Boundary value problems for partial differential equations, II

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1 Laplace's equation

The equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \tag{1}$$

for a function $u(x_1, \ldots, u_n)$ of n variables is called *Laplace's equation.* Typically one wants to find a solution u in a given open set $\Omega \subset \mathbb{R}^n$, with an additional boundary condition of the form

$$u(x) = f(x) \quad x \in \partial\Omega,$$

where f is a given function on the boundary. What we are looking for is thus a continuous function on the closure $\bar{\Omega}$, which satisfies the Laplace equation in Ω and the boundary condition on $\partial\Omega$. We will describe the solution to this problem for $n = 2$ and Ω a circular disk.

We choose the radius of the disk to be 1 for simplicity,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\},$$

As before, with the circular nature of Ω, it is convenient to use polar coordinates

$$(x, y) = (r\cos\theta, r\sin\theta),$$

where $0 \le r \le 1$ and $\theta \in \mathbb{R}$. The Laplace equation (1) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2}$$

and the boundary condition becomes

$$u(1, \theta) = f(\theta) \tag{3}$$

for all θ, where the given function on $\partial\Omega$ is represented as $(\cos\theta, \sin\theta) \mapsto f(\theta)$ with a 2π-periodic function f.

We apply the method of separation of variables and look first for solutions of the product form $u(r, \theta) = R(r)\Theta(\theta)$. Substitution in the differential equation and division by $R(r)\Theta(\theta)$ yields

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = 0.$$

Hence

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)},$$

and we see that both sides of this equation must be independent of both r and θ, that is equal to some constant λ. It follows that R and Θ must satisfy the ordinary differential equations

$$r^2R'' + rR' - \lambda R = 0, \quad \Theta'' + \lambda\Theta = 0.$$

Furthermore, we must request that Θ is 2π-periodic. The differential equation for Θ is easily solved, and we see that it has periodic solutions only if $\lambda \geq 0$. For $\lambda > 0$ the solutions are all linear combinations of $e^{i\sqrt{\lambda}\theta}$ and $e^{-i\sqrt{\lambda}\theta}$, and they are 2π-periodic if and only if λ is the square of an integer. Let $\lambda_n = n^2$ for $n \in \mathbb{Z}$, then we conclude that Θ is a linear combination of $e^{in\theta}$ and $e^{-in\theta}$ if $\lambda = \lambda_n \neq 0$. For $\lambda = 0$ the only 2π-periodic solutions are the constant functions, which we express as multiples of $e^{i0\theta}$ in order to get the uniform expression $e^{in\theta}$, $n \in \mathbb{Z}$, for the basic 2π-periodic solutions.

We now turn to the equation for R with $\lambda = \lambda_n$,

$$r^2R'' + rR' - n^2R = 0.$$

This equation is easily solved with the change of variable $r = e^s$, by which the equation for $S(s) = R(e^s)$ becomes

$$S'' - n^2S = 0.$$

If $n \neq 0$ the general solution is a linear combination of $s \mapsto e^{ns}$ and $s \mapsto e^{-ns}$, and if $n = 0$ it is a linear combination of $s \mapsto 1$ and $s \mapsto s$. Hence the general solution of the equation for R is a linear combination of $r \mapsto r^n$ and $r \mapsto r^{-n}$ if $n \neq 0$, and of $r \mapsto 1$ and $r \mapsto \ln r$ if $n = 0$. The function $R(r)$ must be continuous at $r = 0$ if $R(r)\Theta(\theta)$ is to represent a continuous function on the disk, and this excludes the negative powers $r^{-|n|}$ and $\ln r$. All together, we find that $R(r)$ is a multiple of $r^{|n|}$ for all $n \in \mathbb{Z}$.

In conclusion, we have the separated solutions

$$u(r,\theta) = r^{|n|}e^{in\theta}$$

and a reasonable Ansatz for the general solution appears to be

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta}$$

for some coefficients $c_n \in \mathbb{C}$. These coefficients can be determined from the boundary condition at $r = 1$, where we obtain

$$\sum_{n\in\mathbb{Z}} c_n e^{in\theta} = f(\theta).$$

We recognize this as the Fourier series of f, and hence we anticipate the c_n to be exactly the Fourier coefficients for this function.

The analysis above motivates the following theorem, which we shall now prove.

