Representations of $\mathrm{SU}(n)$

Assignment due April 13, 2015 (counts for 50 % of the grade)

In these exercises, "representation" stands for "continuous representation" everywhere.

Let $n \in \mathbb{N}$ with $n \geq 2$ and consider the connected compact Lie group $G = \mathrm{SU}(n)$ and its real Lie algebra $\mathfrak{g} = \mathfrak{su}(n) \subset \mathfrak{sl}(n, \mathbb{C})$.

For $1 \leq i, j \leq n$ we denote by E_{ij} the matrix in $\mathfrak{gl}(n, \mathbb{C})$ with matrix elements

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad (k, l = 1, \dots, n),$$

that is, it has 1 in the entry of row i and column j, and 0 everywhere else.

(i) Show that $\mathfrak{g} + i\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g} \cap i\mathfrak{g} = \{0\}$, and conclude that the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$.

(ii) Show that the $(n-1)$-dimensional linear space $\mathfrak{t}$ consisting of all diagonal matrices

$$\mathrm{diag}(z_1, \dots, z_n) := \sum_{k=1}^{n} z_k E_{kk}$$

in $\mathfrak{g}$ is a maximal torus.

(iii) Define for $k = 1, \dots, n$ a linear map by

$$\epsilon_k : \mathfrak{t} \to \mathbb{R}, \quad \epsilon_k(\mathrm{diag}(z_1, \dots, z_n)) = z_k.$$

Show that for each pair (i, j) with $i \neq j$ the linear subspace $\mathbb{C}E_{ij}$ is a root space of $\mathfrak{g}_{\mathbb{C}}$ and determine the corresponding root α_{ij} in terms of the ϵ_k. Show that

$$R = \{\alpha_{ij} \mid i \neq j\}$$

is the complete set of roots.

(iv) Let $C = \{X \in i\mathfrak{t} \mid X = \mathrm{diag}(x_1, \dots, x_n) \text{ with } x_1 > \dots > x_n\}$. Show that C is a Weyl chamber, and determine the corresponding sets R^+ and R^- of positive and negative roots. Determine also the subalgebras $\mathfrak{g}_{\mathbb{C}}^+$ and $\mathfrak{g}_{\mathbb{C}}^-$ of $\mathfrak{g}_{\mathbb{C}}$.

(v) Let $\mathfrak{n} := \mathfrak{g}_{\mathbb{C}}^+$. For (π, V) a finite dimensional representation of G on a complex vector space V, let

$$V^{\mathfrak{n}} = \{v \in V \mid \pi_{*,\mathbb{C}}(X)v = 0, \forall X \in \mathfrak{n}\}$$

be the space of vectors annihilated by $\mathfrak{n}$.

Show that if $\dim V^{\mathfrak{n}} = 1$ then π is irreducible.

(vi) Consider the representation σ of G on $E = \mathbb{C}^n$ given by standard matrix multiplication $\sigma(g)v = gv$ for $v \in E$. Show that the representation is irreducible and determine its highest weight.

(vii) Consider the adjoint representation Ad of G on $\mathfrak{g}$ and its complexification $\mathfrak{g}_\mathbb{C}$. Show that the representation is irreducible and determine its highest weight.

(viii) Let $M = \mathfrak{gl}(n, \mathbb{C})$ and define $\pi(g)A = gAg^t$ for $g \in G$ and $A \in M$. Show that π is a representation of G, and determine the derived representation π_* of $\mathfrak{g}$.

For the rest of these exercises, π will denote the representation just defined.

(ix) Find all non-zero weight spaces M_λ for π and the corresponding weights $\lambda \in \mathfrak{t}^*_\mathbb{C}$ in terms of the ϵ_k.

(x) Show that $M^\mathfrak{n} = \text{Span}\{E_{11}, E_{12} - E_{21}\}$. *Hint.* Show first that a lot of entries in A must be zero if $\pi_*(E_{1j})(A) = 0$ for $j = 2, \dots, n$.

(xi) Decompose M as the direct sum of the subspace $\text{Sym}(n, \mathbb{C})$ of symmetric matrices and the subspace $\text{Skew}(n, \mathbb{C})$ of anti-symmetric matrices. Show that these subspaces are G-invariant and irreducible, and determine the highest weights of the restrictions π_{Sym} and π_{Skew} of π to them.

(xii) Let $m \in \mathbb{N}$ and let $\mathcal{P}_m$ denote the space of complex-valued homogeneous polynomials of degree m of n complex variables $x_1, \dots, x_n$. The space is spanned by the polynomials of the form

$$P_l(x) = x_1^{l_1} \cdots x_n^{l_n}, \qquad x \in \mathbb{R}^d,$$

where $l = (l_1, \dots, l_n) \in \mathbb{N}_0$ with $l_1 + \cdots + l_n = m$.

For $P \in \mathcal{P}_m$ and $g \in G$ we define

$$\rho_m(g)P(x) = P(g^t\, x)$$

for $x \in \mathbb{R}^n$ (where g^t denotes the transpose of g).

Show that ρ_m is a representation of G for each m.

(xiii) Define a linear map T from M to $\mathcal{P}_2$ by $TA(x) = x^t\, A\, x$ for $A \in M$. Show that T is equivariant. Determine its kernel and range. We can conclude that two representations are equivalent. Which?

(xiv) Show that ρ_m is irreducible for all m.