The root system of $\mathfrak{so}(7)$.

Recall that $\mathfrak{so}(7)$ is the Lie subalgebra of Mat$(7, \mathbf{R})$ consisting of the anti-symmetric matrices. Let $v : \mathbf{R}^3 \to \mathfrak{so}(7)$ be the map given by

$$v(x) = \begin{pmatrix} 0 & -x_1 & & & & & \\ x_1 & 0 & & & & & \\ & & 0 & -x_2 & & & \\ & & x_2 & 0 & & & \\ & & & & 0 & -x_3 & \\ & & & & x_3 & 0 & \\ & & & & & & 0 \end{pmatrix} \qquad (x \in \mathbf{R}^3).$$

(The entries left blank are to be understood as zero's.)

(i) Prove that $\mathfrak{so}(7)$ is a compact Lie algebra with trivial center.

(ii) Let $\mathfrak{t} = \{v(x) : x \in \mathbf{R}^3\}$. Prove that $\mathfrak{t}$ is a maximal torus of $\mathfrak{so}(7)$.

We define

$$\mathfrak{so}(7; \mathbf{C}) = \{X \in \text{Mat}(7, \mathbf{C}) : X^t = -X\}.$$

(iii) Prove that $\mathfrak{so}(7, \mathbf{C})$ is a Lie subalgebra of Mat$(7, \mathbf{C})$. Show that $\mathfrak{so}(7; \mathbf{C})$ is the complexification of $\mathfrak{so}(7)$.

For $j = 1, 2, 3$, let $e_j \in i\mathfrak{t}^*$ be the real linear map $\mathfrak{t} \to i\mathbf{R}$ given by

$$e_j\big(v(x)\big) = ix_j.$$

(iv) Prove that the set of roots $R = R(\mathfrak{so}(7, \mathbf{C}), \mathfrak{t})$ of $\mathfrak{so}(7, \mathbf{C})$ with respect to $\mathfrak{t}$ is equal to

$$\{\pm e_j \pm e_k : 1 \leq j < k \leq 3\} \cup \{\pm e_j : 1 \leq j \leq 3\}.$$

Determine the corresponding root spaces.

(v) Verify explicitly that R is indeed a root system. Determine for every $\alpha \in R$ the reflection s_α.

(vi) Show that the Weyl group W of R equals the group of all permutations and sign changes of the set $\{e_j : 1 \leq j \leq 3\}$.

(vii) Determine a fundamental system S for R.

(viii) Prove that the set $\{s_\alpha : \alpha \in S\}$ generates W.

(ix) Determine explicitly a W-invariant inner product on $i\mathfrak{t}^*$.

(x) Determine the Cartan integers associated to S.

(xi) Determine the Dynkin diagram of $\mathfrak{so}(7)$.