Representation of cointegrated autoregressive processes with application to fractional processes

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Short title:

Representation of cointegrated autoregressive processes

Abstract

We analyze vector autoregressive processes using the matrix valued characteristic polynomial. The purpose of this paper is to give a survey of the mathematical results on inversion of a matrix polynomial in case there are unstable roots, to study integrated and cointegrated processes.

The new results are in the $I(2)$ representation, which contains explicit formulas for the first two terms and a useful property of the third. We define a new error correction model for fractional processes and derive a representation of the solution.

Keywords: Granger representation, error correction models, integration of order 1 and 2, fractional autoregressive model,

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1 Introduction

The integration and cointegration properties of autoregressive processes are studied by analyzing the inverse function of the matrix polynomial generating the coefficients. This is an well known technique for stationary processes, and for $I(1)$ processes this was also the approach in the original proof of the Granger Representation Theorem, see Engle and Granger (1987). The purpose of this paper is to give a survey of mathematical results on the inverse of a matrix polynomial. This is given in section 2. In section 3 we then discuss the application of the results to the well known models for $I(1)$, $I(2)$, explosive, and seasonally integrated processes.

The $I(2)$ representation is extended to give the first two terms explicitly and a useful property of the third, which implies that the cointegrating relations are $I(0)$ not just stationary. As an application of the representation results, we consider in section 4 a new error correction model for fractionally cointegrated processes and derive a representation of the solution, see Granger (1986).

2 Expansion and inversion of matrix polynomials

To establish some notation consider the $p-$dimensional autoregressive process X_t given by

$$X_t = \sum_{i=1}^{k} \Pi_i X_{t-i} + \varepsilon_t, \; t = 1, \ldots, T. \tag{1}$$

The solution X_t is a function of initial values and the shocks ε_t, which we assume to be a sequence of i.i.d. variables with mean zero and variance Ω. We define the characteristic polynomial $\Pi(z)$ as the $p \times p$ matrix function

$$\Pi(z) = I_p - \sum_{i=1}^{k} \Pi_i z^i,$$

and the characteristic equation by $|\Pi(z)| = 0$, where $|\Pi(z)| = \det(\Pi(z))$.

Let the roots of the characteristic equation be $z_1, \ldots, z_N$. We call a root z_i stable if $|z_i| > 1$, unstable if $|z_i| \leq 1$ and explosive if $|z_i| < 1$. Thus

$$\rho_{stable} = \min\{|z_i| : z_i \text{ stable}\} > 1$$

because it is the minimum of a finite set of numbers greater than one. Note that $\Pi(0) = I_p$ implies $z_i \neq 0$. The expression

$$\Pi^{-1}(z) = \frac{Adj(\Pi(z))}{|\Pi(z)|}, \; z \neq \{z_1, \ldots, z_N\}$$

shows that $\Pi(z)^{-1}$ has a pole at $z = \{z_1, \ldots, z_N\}$, since $|\Pi(z_m)| = 0$. If the pole is of order one it has the form $C_m(1 - z/z_m)^{-1}$, so that the function $H(z)$ defined by

$$H(z) = \Pi(z)^{-1} - C_m \frac{1}{1 - z/z_m}, \; z \neq \{z_1, \ldots, z_N\}$$

has no pole at z_m and has a convergent power series in a neighborhood of $z = z_m$.

We first prove two theorems on expansion of matrix polynomials, which translated into results about the autoregressive process (1) will give the error correction formulations of the autoregressive equations. Note that the points of expansion need not be roots of the polynomial. The first result can be interpreted as an interpolation formula, due to Legendre, which finds a polynomial which at $n+1$ points takes prescribed values.

Theorem 1 *Let* $z_i,\ i = 0, 1, \ldots, n$ *be distinct and* $z_0 = 0$*. The matrix polynomial* $\Pi(z)$ *has the expansion around* $z_0, z_1, \ldots, z_n$

$$\Pi(z) = I_p p(z) + \sum_{m=1}^{n} \Pi(z_m) \frac{p_m(z)}{p_m(z_m)} \frac{z}{z_m} + p(z) z \Pi^*(z),$$

where $\Pi^*(z)$ *is a matrix polynomial and*

$$p(z) = \prod_{i=1}^{n} (1 - z/z_i), \quad p_m(z) = \prod_{\substack{i=1 \\ i \neq m}}^{n} (1 - z/z_i). \tag{2}$$

PROOF. The matrix polynomial

$$R(z) = \Pi(z) - I_p p(z) - \sum_{m=1}^{n} \Pi(z_m) \frac{p_m(z)}{p_m(z_m)} \frac{z}{z_m},$$

is zero for $z = z_j$:

$$R(z_j) = \Pi(z_j) - I_p p(z_j) - \sum_{m=1}^{n} \Pi(z_m) \frac{p_m(z_j)}{p_m(z_m)} \frac{z_j}{z_m} = 0, \quad j = 0, \ldots, n,$$

because $\Pi(0) = I_p,\ p(0) = 1,\ p(z_j) = 0$ and $p_m(z_j) = p_m(z_m) 1_{\{j=m\}}, j = 1, \ldots, m.$ Hence each entry of $R(z)$ has $p(z)z$ as a factor so that $R(z) = p(z) z \Pi^*(z)$ for some matrix polynomial $\Pi^*(z)$. ■

Next we give an expansion around two points, where we can prescribe the values of the function and the first derivative at one of the points. We denote by $\dot{\Pi}(z)$ the derivative and by $\ddot{\Pi}(z)$ the second derivative with respec to to z. A similar result can be given for more than two points and higher derivatives, Johansen (1931).

Theorem 2 *Let $z_1 \neq z_2$. The polynomial has the expansion*

$$\begin{aligned}\Pi(z) &= \frac{(z-z_1)^2}{(z_2-z_1)^2}\Pi(z_2) + (1 - \frac{(z-z_1)^2}{(z_2-z_1)^2})\Pi(z_1) \\ &\quad +(z-z_1)\frac{(z-z_2)}{(z_1-z_2)}\dot{\Pi}(z_1) + (z-z_1)^2(z-z_2)\Pi^*(z)\end{aligned}$$

for some matrix polynomial $\Pi^*(z)$.

PROOF. The matrix polynomial

$$R(z) = \Pi(z) - \frac{(z-z_1)^2}{(z_2-z_1)^2}\Pi(z_2) - (1 - \frac{(z-z_1)^2}{(z_2-z_1)^2})\Pi(z_1) - (z-z_1)\frac{(z-z_2)}{(z_1-z_2)}\dot{\Pi}(z_1),$$

satisfies $R(z_1) = R(z_2) = 0$ and $\dot{R}(z_1) = 0$. Thus each entry of $R(z)$ has the factor $(z-z_2)(z-z_1)^2$, so that $R(z) = (z-z_2)(z-z_1)^2\Pi^*(z)$, as was to be proved. ■

Corresponding to the two types of expansions in Theorems 1 and 2, we next give a representation of the inverse polynomial under various assumptions on the roots. We apply the notation

$$\Pi(z) = \Pi_m + (z-z_m)\dot{\Pi}_m + \frac{1}{2}(z-z_m)^2\ddot{\Pi}_m + (z-z_m)^3\Pi^*(z), \tag{3}$$

and in case the root is $z = 1$, we leave out the subscript m.

These results have essentially been derived before, see Johansen and Schaumburg (1998) and Nielsen (2005b). We give the extra term C^* in the expansion (7), see Rahbek and Mosconi (1999) for the $I(1)$ model, and as a new result a complete expression for C_1 in (13) and a partial result on C_0, (14), for the $I(2)$ model. For a $p \times r$ matrix of rank $r(< p)$ we use the notation $\bar{a} = a(a'a)^{-1}$, and $a_\perp$ is a $p \times (p-r)$ matrix of rank $p - r$ for which $a'a_\perp = 0$. We say that the root z_m has multiplicity n, if $\det(\Pi(z)) = (1 - z/z_m)^n g(z),\ g(z_m) \neq 0$.

