq 2)$$

and $b_1 = 0$.

(d) Conclude that $b_k = 0$ when k is odd, and show that

$$|b_{2n}| \leq \frac{1}{(n!)^2}$$

for $n = 1, 2, 3, \ldots$. Conclude that y exists, that y solves the inhomogeneous equation $\mathrm{B}\, y = -2tx'$, and finally that $\ln(t)J_0 + y$ solves the Bessel equation with $p = 0$.

(e) Determine the behaviour of $\ln(t)J_0(t) + y(t)$ for $t \to 0$, and show that as a function of t, it is linearly independent of J_0. Determine the complete solution of the Bessel equation for $p = 0$ on $(0, \infty)$.

(f) Determine also the complete solution on $(-\infty, 0)$. (Hint: apply the transformation $t \mapsto -t$).