The equivalence of two parametrizations for the $I(2)$ model and an example of the conditions for asymptotic χ^2 inference

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1 The parametrization of Boswijk.

We here discuss the parametrization introduced by Boswijk, which normalizes β on a matrix c, and η on a matrix d. We first prove the following lemma

Lemma 1 *Without loss of generality we can assume that the parameters satisfy the normalizations*

$$c'\beta = I_r, c_\perp'\beta_\perp = I_{p-r}, d'\eta = I_s, d_\perp'\eta_\perp = I_{p-r-s}, \tag{1}$$

where the normalization matrices satisfy the conditions

$$c'c = I_r, c_\perp'c_\perp = I_{p-r}, d'd = I_s, d_\perp'd_\perp = I_{p-r-s}. \tag{2}$$

PROOF. We choose a value of the parameters $\alpha^0, \beta^0, \Gamma^0, \xi^0, \eta^0$ satisfying the $I(2)$ condition. Because $\text{rank}(\beta^0) = r$, we can find a permutation matrices S, so that with $c' = (I_r, 0)S$ we have $|c'\beta^0| \neq 0$. Similarly we can find a permutation matrix S_0 so that with $d' = (I_s, 0)S_0$ we get $|d'\eta^0| \neq 0$. For this choice of S and S_0, consider the subset

$$\Theta_0 = \{\beta, \eta, \ldots : |c'\beta| \neq 0, |d'\eta| \neq 0\}.$$

This is not the full parameter space, but the complement of Θ_0, where either $|c'\beta| = 0$ or $|d'\eta| = 0$, has Lebesgue measure zero. Thus "without loss of generality" we can take $(\beta, \eta) \in \Theta_0$.

We further define

$$c_\perp = S'\begin{pmatrix} 0 \\ I_{p-r} \end{pmatrix}, d_\perp = S_0'\begin{pmatrix} 0 \\ I_{p-r-s} \end{pmatrix}$$

so that (2) holds, and apply these to define the normalized parameters

$$\beta_c = \beta(c'\beta)^{-1},\ \alpha_c = \alpha\beta'c,\ \eta_d = \eta\,(d'\eta)^{-1},\ \xi_d = \xi\eta'd,$$
$$\beta_{c\perp} = (I_p - c\beta_c')c_\perp,\ \eta_{d\perp} = (I_{p-r} - d\eta_d')d_\perp$$

so that (1) holds. This completes the proof of Lemma 2 ∎

In the following we assume that Lemma 1 is satisfied and call the parameters $\alpha_c, \beta_c, \Gamma, \xi_d$, and η_d, to indicate the way they are normalized. We let $c_1 = c_\perp d$ and $c_2 = c_\perp d_\perp$ and introduce the parameters A_i, see Boswijk (2000, page 898), as

$$A_1 = -\beta_c' c_1,\ A_2 = -\beta_c' c_2,\ A_3 = -\bar{\alpha}_c' \Gamma \beta_{c2},\ A_4 = -\eta_d' d_\perp.$$

The relation between the parameters $\beta_c, \beta_{c\perp}, \eta_d, \eta_{d\perp}$, and the parameters A_i is given by

$$\begin{aligned} \beta_c' &= c' - A_1 c_1' - A_2 c_2' = c' - (A_1 d' + A_2 d_\perp')c_\perp', \\ \beta_{c\perp} &= c_\perp + c(A_1 d' + A_2 d_\perp'), \\ \eta_d' &= d' - A_4 d_\perp', \\ \eta_{d\perp} &= d_\perp + dA_4, \end{aligned} \tag{3}$$

from which we find

$$\begin{aligned} &\beta_{c2} = \beta_{c\perp}\eta_{d\perp} = c(A_1A_4 + A_2) + c_1 A_4 + c_2, \\ &\beta_{c1} = \beta_{c\perp}(\beta_{c\perp}'\beta_{c\perp})^{-1}\eta_d = (c_\perp + c(A_1 d' + A_2 d_\perp'))(\beta_{c\perp}'\beta_{c\perp})^{-1}\eta_d, \\ &\tau_c = \beta_{2c\perp} = (c - c_2(A_4'A_1' + A_2'), c_1 - c_2 A_4'). \end{aligned} \tag{4}$$

In the present paper we have normalized the parameters differently. We assume that β and τ are normalized on $\bar{\beta}^0$ and $\bar{\tau}^0$. Thus our parameters can be recovered from $\alpha_c, \beta_c, \ldots$ or A_i as follows:

$$\begin{aligned} &\beta = \beta_c(\bar{\beta}_c^{0\prime}\beta_c)^{-1} = (c - c_1A_1' - c_2A_2')(\bar{\beta}_c^{0\prime}\beta_c)^{-1}, \\ &\tau = \tau_c(\bar{\tau}_c^{0\prime}\tau_c)^{-1} = (c - c_2(A_4'A_1' + A_2'), c_1 - c_2A_4')(\bar{\tau}_c^{0\prime}\tau_c)^{-1}. \end{aligned}$$