Theorem 3 *Let z_0 be a root of the characteristic polynomial of multiplicity* n *and let* $\Pi_0 = -\alpha\beta'$ *of rank* r. *If the $I(1)$ condition*

$$|\alpha_\perp' \dot{\Pi}_0 \beta_\perp| \neq 0 \tag{4}$$

is satisfied, then the multiplicity n *equals* $\mathrm{p} - \mathrm{r}$ *and*

$$\Pi^{-1}(z) = C\frac{1}{1 - z/z_0} + C^* + (1 - z/z_0)H(z),\ 0 < |z - z_0| < \delta, \tag{5}$$

for some $\delta > 0$, where $H(z)$ is regular and has no singularity at $z = z_0$ and

$$C = -z_0^{-1}\beta_\perp(\alpha_\perp'\dot{\Pi}_0\beta_\perp)^{-1}\alpha_\perp', \tag{6}$$

$$C^* = -\bar{\beta}\bar{\alpha}' + C\dot{\Pi}_0\bar{\beta}\bar{\alpha}' + \bar{\beta}\bar{\alpha}'\dot{\Pi}_0 C - C(\dot{\Pi}_0\bar{\beta}\bar{\alpha}'\dot{\Pi}_0 + \frac{1}{2}\ddot{\Pi}_0)C. \tag{7}$$

The proof is given in the Appendix. Note that the root z_0 need not be unstable, and that there may be other roots in the characteristic equation. This implies that we can only expand $H(z)$ in a neighborhood of z_0. When we apply the result to autoregressive processes we need to take into account the position of the other roots:

Corollary 4 *Let* $\{z_m, m = 1, \ldots, n\}$ *be* n *roots of the characteristic polynomial of multiplicity* $\mathrm{n}_m, m = 1, \ldots, n$ *and let* $\Pi(z_m) = -\alpha_m \beta_m'$ *of rank* r_m*. If the* $I(1)$ *condition (4) is satisfied for all* m*, then* $\mathrm{n}_m = \mathrm{p} - \mathrm{r}_m$ *and*

$$\Pi^{-1}(z) = \sum_{m=1}^{n} C_m \frac{1}{1 - z/z_m} + K(z), \tag{8}$$

for some $\delta > 0$*, where* $K(z)$ *is has no singularities for* $z = z_1, \ldots, z_n$ *and* C_m *is given by (6) with* $(\alpha, \beta, z_0, \dot{\Pi}_0)$ *replaced by* $(\alpha_m, \beta_m, z_m, \dot{\Pi}_m)$*.*

PROOF. By repeated application of Theorem 3 we can remove the poles z_m, $m = 1, \ldots, n$. ■

If we remove all the unstable roots, then $H(z)$ is regular on $|z| < \rho_{stable}$, and we can hence expand around $z = 0$, with exponentially decreasing coefficients because $\rho_{stable} > 1$. These coefficients will be used to define a stationary process in the proof of Granger's representation theorem.

Finally let us consider the case where the $I(1)$ condition (4) fails. For notational reasons we formulate the $I(2)$ result in Theorem 5 only for the root $z = 1$. We let $\Pi(1) = -\alpha\beta'$ have rank $r < p$, but also assume that $\alpha_\perp' \dot{\Pi} \beta_\perp = \xi\eta'$, has reduced rank $s < p - r$, so that the $I(1)$ condition (4) is not satisfied. We define the mutually orthogonal directions

$$(\beta, \beta_1, \beta_2) = (\beta, \bar{\beta}_\perp \eta, \beta_\perp \eta_\perp) \text{ and } (\alpha, \alpha_1, \alpha_2) = (\alpha, \bar{\alpha}_\perp \xi, \alpha_\perp \xi_\perp). \tag{9}$$

Theorem 5 *Let $z = 1$ be a root so that $\Pi(1) = -\alpha\beta'$ of rank* r, *and assume that $\alpha'_\perp \dot{\Pi} \beta_\perp = \xi\eta'$ of rank s. Then, if the $I(2)$ condition*

$$|\alpha'_2[\dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi} + \frac{1}{2}\ddot{\Pi}]\beta_2| \neq 0 \tag{10}$$

is satisfied at $z = 1$, it holds that

$$\Pi^{-1}(z) = C_2 \frac{1}{(1-z)^2} + C_1 \frac{1}{1-z} + C_0 + (1-z)H(z),\ 0 < |z-1| < \delta, \tag{11}$$

where $H(z)$ is regular for $|z-1| < \delta$, and C_1 and C_2 are given by

$$C_2 = \beta_2(\alpha'_2\theta\beta_2)^{-1}\alpha'_2, \tag{12}$$

$$\begin{aligned} C_1 &= -\bar{\beta}_1\bar{\alpha}'_1 + [\bar{\beta}_1\bar{\alpha}'_1\theta - \bar{\beta}\bar{\alpha}'\dot{\Pi}]C_2 + C_2[\theta\bar{\beta}_1\bar{\alpha}'_1 - \dot{\Pi}\bar{\beta}\bar{\alpha}'] \\ &\quad + C_2[\dot{\Pi}\bar{\beta}\bar{\alpha}'\theta + \theta\bar{\beta}\bar{\alpha}'\dot{\Pi} - \theta\bar{\beta}_1\bar{\alpha}'_1\theta - \dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi} + \frac{1}{6}\dddot{\Pi}]C_2, \end{aligned} \tag{13}$$

where $\theta = \dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi} + \frac{1}{2}\ddot{\Pi}$. Finally C_0 satisfies

$$\beta' C_0 \alpha = -I_r + \bar{\alpha}'\Gamma C_2 \Gamma \bar{\beta}. \tag{14}$$

Note that C_1 satisfies

$$\beta' C_1 = -\bar{\alpha}'\dot{\Pi}C_2, \qquad C_1\alpha = -C_2\dot{\Pi}\bar{\beta}, \tag{15}$$

$$\beta_1' C_1 = -\bar{\alpha}_1'(I_p - \theta C_2), \quad \beta_1' C_1 \alpha_1 = -I_s. \tag{16}$$

The proof is given in the Appendix. Note that in (36) of the proof appears the relation

$$|\Pi(z)| = (1-z)^{p-r}|K(z)||A'B|^{-1}.$$

This is relation (i) $(\det(G(\lambda)) = \lambda^r g(\lambda),\ \lambda = 1-z)$ from Lemma 1 in Engle and Granger (1987)[2]. The $I(1)$ condition is equivalent to $g(0) = |K||A'B|^{-1} \neq 0$, or that the multiplicity of the root is the dimension minus the rank of Π.

Thus the idea of applying a general theorem about inversion of polynomials to discuss the solution of the error correction model is found in Engle and Granger (1987). The approach taken here gives some information on the conditions under which the matrix function can be inverted and leads to the explicit formulae for some of the coefficients. We start with the error correction model which we usually fit to data, and then find conditions under which the solution is $I(1)$ or $I(2)$. The approach taken by Granger was to start with an $I(1)$ process, which exhibits cointegration, and then derive the (infinite lag) error correction model that it satisfies. In both cases the proof of the result involves the inversion of the matrix function. For $I(d)$ processes similar

[2] A counter example to Lemma 1 in the paper by Engle and Granger (1987) is given by taking $G(\lambda) = diag(1+\lambda, \lambda^2, \lambda^2)$. What is missing in Lemma 1 is a condition corresponding to the $I(1)$ condition of cointegration (see Johansen 1996, Theorem 4.5.)

results were developed by la Cour (1998). A systematic exposition of the mathematics behind the representations for $I(1)$ and $I(2)$ processes is given by Faliva and Zoia (2006).

