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The interpretation of cointegrating coefficients in the cointegrated vector autoregressive model

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Abstract

Regression coefficients are interpreted by a counterfactual experiment or "design of experiment". For simultaneous equations this experiment can be implemented if the coefficients are identified, and the implementation throws some light on the role of instruments and the method of two stage and indirect least squares. The contribution of this paper is to show that another counterfactual experiment can be conducted in the vector autoregressive model in order to implement the interpretation of the coefficients of an identified cointegrating relation. The idea is to use the dynamics of the model to implement a long-run change by changing the current values. Thus instead of using the exogenous variables to produce a change in the endogenous variables, we change the initial values to obtain a change in the long-run values. The counterfactual experiment can be conducted precisely when the cointegrating relation is identified.

Keywords: Counterfactual experiment, causality, cointegrating relation

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1 Introduction

The purpose of this note is to discuss the interpretation of cointegrating coefficients as elasticities. Regression coefficients are usually interpreted by a

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counterfactual experiment or thought experiment, which does not make immediate sense in the cointegrated VAR model, because we do not have the dichotomy of endogenous and exogenous variables.

The basic idea in this paper is to consider another thought experiment which makes a distinction between what happens in the long run and the short run using the dynamics of the cointegrated model. Thus the dichotomy here is that of current and long-run values. The note is inspired by Lütkepohl (1994), who pointed out that there are problems related to the interpretation of cointegrating coefficients as elasticities.

In Section 2 we discuss by some examples the well known problem of interpreting regressions coefficients, and in Section 3 we then turn to cointegrating coefficients. In the Appendix we have collected a few results on the companion form of the cointegrated vector autoregressive model.

2 The interpretation of regression coefficients

The idea of using counterfactuals as a basis for the concept of causality is of course well known in statistics and econometrics. It was used by Haavelmo (1944), Rubin (1974), Holland (1986), see the survey by Angrist, Imbens and Rubin (1996) and the monograph by Hoover (2001). We discuss here the interpretation of the coefficients in linear regression models, by considering some simple examples.

Example 1. Consider first the univariate regression model

$$y_t = \gamma_1 x_{1t} + \gamma_2 x_{2t} + \varepsilon_t,$$

where we assume that the errors ε_t are independent of the regressors x_t. The usual interpretation of the regression coefficient γ_1, say, is that γ_1 is the causal effect of x_1 on y, keeping x_2 fixed. Thus in order to interpret the coefficient γ_1 we conduct the thought experiment of changing the system by changing x_1, without changing x_2, and observing the change in y. It is implicit in the thought experiment that there is no other relation between y and x, and that we can disregard relations (or correlation) between the $x's$ and finally that we can manipulate x without changing ε (independence between x and ε).

Note that since

$$E(y_t|x_t) = \gamma_1 x_{1t} + \gamma_2 x_{2t},$$

it follows that

$$\frac{\partial}{\partial x_{1t}} E(y_t|x_t) = \gamma_1,$$

which is a convenient formulation of the role of γ_1 as a partial derivative and the thought experiment above describes the definition of a partial derivative.

Example 2. As another example we consider a system of simultaneous equations, which is constructed to illustrate various situations

$$\begin{array}{cccccccc} y_{1t} & -\phi_2 y_{2t} & & = & \gamma_{11} x_{1t} & & +\varepsilon_{1t}, \\ & y_{2t} & -\phi_3 y_{3t} & = & & \gamma_{22} x_{2t} & +\varepsilon_{2t}, \\ & y_{2t} & -\phi_4 y_{3t} & = & \gamma_{31} x_{1t} & +\gamma_{32} x_{2t} & +\varepsilon_{3t}. \end{array} \tag{1}$$

In order to make sure that y is a function of x, and hence endogenous, we assume that $\phi_3 \neq \phi_4$, so that y can be solved for x.
The two first equation are uniquely identified by the zero restrictions if the parameters satisfy Wald's condition

$$\left| \begin{pmatrix} \phi_3 & \gamma_{22} \\ \phi_4 & \gamma_{32} \end{pmatrix} \right| \neq 0, \left| \begin{pmatrix} 1 & \gamma_{11} \\ 0 & \gamma_{31} \end{pmatrix} \right| \neq 0,$$