Note that $\beta^0 = \beta_c^0$, and we choose $\beta_1 = \beta_{c1}$ and $\beta_2 = \beta_{c2}$. We can now prove

Lemma 2 *The first order approximation of the parameters (B_0, B_1, B_2, C) in terms of A_i are given by*

$$\begin{aligned} B_0 &\sim \left(\beta_2^{0\prime}\beta_2^0\right)^{-1}[(A_3' - A_3^{0\prime}) - (A_4' - A_4^{0\prime})(A_1^{0\prime}c' + c_1')\psi^0] \\ B_1 &\sim -A_1' + A_1^{0\prime} \\ B_2 &\sim -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4^{0\prime}(A_1' - A_1^{0\prime}) + (A_2' - A_2^{0\prime})) \\ C &\sim -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4' - A_4^{0\prime})\rho_\perp^{0\prime}\rho_\perp^0 \end{aligned} \tag{5}$$

where $\sim$ indicates that terms with higher order have been dropped.

Proof. We find from (36)

$$\begin{aligned} B_2 = \bar{\beta}_2^{0\prime}(\beta - \beta^0) &= -(\beta_2^{0\prime}\beta_2^0)^{-1}(\beta_2^{0\prime}c_1(A_1' - A_1^{0\prime}) + \beta_2^{0\prime}c_2(A_2' - A_2^{0\prime}))(\bar{\beta}_c^{0\prime}\beta_c)^{-1} \\ &= -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4^{0\prime}(A_1' - A_1^{0\prime}) + (A_2' - A_2^{0\prime}))(\bar{\beta}_c^{0\prime}\beta_c)^{-1}, \end{aligned}$$

using the relations $\beta_2^{0\prime}c_2 = I_{p-r-s}$ and $\beta_2^{0\prime}c_1 = A_4^{0\prime}$. It is seen that the parameter $(A_4^{0\prime}(A_1' - A_1^{0\prime}) + (A_2' - A_2^{0\prime}))(\bar{\beta}_c^{0\prime}\beta_c)^{-1}$ is the non-linear combination of A_i parameters which is equivalent to B_2, which is normalized by T^2 when estimated. Because $\bar{\beta}_c^{0\prime}\beta_c \sim I_r$ close to β^0 we find

$$B_2 \sim -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4^{0\prime}(A_1' - A_1^{0\prime}) + (A_2' - A_2^{0\prime}))$$

below we apply this result to replace $A_2' - A_2^{0\prime}$ by $-A_4^{0\prime}(A_1' - A_1^{0\prime})$ in the expansions.

Similarly we find

$$B_1 = \bar{\beta}_1^{0\prime}(\beta - \beta^0) = -(\beta_1^{0\prime}\beta_1^0)^{-1}(\beta_1^{0\prime}c_1(A_1' - A_1^{0\prime}) + \beta_1^{0\prime}c_2(A_2' - A_2^{0\prime}))(\bar{\beta}_c^{0\prime}\beta_c)^{-1}.$$

replacing $A_2' - A_2^{0\prime}$ by $-A_4^{0\prime}(A_1' - A_1^{0\prime})$ and $\bar{\beta}_c^{0\prime}\beta_c$ by I_r, we find

$$B_1 \sim -(\beta_1^{0\prime}\beta_1^0)^{-1}(\beta_1^{0\prime}c_1 - \beta_1^{0\prime}c_2\beta_2^{0\prime}c_1)(A_1' - A_1^{0\prime})$$

From (4) we find

$$\beta_1^{0\prime}c_1 = \eta^{0\prime}(\beta_\perp^{0\prime}\beta_\perp^0)^{-1}d,\ \beta_1^{0\prime}c_2 = \eta^{0\prime}(\beta_\perp^{0\prime}\beta_\perp^0)^{-1}d_\perp,\ \beta_2^{0\prime}c_1 = A_4^{0\prime},$$

so that the coefficient matrix becomes

$$\eta^{0\prime}(\beta_\perp^{0\prime}\beta_\perp^0)^{-1}(d - d_\perp A_4^{0\prime}) = \eta^{0\prime}(\beta_\perp^{0\prime}\beta_\perp^0)^{-1}\eta^0 = \beta_1^{0\prime}\beta_1^0,$$

so that with $\bar{\beta}_c^{0\prime}\beta_c \sim I_r$ we find

$$B_1 \sim -A_1' + A_1^{0\prime}.$$

Next we find an expression for C :

$$\begin{aligned} C &= \bar{\beta}_2^{0\prime}(\tau - \tau^0)\rho_\perp = \bar{\beta}_2^{0\prime}\tau_c(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho_\perp \\ &= -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4'A_1' + A_2' - (A_4^{0\prime}A_1^{0\prime} + A_2^{0\prime}), A_4' - A_4^{0\prime})(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho_\perp \end{aligned}$$