Another type of proof of these results applies the Smith form of a matrix polynomial, which is a reduction of a matrix polynomial to diagonal form using matrix polynomials with determinant one, so that the question of inverting the matrix is reduced to discussing a diagonal matrix, see Engle and Yoo (1991) or Ahn and Reinsel (1994) and the survey by Haldrup and Salmon (1998). In this context, the $I(1)$ condition is most naturally given as the assumption that the number of unit roots of $\Pi(z)$ equals p minus the rank of $\Pi(1)$. The equivalence of the two conditions is given in Johansen (1996, Corollary 4.3), see also Anderson (2003) for a simple proof. Yet another way of considering these results is via the Jordan representation of the companion form, where the $I(1)$ condition is formulated as the absence of Jordan blocks (corresponding to the unit root) of order two in the companion form of the matrix polynomial, see Archontakis (1998), Gregoire (1999), Bauer and Wagner (2005), and Hansen (2005) for further details.

3 The autoregressive model and its solution

The simplest example of the relation between the solution of the autoregressive equations and the inverse of the characteristic polynomial is given by the classical representation of a stationary autoregressive process, see for instance Anderson (1974).

Theorem 6 *If the roots of the characteristic equation are stable, then* $\Pi^{-1}(z)$ *has no singularities for* $|z| < \rho_{stable}$. *The expansion*

$$\Pi^{-1}(z) = \sum_{i=0}^{\infty} C_i z^i, \ |z| < \rho_{stable}$$

defines exponentially decreasing coefficients $\{C_i\}_{i=0}^{\infty}$. *The stationary solution of the equation* $\Pi(L)X_t = \varepsilon_t$ *is given by*

$$X_t = \sum_{i=0}^{\infty} C_i \varepsilon_{t-i}.$$

3.1 Error correction models

The error (or equilibrium) correction formulation is a convenient way of rewriting the autoregressive equations when one takes into account the existence of roots. We apply the results of Theorem 1 and Theorem 2 to formulate two error correction models.

Corollary 7 *Let* $z = z_m, \ m = 1, \ldots, n$ *be distinct roots of* $\Pi(z)$, *and let* $\Pi(z_m) = -\alpha_m \beta'_m$. *Then the process* X_t *satisfies the error correction model:*

$$p(L)X_t = \sum_{m=1}^{n} \alpha_m \beta'_m \frac{p_m(L)}{z_m p_m(z_m)} X_{t-1} + p(L) \sum_{i=1}^{k-n} \Gamma_i X_{t-i} + \varepsilon_t, \tag{17}$$

for some matrix coefficients $\Gamma_i, i = 1, \ldots, k-n$, *where* $p(z)$ *and* $p_m(z)$ *are given by (2).*

PROOF. This follows from Theorem 1 by expanding around $\Pi(z)$ around $0, z_1, \ldots, z_n$.

■

Corollary 8 *Let $z = 1$ be a root so that $\Pi(1) = -\alpha\beta'$. Then the process X_t satisfies the error correction model*

$$\Delta^2 X_t = \alpha\beta' X_{t-1} - \Gamma\Delta X_{t-1} + \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-i} + \varepsilon_t, \tag{18}$$

where $\Gamma = -\dot{\Pi} - \alpha\beta'$, for some matrix coefficients $\Psi_i, i = 1, \ldots, k-2$.

Proof. This follows by choosing $z_1 = 1$ and $z_2 = 0$ in Theorem 2. ∎

Note that the formulation (18) has no assumption about the process being $I(2)$, only that $z = 1$ is a root. We next turn to the solution of the error correction models where we have to be more precise about the roots and the assumptions on the coefficients.

3.2 Granger's representation theorem

The next result is a representation of the solution of $\Pi(L)X_t = \varepsilon_t$. The equations determine the process X_t as a function of the errors ε_i $(i = 1, \ldots, t)$ and the initial values of the process, but the integration properties of X_t depends on the unstable roots.

3.2.1 I(1) processes

We first give a general result for $I(1)$ processes and then apply it to some special cases

Theorem 9 *Let $z_m,\ m = 1, \ldots, n$ be the distinct unstable roots of $\Pi(z)$. Then, if the $I(1)$ condition is satisfied for all the unstable roots, X_t is non-stationary, but $p(L)X_t$ and $p_m(L)\beta_m' X_t$ can be made stationary with mean zero by a suitable choice of initial* *values. In this case, X_t has the representation*

$$X_t = \sum_{m=1}^{n} C_m S_{mt} + \sum_{m=1}^{n} z_m^{-t} A_m + Y_t, \ t = 1, \ldots, T, \tag{19}$$

where $S_{mt} = z_m^{-t} \sum_{i=1}^{t} z_m^i \varepsilon_i$, Y_t is stationary, C_m is given by (6), and finally A_m depends on initial values and are chosen so that $\beta_m' A_m = 0$. If $z_m = 1$ is the only unstable root we find

$$X_t = C \sum_{i=1}^{t} \varepsilon_i + C^* \varepsilon_t + \Delta Z_t + A, \tag{20}$$

where Z_t is stationary, and C and C^* are given by (6) and (7), and $\beta' A = 0$.

PROOF. Under the $I(1)$ condition it follows from (8), by multiplying by $p(z)$, that

$$p(z)\Pi^{-1}(z) = \sum_{m=1}^{n} C_m p_m(z) + p(z)H(z), \ |z| < \rho_{stable},$$

for some regular function H with no singularities for $|z| < \rho_{stable}$ because we have removed all the unstable roots. We then expand $H(z) = \sum_{i=0}^{\infty} C_i^* z^i, \ |z| < \rho_{stable}$ with exponentially decreasing coefficients and define the stationary process $Y_t = \sum_{i=0}^{\infty} C_i^* \varepsilon_{t-i}$.

We find, by multiplying $\Pi(L)X_t = \varepsilon_t$ by $p(L)\Pi^{-1}(L)$, that X_t satisfies the difference equation

$$p(L)X_t = \sum_{m=1}^{n} C_m p_m(L)\varepsilon_t + p(L)Y_t. \tag{21}$$

We first show that $X_t^* = \sum_{m=1}^{n} C_m S_{mt} + Y_t$ is a solution of this equation. Because the process $S_{mt} = z_m^{-t} \sum_{i=1}^{t} z_m^i \varepsilon_i$ is a solution of the equation $(1 - L/z_m)S_{mt} = \varepsilon_t$, and

$p(z) = (1 - z/z_m)p_m(z)$, we get that

$$p(L)X_t^* = \sum_{m=1}^{n} C_m p_m(L)(1 - L/z_m)S_{mt} + p(L)Y_t = \sum_{m=1}^{n} C_m p_m(L)\varepsilon_t + p(L)Y_t.$$

The complete solution is then given by adding the solution X_t^0 to the homogeneous equation $p(L)X_t^0 = 0$, which is given by $X_t^0 = \sum_{m=1}^n z_m^{-t} A_m$ for arbitrary coefficient matrices A_m. Thus $X_t = X_t^* + X_t^0$ represents all possible solutions to (21). We see that $p(L)X_t = p(L)X_t^*$ is stationary for any choice of coefficients A_m, and next turn to the stationarity of $p_k(L)\beta_k' X_t$

$$p_k(L)\beta_k' X_t = \sum_{m=1}^{n} \beta_k' C_m p_k(L) S_{mt} + \sum_{m=1}^{n} p_k(L) z_m^{-t} \beta_k' A_m + p_k(L)\beta_k' Y_t. \tag{22}$$

We use the relation $p_k(z) = p_{km}(z)(1 - z/z_m)$, $p_{km}(z) = \prod_{j \neq k,m}(1 - z/z_j)$ and show that the first term is stationary:

$$\sum_{m=1}^{n} \beta_k' C_m p_k(L) S_{mt} = \sum_{\substack{m=1 \\ m \neq k}}^{n} \beta_k' C_m p_{km}(L)(1 - L/z_m) S_{mt} = \sum_{\substack{m=1 \\ m \neq k}}^{n} \beta_k' C_m p_{km}(L)\varepsilon_t$$

because $\beta_k' C_k = 0$ and $(1 - L/z_m)S_{mt} = \varepsilon_t$. The second term of (22) is

$$p_k(L) z_k^{-t} \beta_k' A_k + \sum_{\substack{m=1 \\ m \neq k}}^{n} p_{km}(L)(1 - L/z_m) z_m^{-t} \beta_k' A_m = p_k(z_k) z_k^{-t} \beta_k' A_k,$$

because $(1 - L/z_m)z_m^{-t} = 0$, and $p_k(L)z_k^{-t} = p_k(z_k)z_k^{-t}$. Hence, if we choose $\beta_k' A_k = 0$, the second term vanishes. Finally the third term of (22) is stationary, so that $p_k(L)\beta_k' X_t$

is stationary with mean zero. If $z_m = 1$ is the only stable root we get from (5) the representation (20) in the same way. ■

We next give some special cases for well known error correction models.