while the third is not identified as there is only one restriction.
We want to interpret or give a meaning to the coefficient ϕ_2, which is a coefficient to an endogenous variable, as the effect of y_2 on y_1 keeping x_1 fixed.
According to Haavelmo (1944, page 6) the equation, and its coefficients, are defined by a "design of experiment" to measure y_1 as a function of y_2 and x_1, by asking the question "what would be the value for y_1 for various values of y_2 and x_1"?
Because the equation is here considered as part of a system which simultaneously determine all y as functions of x, it is natural to consider another experiment that only allows changes in x, and let y follow according to the equations. Thus this second experiment is a way of implementing the changes imagined in the first experiment. We therefore consider an experiment where we keep x_1 fixed and change x_2 in order to produce the desired effect which is 1 on y_2 and ϕ_2 on y_1. We here use that a change in x_2 has no direct influence on y_1, because the coefficient to x_2 is zero. Hence we use x_2 as an instrument to produce an effect in y_2 through equations 2 and 3, which in turn will help the interpretation of ϕ_2.
If we fix x_1 and change x_2 to $x_2 + h$, then y is shifted to a new position y^h satisfying

$$\begin{array}{cccc} y_1^h & -\phi_2 y_2^h & = & \gamma_{11} x_1, \\ y_2^h & -\phi_3 y_3^h & = & \gamma_{22}(x_2 + h), \\ y_2^h & -\phi_4 y_3^h & = & \gamma_{31} x_1 + \gamma_{32}(x_2 + h). \end{array}$$

Comparing the new position y^h to y, we find

$$\begin{array}{l} y_1^h - y_1 - \phi_2(y_2^h - y_2) = 0, \\ y_2^h - y_2 - \phi_3(y_3^h - y_3) = \gamma_{22} h, \\ y_2^h - y_2 - \phi_4(y_3^h - y_3) = \gamma_{32} h. \end{array}$$

The solution is

$$
\begin{aligned}
y_1^h - y_1 &= \phi_2(y_2^h - y_2), \\
y_2^h - y_2 &= h(\phi_4\gamma_{22} - \phi_3\gamma_{32})/(\phi_4 - \phi_3), \\
y_3^h - y_3 &= h(\gamma_{22} - \gamma_{32})/(\phi_4 - \phi_3).
\end{aligned}
$$

Thus by choosing an increment to x_2 of $h = (\phi_4 - \phi_3)/(\phi_4\gamma_{22} - \phi_3\gamma_{32})$ we find the effect on y_2 is $y_2^h - y_2 = 1$ and the effect on y_1 is $y_1^h - y_1 = \phi_2$ as required.

Thus we can conduct the thought experiment of using the instrument x_2 to produce a change of 1 in y_2, disregarding the change in y_3, thereby producing the required change of ϕ_2 in y_1. Note the role of the exogenous variable x_2 as instrument to produce the required effect, and in a sense implement the "design of experiment" of Haavelmo.

The idea of influencing y_1 via y_2, using x_2 as an instrument, and first discussing the effect of x_2 on y_2 (regressing y_2 on x_2), and then use the effect $h(\phi_4\gamma_{22} - \phi_3\gamma_{32})/(\phi_4 - \phi_3)$ (fitted value $\hat{y}_2$) to measure the effect on y_1 (regression of y_1 on $\hat{y}_2$) is clearly the idea behind the 2SLS by Theil (1953*a*,*b*, 1971), even though it this particular just identified case it is just indirect least squares.

Note also that the condition for identification of the first equation is that $\phi_4\gamma_{22} - \gamma_{32}\phi_3 \neq 0$. Thus in the construction of the thought experiment we apply the assumption that the three equations determine the endogenous variables as functions of the exogenous ($\phi_3 \neq \phi_4$), and the assumption that the first equation is identified ($\phi_4\gamma_{22} - \gamma_{32}\phi_3 \neq 0$).