We replace $A_2' - A_2^{0\prime}$ by $-A_4^{0\prime}(A_1' - A_1^{0\prime})$ and find

$$\begin{aligned} C &\sim -(\beta_2^{0\prime}\beta_2^0)^{-1}((A_4' - A_4^{0\prime})A_1', A_4' - A_4^{0\prime})(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho_\perp \\ &= -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4' - A_4^{0\prime})(A_1', I_s)(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho_\perp. \end{aligned}$$

We next want to show that $\rho_\perp' = (A_1', I_s)(\bar{\tau}^{0\prime}\tau_c)^{-1}$. We find from $\beta_c = \tau_c\rho_c$, that

$$\rho_c = (c, c_1)'\beta_c = \begin{pmatrix} I_r \\ -A_1' \end{pmatrix}, \rho_{c\perp} = \begin{pmatrix} A_1 \\ I_s \end{pmatrix}.$$

Similarly we find from $\beta = \tau\rho$, by multiplying by $(c, c_1)'$ that

$$(I_r, -A_1)(\bar{\beta}^{0\prime}\beta_c)^{-1} = (c, c_1)'\beta_c(\bar{\beta}^{0\prime}\beta_c)^{-1} = (c, c_1)'\tau_c(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho = (\bar{\tau}^{0\prime}\tau_c)^{-1}\rho,$$

and hence that

$$\rho_\perp = \left(\tau_c'\bar{\tau}^0\right)^{-1}\begin{pmatrix} A_1 \\ I_s \end{pmatrix}.$$

Thus we find

$$(A_1', I_s)(\bar{\tau}^{0\prime}\tau_c)^{-1}\rho_\perp = \rho_\perp'\rho_\perp \sim \rho_\perp^{0\prime}\rho_\perp^0$$

and the expression for C becomes

$$C \sim -(\beta_2^{0\prime}\beta_2^0)^{-1}(A_4' - A_4^{0\prime})\rho_\perp^{0\prime}\rho_\perp^0.$$

Finally we find B_0 : The relation between A_3 and $B_0 = \bar{\beta}_2^{0\prime}(\psi - \psi^0)$ is given by

$$\begin{aligned} A_3 &= -\bar{\alpha}'\Gamma\beta_2 = \bar{\alpha}'(\alpha\psi' + \bar{\alpha}_\perp\kappa'\tau')\beta_2 = \psi'\beta_2 \\ &= (\psi - \psi^0)'\beta_2^0 + \psi'(\beta_2^0 - \beta_2) + \psi^{0\prime}\beta_2^0 \\ &= B_0'(\beta_2^{0\prime}\beta_2^0) + \psi'(\beta_2^0 - \beta_2) + A_3^0, \end{aligned}$$

so that

$$\begin{aligned} B_0 &= \left(\beta_2^{0\prime}\beta_2^0\right)^{-1}[(A_3' - A_3^{0\prime}) + (\beta_2 - \beta_2^0)'\psi] \\ &\sim \left(\beta_2^{0\prime}\beta_2^0\right)^{-1}[(A_3' - A_3^{0\prime}) - c(A_1A_4 + A_2 - A_1^0A_4^0 - A_2^0) - c_1(A_4 - A_4^0))'\psi^0] \end{aligned}$$

Replacing $A_2 - A_2^0$ by $-(A_1 - A_1^0)A_4^0$ we find

$$\begin{aligned} B_0 &\sim \left(\beta_2^{0\prime}\beta_2^0\right)^{-1}[(A_3' - A_3^{0\prime}) - (cA_1(A_4 - A_4^0) + c_1(A_4 - A_4^0))'\psi^0] \\ &\sim \left(\beta_2^{0\prime}\beta_2^0\right)^{-1}[(A_3' - A_3^{0\prime}) - (A_4 - A_4^0)'(A_1^{0\prime}c' + c_1')\psi^0]. \end{aligned}$$

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2 A simple example.