The $I(1)$ model. When $z = 1$ is the only unstable root we get the usual $I(1)$ model where the error correction form is given by (17) in Corollary 7 for $z_m = 1$, so that $p(L) = 1 - L = \Delta$, and the error correction model becomes

$$\Delta X_{t-1} = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t. \tag{23}$$

The $I(1)$ condition (4) asserts that $|\alpha'_\perp \Gamma \beta_\perp| \neq 0$, where $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$. We find the representation (20). For inference in this model, see for example Johansen (1996). Note that from (20) it follows that ΔX_t and $\beta' X_t = \beta' C^* \varepsilon_t + \beta' \Delta Y_t$ are stationary. In fact $\beta' C^* \alpha = -I_r$, so that $\beta' C^* \neq 0$ and $\beta' X_t$ is $I(0)$, and no linear combination of X_t can be $I(-1)$. Similarly there can not be multi cointegration in the $I(1)$ model in the sense that X_t cannot cointegrate in a non-trivial sense with $\sum_{i=1}^t \beta' X_i$, see Granger and Lee (1990). In case $\gamma'_1 \sum_{i=1}^t \beta' X_i + \gamma'_2 X_t$ were stationary, we would have $\gamma'_1 \beta' C^* + \gamma'_2 C = 0$. Multiplying by α we get $0 = \gamma'_1 \beta' C^* \alpha = -\gamma'_1$, so that $\gamma_1 = 0$. For details see Engsted and Johansen (1999).

The model for seasonal roots. The $I(1)$ model for quarterly seasonal roots is found if there are roots at $z = \pm 1, \pm i$, see Hylleberg, Engle, Granger and Yoo (1990), Ahn and Reinsel (1994), and Schaumburg and Johansen (1998), where inference is discussed. We illustrate here the simpler case of half year data with roots at $z = \pm 1$, see Lee (1992).

We define

$$\Pi(1) = -\alpha_1\beta_1', \quad \Pi(-1) = -\alpha_{-1}\beta_{-1}'.$$

The error correction form of the equation is given by (17) in Corollary 7 with $z_1 = 1, z_2 = -1$. Because

$$p(L) = (1-L)(1+L) = (1-L^2), \ p_1(L) = (1+L), \ p_{-1}(L) = (1-L)$$

we find the error correction equation

$$(1-L^2)X_t = \frac{1}{2}(1+L)\alpha_1\beta_1'X_{t-1} - \frac{1}{2}(1-L)\alpha_{-1}\beta_{-1}'X_{t-1} + (1-L^2)\sum_{i=1}^{k-2}\Pi_i^* X_{t-i} + \varepsilon_t$$

and the solution, if the $I(1)$ condition (4) is satisfied at $z = \pm 1$, is given by (19), which in this case becomes

$$X_t = C_1\sum_{i=1}^{t}\varepsilon_i + C_{-1}(-1)^t\sum_{i=1}^{t}(-1)^i\varepsilon_i + A_1 + (-1)^t A_{-1} + Y_t,$$

where C_1 and C_{-1} are given by (6) for $z_m = \pm 1$, and $\beta_1' A_1 = 0$ and $\beta_{-1}' A_{-1} = 0$.

The non-stationary behavior of the process X_t is governed by the two processes

$$S_{1t} = \sum_{i=1}^{t}\varepsilon_i \text{ and } S_{-1t} = (-1)^t\sum_{i=1}^{t}(-1)^i\varepsilon_i.$$

The first is a random walk, and $\sum_{i=1}^{t}(-1)^i\varepsilon_i$ is also a random walk, if ε_t has a symmetric distribution, but the factor $(-1)^t$ shows that S_{-1t} is not a random walk. It is a non-stationary process that satisfies the basic equation $(1+L)S_{-1t} = \varepsilon_t$, so that yearly aggregates of $S_t^{(-1)}$ are stationary, whereas $(1-L)S_{1t} = \varepsilon_t$ shows that the difference within a year of S_{1t} is stationary.

The interpretation of cointegration in this case is not so simple. What we see is a process, which is non-stationary and even the difference $X_t - X_{t-1}$ is non-stationary. If we take instead the yearly data $X_t^{year} = X_t + X_{t-1}$, then the difference

$$\Delta X_t^{year} = X_t + X_{t-1} - (X_{t-1} + X_{t-2}) = X_t - X_{t-2} = (1 - L^2)X_t$$

is stationary. Cointegration means that even though $X_t - X_{t-1}$ is non-stationary, there are linear combinations β_{-1} for which $\beta'_{-1}(X_t - X_{t-1})$ is stationary. Similarly there are combinations β_1 for which $\beta'_1(X_t + X_{t-1})$ is stationary.

Models for explosive roots. If the characteristic polynomial has one real explosive root $\lambda < 1$, and a root at $z = 1$, then the error correction model is found from (17) by choosing $z_1 = 1,\ z_2 = \lambda$ gives

$$\begin{aligned} p(L) &= (1-L)(1-\lambda^{-1}L) = \Delta_1\Delta_\lambda, \\ p_1(L) &= (1-\lambda^{-1}L), \quad p_1(1) = 1-\lambda^{-1}, \\ p_\lambda(L) &= (1-L), \quad p_\lambda(\lambda) = 1-\lambda. \end{aligned}$$

The error correction model is given by

$$\Delta_1\Delta_\lambda X_t = \frac{1}{(1-1/\lambda)}\alpha_1\beta_1'\Delta_\lambda X_{t-1} + \frac{1}{\lambda(1-\lambda)}\alpha_\lambda\beta_\lambda'\Delta_1 X_{t-1} + \sum_{i=1}^{k-2}\Pi_i^*\Delta_1\Delta_\lambda X_{t-i} + \varepsilon_t.$$

The process has the representation, see (19),

$$X_t = C_1\sum_{i=1}^{t}\varepsilon_i + C_\lambda\lambda^{-t}\sum_{i=1}^{t}\lambda^i\varepsilon_i + \lambda^{-t}A_\lambda + A_1 + Y_t,$$

provided the $I(1)$ condition (4) is satisfied at $z = 1$ and $z = \lambda$. Note that the process is non-stationary because of the random walk, but this time there is also an explosive process $\lambda^{-t}\sum_{i=0}^{t}\lambda^i\varepsilon_i$. The sum $\sum_{i=0}^{t}\lambda^i\varepsilon_i$ actually converges for $t \to \infty$, and the explosiveness is due to the factor $\lambda^{-t} \to \infty$. Note that $\beta_1'X_t$ is not stationary. It is explosive and only $(1 - L/\lambda)\beta_1'X_t$ is stationary. Similarly $\beta_\lambda'X_t$ is not stationary but $I(1)$. For inference the main references are the papers by Nielsen (2002, 2005a, 2005b), see also Anderson (1959).

3.2.2 I(2) processes

We next turn to $I(2)$ processes and assume that $z = 1$ is a root, so that $\Pi = -\alpha\beta'$ of rank r, but $\alpha_\perp'\dot{\Pi}\beta_\perp = \xi\eta'$ of rank s, so that the $I(1)$ condition is not satisfied. This determines orthogonal directions $(\beta, \beta_1, \beta_2)$ and $(\alpha, \alpha_1, \alpha_2)$, which are needed in the formulation of the results, see (9).