Another way of expressing the interpretation of ϕ_2, is to note that from (1) it follows that

$$E(y_{1t}|x_{1t}, x_{2t}) = \phi_2 E(y_{2t}|x_{1t}, x_{2t}) + \gamma_{11}x_{1t}.$$

Here $E(y_{1t}|x_{1t}, x_{2t})$ is considered a function of (x_{1t}, x_{2t}) through $E(y_{2t}|x_{1t}, x_{2t})$ and x_{1t}, so that

$$\phi_2 = \frac{\partial E(y_{1t}|x_{1t}, x_{2t})}{\partial E(y_{2t}|x_{1t}, x_{2t})}.$$

The counterfactual experiment then shows how one could in principle implement the change in $E(y_{2t}|x_{1t}, x_{2t})$ keeping x_{1t} fixed, using x_{2t} as an instrument. Note that if x_{2t} had a non-zero coefficient in the equation for y_{1t}, the expectation would be a function of x_{1t} and x_{2t} as well as of $E(y_{2t}|x_{1t}, x_{2t})$, and the counterfactual experiment could not be performed, as one cannot keep x_{1t} and x_{2t} fixed and change $E(y_{2t}|x_{1t}, x_{2t})$. In this case, however, the parameter ϕ_2 would not be uniquely identified, even though it would still have a meaning according to the original "design of experiment".

3 The interpretation of cointegrating coefficients

For the rest of the paper we consider the cointegrated vector autoregressive model in n dimensions with i.i.d. errors as given by the equations

$$\Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \varepsilon_t, t = 1, \ldots, T, \tag{2}$$

where we have left out deterministic terms to simplify the notation. Here α and β are $n \times r$ matrices of full rank. We assume that the process is $I(1)$ so that $|\alpha'_\perp(I_n - \sum_{i=1}^{k-1}\Gamma_i)\beta_\perp| \neq 0$, see Johansen (1996, Theorem 4.2). For $\Gamma = I_n - \sum_{i=1}^{k-1}\Gamma_i$, we define

$$C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp,$$

and find from the solution to the error correction model in the companion form, see (8) in Appendix, the long-run value $x_{\infty|t}$ as a function of current values $(x_t, \ldots, x_{t-k+1})$ as given by

$$x_{\infty|t} = \lim_{h\to\infty} E(x_{t+h}|x_t, \ldots, x_{t-k+1}) = C(x_t - \sum_{i=1}^{k-1}\Gamma_i x_{t-i}). \tag{3}$$

See also Bedini and Mosconi (2000) for a discussion and interpretation of the long-run value.

We first prove a simple result which defines two different thought experiments

Proposition 1 *By adding* h *to all current values* $x_{t-i}, i = 0, \ldots, k-1$, *the long-run value is changed by* $C\Gamma\text{h} \in sp(\beta_\perp)$. *By adding* h *to* x_t *the long-run value is changed by* $C\text{h} \in sp(\beta_\perp)$.
Conversely a given long-run change $\text{k} \in sp(\beta_\perp)$ *can be achieved by either adding* k *to all current values or by adding* Γk *to* x_t.

Proof. It is seen from (3) and the definition of Γ, that the effect of the two types of changes are as indicated and proportional to $\beta_\perp$.
If on the other hand $k \in \text{sp}(\beta_\perp)$ is the desired effect in the long run, then we write $k = \beta_\perp\psi$, for some $\psi \in R^{n-r}$, and find that a simultaneous change to all current values by k will lead to the long-run change

$$C\Gamma k = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp\Gamma\beta_\perp\psi = \beta_\perp\psi = k.$$

If we only change x_t by adding Γk, we find the change $C\Gamma k = k$. ■
In the cointegrated vector autoregressive model there are no exogenous and endogenous variables and there is no natural way in which we can pick out

some variables and interpret the others as functions of them. If we, nevertheless, solve the stationary relations

$$\beta' x_t = \beta_1' x_{1t} + \beta_2' x_{2t} = u_t$$

for x_{1t}, which we can do provided the $r \times r$ matrix β_1 has full rank, or equivalently that x_{2t} does not cointegrate, then we find

$$x_{1t} = \gamma' x_{2t} + v_t,$$

where $v_t = \beta_1'^{-1} u_t$ is stationary and $\gamma' = -\beta_1'^{-1}\beta_2'$. This is the usual regression formulation, and the idea behind the regression estimator introduced by Engle and Granger (1987), and the triangular representation of Phillips (1991). The regressor x_{2t}, however, is not independent of v_t, and the usual interpretation from section 2 has to be slightly modified. For large t the correlation between x_t and u_t is small and the process is approximately given by a random walk: $x_t \approx C\sum_{t=1}^{t}\varepsilon_i$, which again is approximately $N_n(0, t\Sigma)$ with the singular long-run covariance matrix $\Sigma = C\Omega C'$. Because $\beta' C = 0$, we find from

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}' \begin{pmatrix} \Sigma_{11} \, \Sigma_{12} \\ \Sigma_{21} \, \Sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}'$$

that

$$\beta_1' \Sigma_{12} + \beta_2' \Sigma_{22} = 0.$$

If x_{2t} does not cointegrate then Σ_{22} has full rank and we find

$$\gamma' = -\beta_1^{-1\prime} \beta_2' = \Sigma_{12}\Sigma_{22}^{-1},$$

and hence the coefficient γ' has the interpretation as a regression coefficient derived from the singular long-run covariance matrix Σ.