Let us consider a very stylized example to see what the conditions imply. We let θ_2 bivariate and θ_1 and θ_0 be univariate. We define

$$\begin{aligned} \theta_0(\phi_0, \phi_1, \phi_2) &= \phi_0 \\ \theta_1(\phi_0, \phi_1, \phi_2) &= \phi_1 + d\phi_2 \\ \theta_2(\phi_0, \phi_1, \phi_2) &= \left\{ \begin{array}{l} \phi_2 + a\phi_1^2 + b\phi_1 \\ c\phi_0^2 \end{array} \right. \end{aligned}$$

We assume that $\phi^0 = 0$, and consider the behaviour of θ_i under the local alternative $\phi_0 = T^{-1/2}\eta_0, \phi_1 = T^{-1}\eta_1, \phi_2 = T^{-2}\eta_2$. We normalize θ_i and find

$$\begin{aligned} T^{1/2}\theta_0(T^{-1/2}\eta_0, T^{-1}\eta_1, T^{-2}\eta_2) &= \eta_0 \\ T\theta_1(T^{-1/2}\eta_0, T^{-1}\eta_1, T^{-2}\eta_2) &= \eta_1 + dT^{-1}\eta_2 \\ T^2\theta_2(T^{-1/2}\eta_0, T^{-1}\eta_1, T^{-2}\eta_2) &= \left\{ \begin{array}{l} \eta_2 + a\eta_1^2 + Tb\eta_1 \\ Tc\eta_0^2 \end{array} \right. \end{aligned}$$

Obviously the limit only exists if $b = \partial(\theta_2)_1/\partial\phi_1 = 0$ and $c = \frac{1}{2}\partial^2(\theta_2)_2/\partial\phi_0^2 = 0$. The full set of conditions are collected in (20) and (21). Note that it had no consequences that θ_1 depended on ϕ_2, because the terms with $\partial\theta_1/\partial\phi_2$ disappeared in the limit of a local alternative. If in fact $a = b = 0$, then the limit becomes

$$D_T^{-1} \cdot \theta(D_T \cdot \eta) \to \left(\eta_0, \eta_1, \eta_2 + a\eta_1^2, 0\right)'.$$

Obviously the limit of the likelihood function, which is quadratic in θ_2, will be of degree four in η_1, and if we want a quadratic likelihood function in the limit, we have to assume that also $a = \frac{1}{2}\partial^2(\theta_2)_1/\partial\phi_1^2 = 0$, where the full set of conditions is collected in (21).

Note that the conditions on a and b can always be achieved by a reparametrization. Simply define

$$\tilde{\phi}_2 = \phi_2 + a\phi_1^2 + b\phi_1,\ \tilde{\phi}_1 = \phi_1,\ \tilde{\phi}_0 = \phi_0.$$

Thus for this particular example we need $a = b = 0$, in order to achieve a finite limit of the likelihood function under the local alternative and $c = 0$ to get a limit that is quadratic.

In order to investigate the condition (16), we have to evaluate the derivative, not at the true value but at the local value given by the $(T^{-1/2}\psi_0, T^{-1}\psi_1, T^{-2}\psi_2)$. Let us first find the derivative of $\theta = (\theta_0, \theta_1, \theta_{21}, \theta_{22})'$ at the point $\phi = (\phi_0, \phi_1, \phi_2)'$ in the direction $D_T \cdot \eta$:

$$\frac{d\theta}{d\phi'}(D_T^{-1} \cdot \eta) = \begin{pmatrix} \eta_0 \\ \eta_1 + dT^{-1}\eta_2 \\ (2a\phi_1 + b)T\eta_1 + \eta_2 \\ 2c\phi_0 T^{3/2}\eta_0 \end{pmatrix}$$

Hence at the point $(\phi_0, \phi_1, \phi_2) = (T^{-1/2}\psi_0, T^{-1}\psi_1, T^{-2}\psi_2)$ we find

$$D_T^{-1} \cdot \frac{d\theta}{d\phi'}(D_T \cdot \eta) = \begin{pmatrix} \eta_0 \\ \eta_1 + dT^{-1}\eta_2 \\ (2a\psi_1 + Tb)\eta_1 + \eta_2 \\ 2cT\psi_0\eta_0 \end{pmatrix}$$

We see again that in order for the limit to exist, we need the conditions $b = c = 0$, and for the limit to be independent of ψ we must have $a = 0$.

Thus for this simple example we get that condition (16) is equivalent to the conditions given in (20) and (21)

References

[1] Boswijk, H. P. (2000). Mixed Normality and Ancillarity in $I(2)$ Systems. *Econometric Theory* 16, 878-904.

[2] Johansen, S. (1997). Likelihood analysis of the $I(2)$ model. *Scandinavian Journal of Statistics* 24, 433-462.

[3] Johansen (2004). Statistical analysis of hypotheses on the cointegrating relations in the $I(2)$ model. forthcoming *Journal of Econometrics*