Theorem 10 *Let $z = 1$ be the only unstable root, and assume that the $I(1)$ condition (4) does not hold. If the $I(2)$ condition (10) is satisfied for $z = 1$, then X_t is non-*

stationary, but $\Delta^2 X_t$, $(\beta, \beta_1)'\Delta X_t$, *and* $\beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t$ *can be made* $I(0)$ *by a suitable choice of initial values, and then* X_t *has the representation*

$$X_t = C_2 \sum_{i=1}^{t} \sum_{j=1}^{i} \varepsilon_j + C_1 \sum_{i=1}^{t} \varepsilon_i + Y_t + A + Bt, \tag{24}$$

where C_1 *and* C_2 *are given by (12) and (13),* Y_t *is* $I(0)$ *and* A *and* B *depend on initial values and are chosen so that* $\beta' A + \bar{\alpha}'\dot{\Pi} B = 0$ *and* $(\beta, \beta_1)' B = 0$*. The linear combination* $\beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t$ *is called the polynomial cointegrating relation.*

PROOF. We follow the proof of Theorem 9. From (11) we find

$$(1-z)^2 \Pi^{-1}(z) = C_2 + C_1(1-z) + (1-z)^2 H(z), \ |z| < \rho_{stable},$$

where $H(z)$ is a regular function with no singularities for $|z| < \rho_{stable}$, and hence given by $H(z) = \sum_{i=0}^{\infty} C_i^* z^i, |z| < \rho_{stable}$, where C_i^* are exponentially decreasing. We define the stationary process $Y_t = \sum_{i=0}^{\infty} C_i^* \varepsilon_{t-i}$, and note that $C_0 = \sum_{i=0}^{\infty} C_i^* \neq 0$ because of (14), so that Y_t is $I(0)$. Next apply $\Delta^2 \Pi^{-1}(L)$ to $\Pi(L) X_t = \varepsilon_t$. We then find the difference equation

$$\Delta^2 X_t = C_2 \varepsilon_t + C_1 \Delta\varepsilon_t + \Delta^2 Y_t, \ t = 1, \dots, T.$$

The process

$$X_t^* = C_2 \sum_{i=1}^{t} \sum_{j=1}^{i} \varepsilon_j + C_1 \sum_{i=1}^{t} \varepsilon_i + Y_t$$

is clearly a solution, and the complete solution is found by adding the solution to the homogenous equation $\Delta^2 X_t^0 = 0$, which is given by $X_t^0 = A + Bt$, that is, any solution has the form $X_t = X_t^* + A + Bt$. It follows that $\Delta^2 X_t = \Delta^2 X_t^*$ is stationary with mean zero for any choice of A and B. We want to choose A and B so that $(\beta, \beta_1)'\Delta X_t$, and $\beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t$ become stationary with mean zero. We find

$$(\beta, \beta_1)'\Delta X_t = (\beta, \beta_1)' C_1 \varepsilon_t + (\beta, \beta_1)'\Delta Y_t + (\beta, \beta_1)' B,$$

which is stationary if B is chosen so that $(\beta, \beta_1)' B = 0$. Moreover it is $I(0)$ because, by (16), $(\beta, \beta_1)' C_1 \alpha_1 = (0, -I_s) \neq 0$. Next

$$\begin{aligned} \beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t &= \beta' C_1 \sum_{i=1}^t \varepsilon_i + \beta' Y_t + \beta' A + \beta' Bt + \bar{\alpha}'\dot{\Pi}(C_2 \sum_{i=1}^t \varepsilon_i + C_1 \varepsilon_t + \Delta Y_t + B) \\ &= \beta' Y_t + \bar{\alpha}'\dot{\Pi} C_1 \varepsilon_t + \bar{\alpha}'\dot{\Pi}\Delta Y_t + \beta' A + \bar{\alpha}'\dot{\Pi} B, \end{aligned}$$

because $\beta' B = 0$ and $\beta' C_1 + \bar{\alpha}'\dot{\Pi} C_2 = 0$, see (15). We next choose $\beta' A + \bar{\alpha}'\dot{\Pi} B = 0$, that is, $A = -\bar{\beta}\bar{\alpha}'\dot{\Pi} B$. This completes the proof of the representation (24).

Note that $\Delta^2 X_t$ is $I(0)$ because $C_2 \neq 0$, $(\beta, \beta_1)'\Delta X_t$ is $I(0)$ because $(\beta, \beta_1)' C_1 \neq 0$, and finally $\beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t$ is $I(0)$ because the sum of the coefficients is $\beta' C_0 + \bar{\alpha}'\dot{\Pi} C_1 \neq 0$, because (14) and (15) show that

$$\begin{aligned} \beta' C_0 \alpha + \bar{\alpha}'\dot{\Pi} C_1 \alpha &= -I_r + \bar{\alpha}'\Gamma C_2 \Gamma \bar{\beta} - \bar{\alpha}'\dot{\Pi} C_2 \dot{\Pi}\bar{\beta} \\ &= -I_r + \bar{\alpha}'\Gamma C_2 \Gamma \bar{\beta} - \bar{\alpha}'\Gamma C_2 \Gamma \bar{\beta} = -I_r. \end{aligned}$$

■

For $I(2)$ processes we usually apply the error correction model (18), where Γ satisfies $\alpha_\perp'\Gamma\beta_\perp = \xi\eta'$. Inference can be found in Boswijk (2000), Johansen (1997, 2006) and Paruolo (1996, 2000). Note that the representations (19) and (24) are chosen so that it follows directly from (19) that ΔX_t and $\beta' X_t$ are stationary, and from (24) it is seen that $\Delta^2 X_t, (\beta, \beta_1)'\Delta X_t$ and $\beta' X_t + \bar{\alpha}'\dot{\Pi}\Delta X_t$ are stationary.

4 Fractional processes

In this section we discuss some models for cofractional processes that have been analyzed in the literature and suggest a modification of the model by Granger (1986), which admits a feasible representation theory. We present here the first results for this new model and they will be followed up in a companion paper Johansen (2007), with the subsequent aim of developing model based inference for fractional and cofractional processes based upon the model.

4.1 Basic definitions

The concept of the fractional difference was introduced by Granger and Joyeux (1980) and Hosking (1981) to discuss processes with a long memory. The fractional operator is defined by the expansion

$$\Delta^d = (1-L)^d = \sum_{k=0}^{\infty} (-1)^k \binom{-d}{k} L^k,$$

and we shall use the definition

$$\Delta_+^d = \sum_{k=0}^{t-1} (-1)^k \binom{-d}{k} L^k,$$

so that $\Delta_+^d X_t = \Delta^d X_t 1\{t \geq 0\}$, see for instance Robinson and Iacone (2005). We denote by $I(0)_+$ the class of asymptotically stationary processes of the form

$$X_t = 1\{t \geq 0\} \sum_{i=0}^{t-1} C_i \varepsilon_{t-i},$$

where $0 < tr(\sum_{i=0}^{\infty} C_i)'(\sum_{i=0}^{\infty} C_i) < \infty$.

Definition 11 *We say that Y_t is fractional of order d, and write $Y_t \in \mathcal{F}(d)$ if $\Delta_+^d Y_t - \mu_t \in I(0)_+$ for some function, μ_t, of the initial values.*

Definition 12 *If $Y_t \in \mathcal{F}(d)$ with $d > 0$ and there exists a linear combination $\beta \neq 0$ so that $\beta' Y_t \in \mathcal{F}(d-b)$ for $0 < b \leq d$ we call Y_t cofractional, $\mathcal{F}(d,b)$, with cofractional vector β, and the number of linearly independent such vectors is the cofractional rank.*

4.2 Some models for cofractional processes

A parametric model for cofractional processes, which satisfactorily describes the variation of the data, should ideally have the properties

- The model is a dynamic model that generates $\mathcal{F}(d,b)$ processes
- The cofractional relations as well as the adjustment towards these are modelled

- The models for different cofractional rank should be nested
- A representation theory for the solution should allow a discussion of the properties of the process, depending on the parameter values

If we can find such a model we can conduct inference by deriving estimators and tests of relevant hypotheses on long-run and short-run parameters, based upon the Gaussian likelihood function.