There is, however, a dynamic structure in the vector autoregressive model, which makes a distinction between short run and long run or between current values and long-run values see Proposition 1. We apply this to consider a different type of thought experiment, and use the initial values to implement a change in the long-run value. This thought experiment then gives the possibility to implement the changes in the long run needed for the interpretation of the coefficients instead of, as before, to use a change in the exogenous variables to produce a change in the endogenous variables. We first consider two examples and then give a general result.

Example 3. Suppose the dimension $n = 3$, and

$$\beta' x_t = \beta_1 x_{1t} + \beta_2 x_{2t} + \beta_3 x_{3t} = u_t$$

is the only cointegrating relation in model (2), so that u_t is stationary, whereas in general the components x_{it} are non-stationary. If $\beta_1 \neq 0$, we identify the coefficients by solving for x_{1t}, and find

$$x_{1t} = \gamma_2 x_{2t} + \gamma_3 x_{3t} + v_t. \tag{4}$$

with $\gamma_i = -\beta_i/\beta_1$. We would like to interpret the coefficient γ_2 as the long-run effect of x_2 on x_1, keeping x_3 fixed. That is, we would like to change x_2 by 1, and x_3 by 0, and observe a long-run change (γ_2) in x_1. Because the relation is a long-run relation, that is, can be formulated as $\beta' x_{\infty|t} = 0$, it is natural to think of these changes as changes in the long-run values and the idea is to accomplish these changes by a suitable change in the current values, as formulated by Proposition 1.

We therefore introduce a change to x_t so that in the long run x_2 is shifted by 1 and x_3 is kept constant, while x_1 is shifted by γ_2. That is, we want to change $(x_1, x_2, x_3)_{\infty|t}$ to $(x_1, x_2, x_3)_{\infty|t} + (\gamma_2, 1, 0)$. Now notice that the vector $k = (\gamma_2, 1, 0)'$ is orthogonal to $\beta = (1, -\gamma_2, -\gamma_3)'$:

$$k'\beta = (\gamma_2, 1, 0)\beta = \gamma_2 - \gamma_2 = 0,$$

which means that the desired long-run change is a vector in the space spanned by the columns of C, that is sp$(\beta_\perp)$. In general we can ask the question if it is possible to find a vector h and shift x_t to $x_t + h$ so that the long-run value is shifted from $x_{\infty|t}$ to $x_{\infty|t} + k$, for a given vector k in sp$(\beta_\perp)$. This is exactly the content of Proposition 1, where the relation between initial change and long-run change is given, and it is seen that we can choose $h = \Gamma k$, and shift x_t to $x_t + \Gamma k$.

We use this result to interpret the coefficient γ_2 in the identified cointegrating relation (4) as the effect of x_2 on x_1, in the sense that a change of x_t by $\Gamma(\gamma_2, 1, 0)'$ will lead to a long-run change of the system where $x_{1\infty|t}$ is changed to $x_{1\infty|t} + \gamma_2$, $x_{2\infty|t}$ to $x_{2\infty|t} + 1$, and $x_{3\infty|t}$ is kept fixed, as desired. In this sense the coefficients of a cointegrating relation can be considered long-run elasticities, provided the variables are measured in logs.