1.1 Theorem. *Consider the Laplace equation (2) in polar coordinates with the boundary condition (3), and assume that f is 2π-periodic, continuous and piecewise C^1. Let $(c_n)_{n\in\mathbb{Z}}$ be its Fourier coefficients and define*

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta} \tag{4}$$

for $0 \le r \le 1$ and $\theta \in \mathbb{R}$. Then the series (4) converges uniformly and absolutely on $\{(r,\theta) \mid 0 \le r \le 1\}$ and its sum

- *is continuous on $\bar{\Omega}$*
- *belongs to $C^2(\Omega)$*
- *satisfies the Laplace equation in Ω*
- *satisfies the boundary condition on $\partial\Omega$.*

Furthermore, the sum of (4) is the only function on $\bar{\Omega}$ with these properties.

Proof. The conditions on f imply (see Solovej's notes) that

$$\sum_{n\in Z} |c_n| < \infty \tag{5}$$

and that the last item above is valid. It follows from Weierstrass that (4) converges uniformly and absolutely on Ω since it is majorized by (5). The first item follows.

To prove the other statements we first note that since $re^{i\theta} = x + iy$ and $re^{-i\theta} = x - iy$, the series (4) can be written in the original cartesian coordinates as follows

$$u(x,y) = c_0 + \sum_{n\in\mathbb{Z},n>0} c_n(x+iy)^{|n|} + \sum_{n\in\mathbb{Z},n<0} c_n(x-iy)^{|n|} \tag{6}$$

The convergence (5) is not strong enough to majorize the termwise differentiated series simultaneously on all of Ω, since each differentiation with respect to x or y will produce an extra factor n in the series. For example, by applying d/dx termwise to (6) we obtain the series

$$\sum_{n\in\mathbb{Z},n>0} c_n|n|(x+iy)^{|n|-1} + \sum_{n\in\mathbb{Z},n<0} c_n|n|(x-iy)^{|n|-1}. \tag{7}$$

To obtain the necessary majorization we proceed as follows. Let $0 < \rho < 1$ and consider

$$\Omega_\rho = \{(x,y) \mid x^2 + y^2 < \rho^2\}.$$

On this set we have $|x \pm iy| < \rho$ and hence the series (7) are dominated by

$$\sum_{n\in\mathbb{Z},n\neq 0} |c_n||n|\rho^{|n|-1}.$$

The dominating sum converges, since

$$C_\rho := \sup_{n\in\mathbb{N}} n\rho^{n-1} < \infty$$

for $0 < \rho < 1$ so that

$$\sum_{n\in\mathbb{Z},n\neq 0} |c_n||n|\rho^{|n|-1} \leq C_\rho \sum_{n\in\mathbb{Z},n\neq 0} |c_n|.$$

It follows that $u(x, y)$ is differentiable in Ω_ρ as a function of x, with a derivative which is obtained by termwise differentiation. Since ρ was arbitrary, we arrive at the same conclusion for all of Ω.

A similar argument applies to differentiation with respect to y, and also to consequtive differentiations with x and/or y. It follows that the sum is two times (in fact infinitely often) differentatible in the open disk, and that all differentiations can be done termwise in the series. This proves the second item. Since termwise differentiation has been justified, and since each term is already known to satisfy Laplace's equation (by the analysis preceding the theorem), the third item follows as well.

Only the asserted uniqueness remains to be established. Assume $v : \Omega \to \mathbb{C}$ is a function which also satisfies all items marked $\bullet$. For $0 < r \leq 1$ we let $c_n(r)$ denote the n-th Fourier coefficient

$$c_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r\cos\theta, r\sin\theta)e^{-in\theta}\,d\theta$$

of $\theta \mapsto v(r\cos\theta, r\sin\theta)$. Then

$$v(r,\theta) = \sum_{n=-\infty}^{\infty} c_n(r)e^{in\theta}.$$

Since v is a C^2-function, the theorem of differentiation under the integral permits us to conclude that c_n is two times differentiable as a function of $r \in (0, 1)$, and that the derivatives can be determined by differentiation inside the integral. It follows that

$$r^2c_n''(r) + rc_n'(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r})v(r,\theta)e^{-in\theta}\,d\theta.$$

Moreover

$$n^2c_n(r) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta)\frac{\partial^2 e^{-in\theta}}{\partial\theta^2}\,d\theta = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v(r,\theta)}{\partial^2\theta}e^{-in\theta}\,d\theta$$

by partial integration (twice). Hence

$$r^2 c_n''(r) + r c_n'(r) - n^2 c_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}) v(r,\theta) \right] e^{-in\theta}\, d\theta = 0$$

since $v(r,\theta)$ satisfies the Laplace equation.