The first parametric model for cofractional processes $\mathcal{F}(d,b)$ was suggested by Granger (1986) as a modification of the usual error correction model:

$$A^*(L)\Delta^d X_t = -(1-\Delta^b)\Delta^{d-b}\gamma\alpha' X_{t-1} + d(L)\varepsilon_t,$$

Here $A^*(L)$ is a lag polynomial and $d(L)$ is a scalar function. If we set $d(L) = 1$ and change X_{t-1} to X_t and $-\gamma\alpha'$ to $\alpha\beta'$, we get the model

$$A^*(L)\Delta^d X_t = (1-\Delta^b)\Delta^{d-b}\alpha\beta' X_t + \varepsilon_t. \tag{25}$$

This is a dynamic parametric model which models both short-run and long-run parameters. The model was applied by Davidson (2002), and Lasak (2005) derives a likelihood ratio test for the hypothesis $r = 0$ when $d = 1$ and finds the asymptotic distribution.

Most authors consider the simpler regression type model, which explicitly models the order of cofractionality:

$$\begin{aligned} X_{1t} &= \gamma' X_{2t} + \Delta_+^{-d+b} u_{1t} \\ X_{2t} &= \Delta_+^{-d} u_{2t}, \end{aligned} \tag{26}$$

where u_t is a general invertible $I(0)$ process. Here X_1 has dimension r and X_2 dimension $p-r$.

There is a large literature, based upon (26), aimed at developing (a semiparametric) estimation theory for fractional and cofractional processes, see for instance Robinson and Marinucci (2001), Chen and Hurvich (2003) or Nielsen and Shimotsu (2006) to mention a few. In this literature the purpose is mainly to conduct inference on the order of fractionality and test for the occurrence of cofractionality.

The relations between (26) and (25) is best seen by defining the matrix $\alpha = (-I_r, 0)'$ and $\beta = (I_r, -\gamma')'$. Then (26) can be written in an error correction form as follows:

$$\Delta^d X_t = (1 - \Delta^b)\Delta^{d-b}\alpha\beta' X_t + v_t, \tag{27}$$

where $v_t = (u_{1t}' + u_{2t}'\gamma, u_{2t}')'$.

For parametric inference in (26), a model for v_t is needed, and Sowell (1989) suggested that v_t could be modelled as an invertible and stationary ARMA model and derived a computable expression for the variance of the fractional stationary process. For this model Dueker and Startz (1998) applied Gaussian maximum likelihood estimation, using the results by Sowell.

Note that for an ARMA error $A(L)v_t = B(L)\varepsilon_t$, with ε_t i.i.d.$(0,\Omega)$, we get the model

$$A(L)\Delta^d X_t = A(L)(1-\Delta^b)\Delta^{d-b}\alpha\beta' X_t + B(L)\varepsilon_t,$$

which is close to the error correction formulation (25).

Breitung and Hassler (2002) suggested to analyze model (27), where α is an unknown parameter and construct test for the rank of α and β based on the assumption that d is known, and v_t is an AR process $(B(L) = I_p)$ with i.i.d. Gaussian errors.

The representation theory for model (26) is easy, as $X_t \in \mathcal{F}(d,b)$, and $X_{1t} - \gamma' X_{2t} = \beta' X_t \in \mathcal{F}(d-b)$. The adjustment α, however, is assumed known, and so is the cofractional rank r, so the models for different cofractional rank are not nested.

The representation theory for Granger's model (25) is not so easy due to the occurrence of both the lag operator and the fractional 'lag' operator $L_b = 1 - \Delta^b$, which appears natural for fractional processes. We therefore propose a different model

$$\Delta^d X_t = (1-\Delta^b)\alpha\beta'\Delta^{d-b} X_t + \sum_{i=1}^{k} \Gamma_i \Delta^d (1-\Delta^b)^i X_t + \varepsilon_t. \tag{28}$$

In this model the polynomial in the lag operator is replaced by a polynomial in the fractional lag operator. This model satisfies the requirements for a parametric model for cofractional processes, as we shall now show that there is a feasible representation theory.

4.3 A representation theorem for cofractional processes

We associate with (28) the (regular) characteristic function

$$\Pi(z) = (1-z)^d I_p - \alpha\beta'(1-(1-z)^b)(1-z)^{d-b} - \sum_{i=1}^{k} \Gamma_i (1-(1-z)^b)^i (1-z)^d$$

but also the polynomial

$$\Pi^*(u) = (1-u)I_p - \alpha\beta' u - \sum_{i=1}^{k} \Gamma_i u^i (1-u),$$

which is related to $\Pi(z)$ via the substitution $u = 1-(1-z)^b$:

$$\Pi^*(u) = (1-z)^{b-d}\Pi(z).$$

Note that $(1-z)^b$ and hence $\Pi(z)$ has a singularity at $z = 1$, unless d and b are nonnegative integers. By introducing the variable $u = 1-(1-z)^b$ the function $(1-z)^{b-d}\Pi(z)$ becomes a polynomial and the conditions on the roots are in terms of this polynomial, which is the characteristic polynomial for the usual $I(1)$ model, see (23).

We want to find the solution of equation (28) under the $I(1)$ condition for $\Pi^*(u)$. A condition for the process to be fractional of order d and cofractional of order $d-b$ is given in Johansen (2007). We define $\Gamma = I_p - \sum_{i=1}^{k} \Gamma_i$.

Theorem 13 *If $u = 1$ is the only unstable root of $|\Pi^*(u)| = 0$, if $\Pi^*(1) = -\alpha\beta'$, and* *if the $I(1)$ condition holds*

$$|\alpha_\perp' \Gamma \beta_\perp| \neq 0, \tag{29}$$

then for $|z| < \delta$,

$$\Pi(z)^{-1} = \sum_{i=0}^{n} D_i (1-z)^{-d+bi} + (1-z)^{-d+b(n+1)} \sum_{i=0}^{\infty} D_i^* z^i, \tag{30}$$

where $n = [(d-b)/b]$ is chosen so that $-d+nb < 0 \leq -d+(n+1)b$. The first two matrices are given by

$$\begin{aligned} D_0 &= \beta_\perp (\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp', & (31) \\ D_1 &= -[\bar{\beta}\bar{\alpha}' + \bar{\beta}\bar{\alpha}'\Gamma D_0 + D_0 \Gamma \bar{\beta}\bar{\alpha}' + D_0(\Gamma\bar{\beta}\bar{\alpha}\Gamma + \sum_{i=1}^{k} i\Gamma_i) D_0]. & (32) \end{aligned}$$

The solution of $\Pi(L)X_t = \varepsilon_t$ has the representation

$$X_t = \sum_{i=0}^{n} D_i \Delta_+^{-d+ib} \varepsilon_t + \Delta_+^{-d+(n+1)b} Y_t^{\prime +} + \mu_t, \quad t = 1, 2, \ldots \tag{33}$$

where the first $n+1$ terms are fractional of order $d, d-b, \ldots, d-nb$ and $Y_t^{\prime +} = \sum_{i=0}^{t-1} D_i^* \varepsilon_{t-i}$. The function μ_t depends on initial values. If $Y_t \in I(0)_+$, it follows that X_t is $\mathcal{F}(d)$ and $\beta' X_t$ is $\mathcal{F}(d-b)$.

PROOF. Under the $I(1)$ condition (29), $\Pi^*(u)$ can be inverted using (5) with $z_0 = 1$, $D_0 = C$, $D_1 = C^*$, so that

$$(1-u)\Pi^*(u)^{-1} = D_0 + D_1(1-u) + (1-u)^2 H(u)$$

is a regular function for $|u| < \rho_{stable}$. We expand as

$$(1-u)\Pi^*(u)^{-1} = \sum_{i=0}^{n} D_i(1-u)^i + (1-u)^{n+1}\sum_{i=0}^{\infty} C_i u^i, \ |u| < \rho_{stable},$$

where n is chosen so that $-d+nb < 0 \le -d+(n+1)b$, and the coefficients C_i are exponentially decreasing. For z small we find that

$$\begin{aligned}(1-z)^d\Pi(z)^{-1} &= \sum_{i=0}^{n} D_i(1-z)^{ib} + (1-z)^{(n+1)b}\sum_{i=0}^{\infty} C_i(1-(1-z)^b)^i \\ &= \sum_{i=0}^{n} D_i(1-z)^{ib} + (1-z)^{(n+1)b}\sum_{i=0}^{\infty} D_i^* z^i, \end{aligned} \qquad (34)$$

which proves (30). Here D_0 and D_1 are given by (6) and (7), and substituting

$$\dot{\Pi} = -\alpha\beta' - \Gamma, \ \ \frac{1}{2}\ddot{\Pi} = \sum_{i=1}^{k} i\Gamma_i$$

we get (31) and (32).