Example 4. Consider next a cointegrated system with four variables and two cointegrating relations, (β_1, β_2), which we identify by solving them for x_1 and x_2

$$x_{1t} = \gamma_{13} x_{3t} + \gamma_{14} x_{4t} + u_{1t},$$
$$x_{2t} = \gamma_{23} x_{3t} + \gamma_{24} x_{4t} + u_{2t}.$$

We want to implement a long-run change in the system, so that we can interpret γ_{13} as the long-run effect of x_3 on x_1 keeping x_4 fixed. Because x_2 does not enter the first relation, we consider a long-run change of the form

$$k'(\lambda) = (\gamma_{13}, \lambda, 1, 0),$$

for some λ, where the value of λ is at our disposal, and we choose it so that $k(\lambda)$ is orthogonal to β_1 and β_2. Note that

$$k'(\lambda)\beta_1 = (\gamma_{13}, \lambda, 1, 0)(1, 0, -\gamma_{13}, -\gamma_{14})' = 0,$$
$$k'(\lambda)\beta_2 = (\gamma_{13}, \lambda, 1, 0)(0, 1, -\gamma_{23}, -\gamma_{24})' = \lambda - \gamma_{23}.$$

Thus by the choice $\lambda = \gamma_{23}$ we make $k(\gamma_{23})$ orthogonal to both cointegrating vectors. Hence we can achieve the long-run change $k = k(\gamma_{23})$ by giving a change to x_t of the form Γk, which by Proposition 1 will give a long-run change of $C\Gamma k = k$. Note that the long-run change also changes x_2, and it is the change of x_2 that achieves that $k(\lambda)$ is orthogonal to both β vectors. Thus the role of the excluded variable x_2 is that of an instrument needed to produce the required effect on other variables namely that the point stays on the attractor set, where the equations are satisfied.
We finally show in Proposition 2 the general result that the identifying restrictions define the instruments, that allow the interpretation of an identified coefficient in a cointegrating relation. We construct a thought experiment that achieves the required counterfactual change in the long-run value by suitable changes in the current values.
We consider the model (2) for the cointegrated $I(1)$ process X_t with r cointegrating vectors. Let β_1 be one of them of the form

$$\beta_1' = (1, \beta_{21}, \ldots, \beta_{s1}, 0, \ldots, 0).$$

That is, normalized on $\beta_{11} = 1$, and restricted by $R'\beta_1 = 0$, where

$$R' = \left(0_{(n-s)\times s}, I_{n-s}\right).$$

We remind that the coefficient β_{21} is identified by the restrictions R, if for any other vector $\beta_1^* \in \text{sp}(\beta)$ satisfying $\beta_{11}^* = 1$ and $R'\beta_1^* = 0$, it holds that $\beta_{21}^* = \beta_{21}$.

Proposition 2 *Let β_1 be a cointegrating vector of the form*

$$\beta_1' = (1, \beta_{21}, \ldots, \beta_{s1}, 0, \ldots, 0).$$

The counterfactual experiment for implementing the definition of $-\beta_{21}$ as the long-run effect of y_2 on y_1 keeping $x_3, \ldots, x_s$ fixed and using $x_{s+1}, \ldots, x_n$ as instruments, can be performed using a shift in x_t, if and only if β_{21} is identified by the zero restrictions.

Proof. Assume first that β_{21} is identified. We want to apply Proposition 1 to find the shift needed to the initial values. In order to apply this result, we construct a vector k of the form

$$k'(\lambda) = (-\beta_{21}, 1, 0, \ldots, 0, \lambda_{s+1}, \ldots, \lambda_n) = k(0)' + \lambda' R'.$$

This vector is by construction orthogonal to β_1 and we want to choose $\lambda' = (\lambda_{s+1}, \ldots, \lambda_n)$ so that it becomes orthogonal to all cointegrating vectors and becomes a vector in $\text{sp}(\beta_\perp)$, in which case it can be implemented as a long-run change by Proposition 1. Thus we want to find λ, so that

$$k(\lambda)'(\beta_2, \ldots, \beta_r) = k(0)'(\beta_2, \ldots, \beta_r) + \lambda' R'(\beta_2, \ldots, \beta_r) = 0. \qquad (5)$$

This equation can be solved for λ if the row vector $k(0)'(\beta_2, \ldots, \beta_r)$ is spanned by the rows of $R'(\beta_2, \ldots, \beta_r)$, or equivalently if for any v so that $R'(\beta_2, \ldots, \beta_r)v = 0$, it holds that $k(0)'(\beta_2, \ldots, \beta_r)v = 0$. Let v be a vector such that $R'(\beta_2, \ldots, \beta_r)v = 0$, and define the vectors $\tilde{\beta}_1 = (\beta_2, \ldots, \beta_r)v$ and