This equation for $c_n(r)$ was solved in our analysis prior to the theorem, and from there we can conclude that $c_n(r)$ is a linear combination of r^n and r^{-n} (respectively of $r^0 = 1$ and $\ln r$ if $n = 0$). The fact that v is continuous at the origin implies that $c_n(r)$ is bounded as $r \searrow 0$, and we conclude (as in the preceding analysis) that $c_n(r)$ is proportional to $r^{|n|}$. Furthermore, from the fact that v is continuous on the compact set $\bar{\Omega}$ we infer that it is uniformly continuous. This implies that the following limit can be taken

$$\lim_{r \nearrow 1} c_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{r \nearrow 1} u(r,\theta) e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta = c_n.$$

We have already seen that $c_n(r)$ is proportional to $r^{|n|}$, and this limit relation then implies that

$$c_n(r) = c_n r^{|n|}.$$

This means that we have established the identity of v with u from (4). $\square$

Exercises

1 Exercise. Consider the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$

on $[0,\pi] \times [0,\infty)$ with the so-called Neumann boundary conditions

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(\pi,t) = 0 \qquad (t > 0)$$

and initial condition

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x) \qquad (x \in [0,\pi])$$

for functions $f \in C^3([0,\pi])$ and $g \in C^2([0,\pi])$, with

$$\frac{df}{dx}(0) = \frac{df}{dx}(\pi) = \frac{d^3 f}{dx^3}(0) = \frac{d^3 f}{dx^3}(\pi) = 0$$

and

$$\frac{dg}{dx}(0) = \frac{dg}{dx}(\pi) = 0.$$

a. Extend the functions f, g, and u (in the first variable), to even functions $\tilde{u}$, $\tilde{f}$ and $\tilde{g}$ on $[-\pi,\pi]$. Prove that $\tilde{f}$ and $\tilde{g}$ are C^3 and C^2 respectively. Under the assumption that $\tilde{u}$ is sufficiently smooth, show that $\tilde{u}$, $\tilde{f}$ and $\tilde{g}$ can be written as cos-series.

b. Why did we take even functions instead of odd in (a)?

c. Determine the coefficients of the cos-series of $\tilde{u}$ in terms of the coefficients of the cos-series of f and g.

d. Prove that the sum of the cos-series with the coefficients determined in (c), is twice differentiable and gives a solution of the boundary value problem.

e. Prove that this solution is unique.

f. Apply the above obtained results to the case where $f = \sin^4$ and $g = \cos^2$.

2 Exercise. Let $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y}$. Verify the following identity of differential operators

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)^2 = (x^2+y^2)\Delta$$

Let the polar coordinate map $\Phi : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$\Phi(r, \theta) = (r \cos\theta, r \sin\theta)\,.$$

Prove that

$$r \frac{\partial}{\partial r}(f \circ \Phi) = \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \circ \Phi$$

and

$$\frac{\partial}{\partial \theta}(f \circ \Phi) = \left(x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right) \circ \Phi$$

for f differentiable.

Use these results to conclude that

$$(\Delta f) \circ \Phi = \frac{1}{r^2} \left(\left(r \frac{\partial}{\partial r} \right)^2 + \frac{\partial^2}{\partial \theta^2} \right) (f \circ \Phi) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (f \circ \Phi)$$

for f two times differentiable.

3 Exercise. Solve the Laplace equation on the unit disk with the boundary value $f(\theta) = \sin(2\theta)$ on the circle.

4 Exercise.

1. Generalize the solution to Laplace's equation on the unit disk so that it allows a disk with arbitrary radius $\delta > 0$.

2. Consider the Laplace equation on an annulus

$$\{(x, y) \in \mathbb{R}^2 \mid \gamma^2 < x^2 + y^2 < \delta^2\}$$

of inner radius $\gamma > 0$ and outer radius $\delta > \gamma$. Solve it for an arbitrary pair of boundary functions f and g (assumed to be continuous and piecewise C^1) on the outer and inner boundary circle, respectively.