In order to prove (33), we write (28) as

$$\varepsilon_t = \Pi(L)X_t = \Pi_+(L)X_t + \Pi_-(L)X_t$$

and multiply by $\Pi(L)_+^{-1}$ to find

$$\Pi(L)_+^{-1}\varepsilon_t = X_t - \mu_t, \qquad (35)$$

where $\mu_t = -\Pi_+(L)^{-1}\Pi_-(L)X_t$ is a function of initial values of X_t. We let $Y_t^+ =$

$\sum_{i=0}^{t-1} D_i^* \varepsilon_{t-i}$ and find the representation (33)

$$X_t = \mu_t + \Pi(L)_+^{-1} \varepsilon_t = \mu_t + \sum_{i=0}^{n} D_i \Delta_+^{-d+ib} \varepsilon_t + \Delta_+^{-d+(n+1)b} Y_t^+.$$

It follows that

$$\Delta_+^d X_t = \Delta_+^d \mu_t + D_0 \varepsilon_t + D_1 \Delta_+^b \varepsilon_t + D_2 \Delta_+^{2b} \varepsilon_t + \cdots + D_n \Delta_+^{nb} \varepsilon_t + \Delta_+^{(n+1)b} Y_t^+.$$

If $Y_t^+ \in I(0)_+$, we see that X_t is fractional of order d because $D_0 \neq 0$, and that $\beta' X_t$ is fractional of order $d - b$ because $\beta' D_0 = 0$, and $\beta' D_1 \neq 0$. ■

The representation in (33) for $d = b = 1$, and hence $n = 0$, is

$$X_t = D_0 \sum_{i=1}^{t} \varepsilon_i + \sum_{i=0}^{t-1} D_i^* \varepsilon_{t-i} + \mu_t, \quad t = 1, 2, \ldots$$

This should be compared with (20), where we have chosen a representation in terms of the stationary process $C^* \varepsilon_t + \Delta Z_t$ and given a more explicit discussion of the role of the initial values, rather than just the term μ_t.

5 Conclusion

This paper discusses the connection between the solution to the autoregressive equations and the inverse of the generating function of the coefficients under different conditions on the coefficient matrices. We apply these results to find representations of cointegrated vector autoregressive processes which are $I(1)$, $I(2)$, seasonally integrated and explosive.

For cofractional processes we suggest a modification of the model suggested by Granger (1986) for which the above results can be applied to give a representation theory.

6 Appendix

Proof of Theorem 3. There is no loss of generality in assuming that $z_0 = 1$, as the polynomial $\Pi(uz_0)$ has a root at $u = 1$, and satisfies the $I(1)$ condition at $u = 1$, if $\Pi(z)$ does at $z = z_0$. Note that the derivative for $u = 1$ is $z_0\dot{\Pi}(z_0)$.

We therefore assume that $z_0 = 1$ and leave out the subscript 0 for notational convenience. Let $\Pi = -\alpha\beta'$ have rank r, and let C and C^* be given by (6) and (7). Note that $\Pi(z)^{-1}$ is a regular function with singularities at the zeros of the characteristic equation. We want to show that the function

$$H(z) = (\Pi(z)^{-1} - C\frac{1}{1-z} - C^*)/(1-z),\ 0 < |z-1| < \delta,$$

extended by continuity at $z = 1$ is regular for $|z-1| < \delta$.

Multiplying (3) by $A = (\bar{\alpha}, \alpha_\perp)$ and $B = (\bar{\beta}, \beta_\perp)$ we obtain

$$\begin{aligned} A'\Pi(z)B &= \begin{pmatrix} -I_r + (z-1)\dot{\Pi}_{00} & (z-1)\dot{\Pi}_{01} \\ (z-1)\dot{\Pi}_{10} & (z-1)\dot{\Pi}_{11} \end{pmatrix} + \frac{1}{2}(z-1)^2 \begin{pmatrix} \ddot{\Pi}_{00} & \ddot{\Pi}_{01} \\ \ddot{\Pi}_{10} & \ddot{\Pi}_{11} \end{pmatrix} \\ &\quad +(z-1)^3 A'\Pi^*(z)B, \end{aligned}$$

where $\dot{\Pi}_{00} = \bar{\alpha}'\dot{\Pi}\bar{\beta}$, $\dot{\Pi}_{10} = \alpha_\perp'\dot{\Pi}\bar{\beta}$, etc. We can factorize $(z-1)$ from the last $p-r$ columns by multiplying from the right by

$$F(z) = \begin{pmatrix} I_r & 0 \\ 0 & (z-1)^{-1}I_{p-r} \end{pmatrix},$$

and we find that

$$K(z) = A'\Pi(z)BF(z) = K + (z-1)\dot{K} + (z-1)^2K^*(z)$$

where

$$K = \begin{pmatrix} -I_r & \dot{\Pi}_{01} \\ 0 & \dot{\Pi}_{11} \end{pmatrix}, \quad \dot{K} = \begin{pmatrix} \dot{\Pi}_{00} & \frac{1}{2}\ddot{\Pi}_{01} \\ \dot{\Pi}_{10} & \frac{1}{2}\ddot{\Pi}_{11} \end{pmatrix}.$$

From

$$|K(z)| = |A'B||\Pi(z)|(1-z)^{-(p-r)}, \quad z \neq 1, \tag{36}$$

it is seen that $K(z)$ has the same roots as $\Pi(z)$, except for $z = 1$, where $|K(1)| = |K| = |\dot{\Pi}_{11}| = |\alpha_\perp'\dot{\Pi}\beta_\perp| \neq 0$, if the $I(1)$ condition (4) is satisfied for $z = 1$.

It follows that the multiplicity of the root at $z = 1$ is $p - r$, and that $K(z)$ is invertible for $|z-1| < \varepsilon$ for some small δ :

$$K^{-1}(z) = K^{-1} - (z-1)K^{-1}\dot{K}K^{-1} + (z-1)^2H_1(z)$$

where $H_1(z)$ is regular for $|z-1|<\delta$, and

$$K^{-1} = \begin{pmatrix} -I_r & \dot{\Pi}_{01}\dot{\Pi}_{11}^{-1} \\ 0 & \dot{\Pi}_{11}^{-1} \end{pmatrix}$$

$$K^{-1}\dot{K}K^{-1} = \begin{pmatrix} \dot{\Pi}_{00} - \dot{\Pi}_{01}\dot{\Pi}_{11}^{-1}\dot{\Pi}_{10} & (-\theta_{01} + \dot{\Pi}_{01}\dot{\Pi}_{11}^{-1}\theta_{11})\dot{\Pi}_{11}^{-1} \\ -\dot{\Pi}_{11}^{-1}\dot{\Pi}_{10} & \dot{\Pi}_{11}^{-1}\theta_{11}\dot{\Pi}_{11}^{-1} \end{pmatrix}$$

with

$$\theta_{ij} = \dot{\Pi}_{i0}\dot{\Pi}_{0j} + \frac{1}{2}\ddot{\Pi}_{ij},\ \ i,j=0,1. \tag{37}$$