$$\beta_1^* = (1 - \tilde{\beta}_{11})\beta_1 + \tilde{\beta}_1.$$

We want to prove that $k(0)'\tilde{\beta}_1 = k(0)'(\beta_2, \ldots, \beta_r)v = 0$.
Note that $R'\beta_1^* = 0$ and $\beta_{11}^* = 1$, so that, since β_{21} is identified, we have

$$\beta_{12} = \beta_{12}^* = (1 - \tilde{\beta}_{11})\beta_{12} + \tilde{\beta}_{12},$$

which implies that

$$\tilde{\beta}_{11}\beta_{12} = \tilde{\beta}_{12}.$$

Hence

$$k(0)'(\beta_2, \ldots, \beta_r)v = k(0)'\tilde{\beta}_1 = -\beta_{12}\tilde{\beta}_{11} + \tilde{\beta}_{12} = 0,$$

as was to be shown. Thus if β_{21} is identified we can find a vector λ which solves (5) and hence Proposition 1 can be applied to implement the desired long-run change.
If on the other hand β_{21} is not identified we can find a vector $\beta_1^* = a\beta_1 + (\beta_2, \ldots, \beta_r)v$ for some $a \in R$ and $v \in R^{r-1}$, with the property that $\beta_{11}^* = 1$, $R'\beta_1^* = 0$, and $\beta_{21}^* \neq \beta_{21}$. We see that $R'(\beta_2, \ldots, \beta_r)v = R'(\beta_1^* - a\beta_1) = 0$, and

$$\begin{aligned} &k(0)'(\beta_2, \ldots, \beta_r)v = k(0)'(\beta_1^* - a\beta_1) \\ &= -\beta_{21}(1 - a) + (\beta_{21}^* - a\beta_{21}) = \beta_{21}^* - \beta_{21} \neq 0, \end{aligned}$$

and hence there cannot be a solution λ to the equation (5). Thus the existence of a λ that solves (5) is necessary and sufficient for the identification of β_{21}, and hence for the construction of the counterfactual experiment. ■
Obviously a similar result can be proved for the interpretation of identified regression coefficients in simultaneous equations, but we give it here only for the case of cointegration.

3.1 What is a positive relation ?

The title of this subsection is inspired by the following example

$$\Delta x_{1t} = -\tfrac{1}{2}(x_{1t-1} - x_{2t-1}) + \varepsilon_{1t},$$
$$\Delta x_{2t} = -\tfrac{1}{4}(x_{1t-1} - x_{2t-1}) + \varepsilon_{2t}.$$

This is a cointegrated $I(1)$ system and $\beta' x_t = x_{1t} - x_{2t}$ is stationary, so that when we solve for x_{1t} we get

$$x_{1t} = x_{2t} + v_t,$$

where v_t is stationary. Thus, there is evidently a one to one (long-run) relation between x_1 and x_2 and it is a simple example of a positive relation. Notice that whereas the first variable, x_{1t}, reacts sensibly to a disequilibrium error, $x_{1t-1} - x_{2t-1}$, by the coefficient $-\frac{1}{2}$ to x_{1t-1}, the second, x_{2t}, seems to correct the wrong way by the (numerically smaller) coefficient $\frac{1}{4}$ to x_{2t-1}. Now imagine that the process at time t is at the equilibrium position $(x_{1t}, x_{2t}) = (1, 1)$, and you give a shock to the system by adding 1 to x_{t1}. Then the new system starts from $(2, 1)$ and the long-run value becomes

$$C \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence a change of x_{1t} of $+1$ from 1 to 2, will lead to a long-run change in x_{2t} of -1, from 1 to 0. This does not look like a positive relation, but of course the relation should be understood as a long-run positive relation, and in fact a shift in x_{1t} of -1 will lead to a long-run change of $+1$ in both variables consistent with the long-run interpretation.

4 Conclusion

We have discussed the interpretation of regression coefficients in simultaneous equations and cointegrating coefficients. Whereas the interpretation of the individual (autonomous) equations or cointegrating relations is in terms of a counterfactual experiment that allows one to vary some variables and keep others fixed, we discuss here how to implement such a change in terms of another counterfactual experiment when the equations form part of a system of simultaneous equations. This experiment consists for regression coefficients in manipulating the exogenous variables to achieve the relevant effect on the endogenous variables. For cointegrating coefficients we manipulate the current values in order to implement the required effect on the long-run values. This last counterfactual experiment can be conducted precisely when the coefficient is identified.

