Exercises Week 6

1) For a Lie algebra $\mathfrak{g}$ one defines a sequence of subspaces by $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$. Show that this is a descending sequence of ideals. The Lie algebra is said to be *nilpotent* if $\mathfrak{g}^k = 0$ for some k. Show as an example that the space of strictly upper triangular real $n \times n$ matrices is a nilpotent Lie algebra.

Let $\mathfrak{g}$ be nilpotent, and let $\mathfrak{h}$ be an ideal. Show that $\mathfrak{g}/\mathfrak{h}$ is nilpotent.

2) Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that $\mathfrak{g}$ has a basis with respect to which $\mathrm{ad}(X)$ is strictly upper trangular for all $X \in \mathfrak{g}$. Show that

$$\det\left(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right) = 1.$$

Let G be a Lie group with nilpotent Lie algebra $\mathfrak{g}$, and assume in addition that $\exp : \mathfrak{g} \to G$ is bijective. Show that then $\exp$ is a diffeomorphism of $\mathfrak{g}$ to G.

Verify by explicit calculation that $\exp$ is bijective for the Lie group of upper triangular $n \times n$ matrices with 1's in the diagonal, when $n = 2$ and $n = 3$ (in fact this holds for all n).

3) Let G be a connected Lie group, H a closed connected central subgroup, and $L = G/H$. Show that if the exponential map of L is surjective, then the exponential map of G is surjective.

Use this to prove that $\exp$ is surjective for all connected Lie groups with a nilpotent Lie algebra (begin with the same statement for a commutative Lie algebra).

Give an example of a Lie group with nilpotent Lie algebra for which $\exp$ is not injective.

4) Let G be a connected Lie group, and let U be a neighborhood of 0 in its Lie algebra for which the restriction of $\exp$ is a diffeomorphism $U \to \exp(U)$. Show that for a function $f \in C(G)$ with support in U

$$\int_G f(g)\, dg = \int_U f(\exp X)\, \det\left(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right)\, dX$$

where dg is a left Haar measure on G and dX is a Lebesgue measure on the vector space $\mathfrak{g}$ (the formula becomes particularly simple for Lie groups with nilpotent Lie algebra).

5) Let G consist of the 2×2 real matrices of the form

$$g(u, v) = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$$

where $u > 0$. Show that G is a non-commutative Lie group. Show that if $g = g(a, b)$, then the differential of the left multiplication map ℓ_g has determinant a^2 with respect to the the coordinates (u, v). Conclude that $u^{-2}\, du\, dv$ is a left Haar measure, where $du\, dv$ is Lebesgue measure on the half plane (combine (19) on page 65 with the formula on top of page 64).

6) Determine the modular function $|\det(\mathrm{Ad}(g)|$ for the group G in the previous exercise, and find a right Haar measure.

7) Let $p > 1$ be fixed and let G_p be the subgroup of the group G from the previous two exercises, consisting of the matrices $\sigma(u, v)$ for which $u = p^k$ for some $k \in \mathbf{Z}$. Find the left and right Haar measures.

8) Let $G = \mathbf{GL}(n, \mathbf{R})$ which we regard as usual as an open subet of the set $M(n, \mathbf{R})$ of all $n \times n$ real matrices. Show that a left and right Haar integral is given by

$$\int_G f(g)\, dg = \int_{M(n,\mathbf{R})} f(X)\, |\det X|^{-n}\, dX$$

where dX denotes a Lebesgue measure on the vector space $M(n, \mathbf{R})$.

9) Let $G \subset \mathbf{GL}(n, \mathbf{R})$ denote the subgroup of all matrices g for which $gg^t = cI$ with a positive scalar $c \in \mathbf{R}_+$, and for which $\det g > 0$. Then $G = \mathbf{SO}(n)D$ where D is the central subgroup of all diagonal matrices rI with $r > 0$. Show that G acts transitively on $\mathbf{R}^n \setminus \{0\}$ by matrix multiplication, and determine a left invariant measure on this homogeneous space.

10) Show that the normalized $\mathbf{SO}(3)$-invariant measure on $S^2 = \mathbf{SO}(3)/\mathbf{SO}(2)$ is obtained from the surface integral given by

$$\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \sin\phi \, d\theta \, d\phi$$

with respect to spherical coordinates, by relating this measure to Lebesgue measure of $\mathbf{R}^3$ (which is known to be rotation invariant) through the standard formula for integration with respect to spherical and radial coordinates (a formula which is easily derived by the calculation of a Jacobian determinant).