Multiplying by $F(z)$ from the left we find

$$F(z)K^{-1}(z) = (z-1)^{-1}M_{-1} + M_0 + (z-1)H_2(z),\ \ z \neq 1$$

with

$$M_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \dot{\Pi}_{11}^{-1} \end{pmatrix},\ M_0 = \begin{pmatrix} -I_r & \dot{\Pi}_{01}\dot{\Pi}_{11}^{-1} \\ \dot{\Pi}_{11}^{-1}\dot{\Pi}_{10} & -\dot{\Pi}_{11}^{-1}\theta_{11}\dot{\Pi}_{11}^{-1} \end{pmatrix}$$

for a function $H_2(z)$ which is regular for $|z-1|<\delta$. Therefore, multiplying by B and A', we find

$$\begin{aligned} \Pi^{-1}(z) &= BF(z)(A'\Pi(z)BF(z))^{-1}A' = BF(z)K(z)^{-1}A' \\ &= (z-1)^{-1}\beta_\perp\dot{\Pi}_{11}^{-1}\alpha_\perp' - \bar{\beta}\bar{\alpha}' + \beta_\perp\dot{\Pi}_{11}^{-1}\dot{\Pi}_{10}\bar{\alpha}' + \bar{\beta}\dot{\Pi}_{01}\dot{\Pi}_{11}^{-1}\alpha_\perp' - \beta_\perp\dot{\Pi}_{11}^{-1}\theta_{11}\dot{\Pi}_{11}^{-1}\alpha_\perp' \\ &\quad +(z-1)BH_2(z)A' \\ &= C(1-z)^{-1} - \bar{\beta}\bar{\alpha}' + C\dot{\Pi}\bar{\beta}\bar{\alpha}' + \bar{\beta}\bar{\alpha}'\dot{\Pi}C - C(\dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi} + \frac{1}{2}\ddot{\Pi})C + (z-1)H_3(z), \end{aligned}$$

where C is given in (6) and $H_3(z)$ is a regular function for $|1-z|<\delta$. Thus the pole at $z=1$ can be removed by subtracting the term $C(1-z)^{-1}$. This proves (5), (6), and (7).⊡

Proof of Theorem 5. We define $A=(\bar{\alpha},\bar{\alpha}_1,\alpha_2)$ and $B=(\bar{\beta},\bar{\beta}_1,\beta_2)$. Because $\Pi=-\alpha\beta'$ and $\alpha'_\perp\dot{\Pi}\beta_\perp=\xi\eta'$ a calculation will show that

$$A'\Pi(z)B \;=\; \begin{pmatrix} -I_r+(z-1)\dot{\Pi}_{00} & (z-1)\dot{\Pi}_{01} & (z-1)\dot{\Pi}_{02} \\ (z-1)\dot{\Pi}_{10} & (z-1)I_s & 0 \\ (z-1)\dot{\Pi}_{20} & 0 & 0 \end{pmatrix}$$
$$+\frac{1}{2}(z-1)^2\{\ddot{\Pi}_{ij}\}+\frac{1}{6}(z-1)^3\{\dddot{\Pi}_{ij}\}+(z-1)^4\Pi^*(z).$$

Next we multiply from the right by the matrix

$$F(z)=\begin{pmatrix} I_r & 0 & (z-1)^{-1}\dot{\Pi}_{02} \\ 0 & (z-1)^{-1}I_s & 0 \\ 0 & 0 & (z-1)^{-2}I_{p-r-s} \end{pmatrix}$$

to get

$$K(z)=A'\Pi(z)BF(z)=K+(z-1)\dot{K}+(z-1)^2\Pi_1^*(z)$$

where

$$K = \begin{pmatrix} -I_r & \dot{\Pi}_{01} & \dot{\Pi}_{00}\dot{\Pi}_{02} + \frac{1}{2}\ddot{\Pi}_{02} \\ 0 & I_s & \dot{\Pi}_{10}\dot{\Pi}_{02} + \frac{1}{2}\ddot{\Pi}_{12} \\ 0 & 0 & \dot{\Pi}_{20}\dot{\Pi}_{02} + \frac{1}{2}\ddot{\Pi}_{22} \end{pmatrix}$$

$$\dot{K} = \begin{pmatrix} \dot{\Pi}_{00} & \frac{1}{2}\ddot{\Pi}_{01} & \frac{1}{2}\ddot{\Pi}_{00}\dot{\Pi}_{02} + \frac{1}{6}\dddot{\Pi}_{02} \\ \dot{\Pi}_{10} & \frac{1}{2}\ddot{\Pi}_{11} & \frac{1}{2}\ddot{\Pi}_{10}\dot{\Pi}_{02} + \frac{1}{6}\dddot{\Pi}_{12} \\ \dot{\Pi}_{20} & \frac{1}{2}\ddot{\Pi}_{21} & \frac{1}{2}\ddot{\Pi}_{20}\dot{\Pi}_{02} + \frac{1}{6}\dddot{\Pi}_{22} \end{pmatrix}.$$

Note that K has full rank because $|\dot{\Pi}_{20}\dot{\Pi}_{02} + \frac{1}{2}\ddot{\Pi}_{22}| = |\alpha_2'(\dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi} + \frac{1}{2}\ddot{\Pi})\beta_2| \neq 0$. For $0 < |z-1| < \delta$

$$K^{-1}(z) = K^{-1} - (z-1)K^{-1}\dot{K}K^{-1} + (z-1)^2 H_1(z),$$

where $H_1(z)$ is regular for $|z-1| < \delta$.

With θ_{ij} given in (37) we find

$$K = \begin{pmatrix} -I_r & \dot{\Pi}_{01} & \theta_{02} \\ 0 & I_s & \theta_{12} \\ 0 & 0 & \theta_{22} \end{pmatrix}, \; K^{-1} = \begin{pmatrix} -I_r & \dot{\Pi}_{01} & (\theta_{02} - \dot{\Pi}_{01}\theta_{12})\theta_{22}^{-1} \\ 0 & I_s & -\theta_{12}\theta_{22}^{-1} \\ 0 & 0 & \theta_{22}^{-1} \end{pmatrix},$$

so that

$$F(z)K^{-1}(z) = (1-z)^{-2}M_{-2} + (1-z)^{-1}M_{-1} + M_0 + (1-z)H_2(z)$$

where $H_2(z)$ is regular for $|z-1|<\delta$ and

$$M_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta_{22}^{-1} \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} 0 & 0 & -\dot{\Pi}_{02}\theta_{22}^{-1} \\ 0 & -I_s & \theta_{12}\theta_{22}^{-1} \\ -\theta_{22}^{-1}\dot{\Pi}_{20} & \theta_{22}^{-1}\theta_{21} & \phi \end{pmatrix},$$

$$\phi = \theta_{22}^{-1}[\dot{\Pi}_{20}\dot{\Pi}_{00}\dot{\Pi}_{02} + \frac{1}{2}\dot{\Pi}_{20}\ddot{\Pi}_{02} + \frac{1}{2}\ddot{\Pi}_{20}\dot{\Pi}_{02} - \theta_{21}\theta_{12} + \frac{1}{6}\dddot{\Pi}_{22}]\theta_{22}^{-1}.$$

The matrix M_0 is rather complicated, but it suffices to find that it has the form

$$M_0 = \begin{pmatrix} -I_r + \dot{\Pi}_{02}\theta_{22}^{-1}\dot{\Pi}_{20} & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

Finally we use

$$\Pi(z)^{-1} = BF(z)(A'\Pi(z)BF(z))^{-1}A' = BF(z)K(z)^{-1}A'$$

to find

$$\begin{aligned} C_2 &= \beta_2\theta_{22}^{-1}\alpha_2', \\ C_1 &= -\bar{\beta}_1\bar{\alpha}_1' + (\bar{\beta}_1\theta_{12} - \bar{\beta}\dot{\Pi}_{02})\theta_{22}^{-1}\alpha_2' + \beta_2\theta_{22}^{-1}(\theta_{21}\bar{\alpha}_1' - \dot{\Pi}_{20}\bar{\alpha}') + \beta_2\phi\alpha_2', \\ \beta' C_0\alpha &= -I_r + \bar{\alpha}'\dot{\Pi}C_2\dot{\Pi}\bar{\beta} \end{aligned}$$

which reduces to (12), (13), and (14). This completes the proof of Theorem 5.⊡

7 References

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