5 Appendix

5.1 The companion form

Model (2) can be expressed as an $AR(1)$ model for the stacked process, see Hansen and Johansen (1998, Exercise 4.7)

$$\tilde{x}_t = (x_t', x_{t-1}', \ldots, x_{t-k+1}')' \tag{6}$$

with errors

$$\tilde{\varepsilon}_t = (\varepsilon_t', 0, \ldots, 0)',$$

and parameters

$$\tilde{\alpha} = \begin{pmatrix} \alpha \, \Gamma_1 \cdots \Gamma_{k-1} \\ 0 \; I_n \cdots \; 0 \\ \vdots \; \vdots \quad \; \vdots \\ 0 \; 0 \; \cdots \; I_n \end{pmatrix}, \tilde{\beta} = \begin{pmatrix} \beta \quad I_n \cdots \quad 0 \\ 0 - I_n \cdots \quad 0 \\ \vdots \quad \vdots \qquad \vdots \\ 0 \quad 0 \cdots - I_n \end{pmatrix}, \tag{7}$$

so that

$$\Delta \tilde{x}_t = \tilde{\alpha} \tilde{\beta}' \tilde{x}_{t-1} + \tilde{\varepsilon}_t$$

with solution

$$\tilde{x}_{t+h} = (I_{kn} + \tilde{\alpha}\tilde{\beta}')^h \tilde{x}_t + \sum_{i=0}^{h-1} (I_{kn} + \tilde{\alpha}\tilde{\beta}')^i \tilde{\varepsilon}_{t+h-i}. \tag{8}$$

Hence

$$E(\tilde{x}_{t+h} | \tilde{x}_t) = (I_{kn} + \tilde{\alpha}\tilde{\beta}')^h \tilde{x}_t. \tag{9}$$

The $I(1)$ condition, Johansen (1996) Theorem 4.2, is equivalent to

$$|eig(I_{r+(k-1)n} + \tilde{\beta}'\tilde{\alpha})| < 1,$$

which implies that $(I_{r+(k-1)n} + \tilde{\beta}'\tilde{\alpha})^h \to 0$. Applying this we find

$$\begin{aligned}(I_{kn} + \tilde{\alpha}\tilde{\beta}')^h \tilde{x}_t &= (I_{kn} + \tilde{\alpha}\tilde{\beta}')^h [\tilde{\beta}_\perp (\tilde{\alpha}_\perp' \tilde{\beta}_\perp)^{-1} \tilde{\alpha}_\perp' + \tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\beta}'] \tilde{x}_t \\ &= \tilde{\beta}_\perp (\tilde{\alpha}_\perp' \tilde{\beta}_\perp)^{-1} \tilde{\alpha}_\perp' \tilde{x}_t + \tilde{\alpha}(I_{r+(k-1)n} + \tilde{\beta}'\tilde{\alpha})^h (\tilde{\beta}'\tilde{\alpha})^{-1} \tilde{\beta}' \tilde{x}_t \\ &\to \tilde{\beta}_\perp (\tilde{\alpha}_\perp' \tilde{\beta}_\perp)^{-1} \tilde{\alpha}_\perp' \tilde{x}_t = \tilde{C} \tilde{x}_t = \tilde{x}_{\infty|t}.\end{aligned}$$

The limit $\tilde{x}_{\infty|t}$ is called the long-run value of $\tilde{x}_{t+h}$ starting at $\tilde{x}_t$.
It is seen that

$$\tilde{\alpha}_\perp' = \alpha_\perp'(I_n, -\Gamma_1, \ldots, -\Gamma_{k-1}), \; \tilde{\beta}_\perp' = \beta_\perp'(I_n, \ldots, I_n)$$

which shows that

$$\tilde{C} = \tilde{\beta}_{\perp} \left(\tilde{\alpha}'_{\perp} \tilde{\beta}_{\perp} \right)^{-1} \tilde{\alpha}'_{\perp} = (I_n, \dots, I_n)' C (I_n, -\Gamma_1, \dots, -\Gamma_{k-1}). \tag{10}$$

For $J' = (I_n, 0, \dots, 0)$, we recover the process x_t by $x_t = J' \tilde{x}_t$, so that the long-run value of x_{t+h} starting at the current values $(x_t, \dots, x_{t-k+1})$ is

$$x_{\infty|t} = C(x_t - \sum_{i=1}^{k-1} \Gamma_i x_{t-i}).$$

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