1 The Lieb-Thirring inequality

We consider the Schrödinger operator

$$H = -\Delta + V$$

defined on $\mathcal{D}(H) = C^2_0(\mathbb{R}^d)$, which is dense in $L^2(\mathbb{R}^d)$. Here $V \in L^2_{\text{loc}}(\mathbb{R}^d)$, such that the operator maps into $L^2(\mathbb{R}^d)$.

Theorem 1 (Lieb-Thirring inequality). If $H$ is as above and the negative part $V_- = \min\{V, 0\} \in L^{\frac{d}{2}+1}(\mathbb{R}^d)$ then $H$ is bounded below and its min-max values $\mu_n$ satisfy the Lieb-Thirring inequality

$$\sum_{n=1}^{\infty} |\mu_n|_- \geq -C_d \int |V_-(x)|^{\frac{d}{2}+1} dx.$$

From the Lieb-Thirring inequality we see that the bottom of the essential spectrum of $H$ must satisfy $\Sigma(H) \geq 0$, because otherwise all $\mu_n \leq \Sigma(H) < 0$ and hence we would have $\sum_{n=1}^{\infty} |\mu_n|_- = -\infty$. It follows that all negative min-max values are eigenvalues and hence the Lieb-Thirring inequality is an estimate on the sum of negative eigenvalues. The proof is based on M. Rumin, *Balanced distribution-energy inequalities and related entropy bounds*, Duke Math. J., 160 (2011), pp. 567-597.

Proof of the Lieb-Thirring Inequality.

Step 1: It is enough to show that

$$\sum_{j=1}^{N} \langle \phi_j, H\phi_j \rangle \geq -C_d \int |V_-(x)|^{\frac{d}{2}+1} dx$$

for all finite orthonormal families $\{\phi_j\}_{j=1}^{N}$ in $C^2_0(\mathbb{R}^d)$.

Indeed, if (1) holds it is clear that $H$ is bounded below (the case $N = 1$). Moreover, it is clear from (1) that $\Sigma(H) \geq 0$. Otherwise, we could find a subspace $M \subseteq C^2_0(\mathbb{R}^d)$ of arbitrarily large dimension $N$ such that $\langle \phi, H\phi \rangle < \Sigma(H)/2 < 0$ for all normalized $\phi \in M$. Choosing any orthonormal basis in $M$ the sum on the left side of (1) would be less than $-N\Sigma(H)/2$ contradicting (1). Since $\Sigma(H) \geq 0$ all negative min-max values are eigenvalues and we can therefore choose an orthonormal family in $C^2_0(\mathbb{R}^d)$ approximating the corresponding eigenvectors such that the left side of (1) approximates arbitrarily well the sum of the negative eigenvalues.
Step 2: We will use the following convention for the Fourier transform of $f \in L^1(\mathbb{R}^d)$

$$\hat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ipx} \, dx.$$  

Then the Fourier transform extends to a unitary map on $L^2(\mathbb{R}^d)$.

For all $e > 0$ and $\phi \in L^2(\mathbb{R}^d)$ we write

$$\phi = \phi^{e,+} + \phi^{e,-}$$

where

$$\hat{\phi}^{e,+}(p) = \begin{cases} \hat{\phi}(p), & p^2 > e \\ 0, & p^2 \leq e \end{cases}, \quad \hat{\phi}^{e,-}(p) = \begin{cases} 0, & p^2 > e \\ \hat{\phi}(p), & p^2 \leq e \end{cases}.$$  

Then since the Fourier transform is unitary we obtain for $\phi \in C^1_0(\mathbb{R}^d)$

$$\int_0^\infty \int_{\mathbb{R}^d} |\phi^{e,+}(x)|^2 \, dx \, de = \int_{\mathbb{R}^d} \int_0^\infty |\hat{\phi}^{e,+}(p)|^2 \, dp \, de = \int_{\mathbb{R}^d} \int_0^{p^2} |\hat{\phi}(p)|^2 \, dp \, de = \int_{\mathbb{R}^d} p^2 |\hat{\phi}(p)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \, dx. \quad (2)$$

Step 3: For any family $\{\phi_j\}_{j=1}^N$ in $L^2(\mathbb{R}^d)$ we have using the triangle inequality in $\mathbb{C}^N$

$$\left( \sum_{j=1}^N |\phi_j^{e,+}(x)|^2 \right)^{1/2} \geq \left[ \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{1/2} - \left( \sum_{j=1}^N |\phi_j^{e,-}(x)|^2 \right)^{1/2} \right]_+ \quad (3)$$

for all $e > 0$. Here we have again used $t_+ = \max\{t, 0\}$.

Step 4: For any orthonormal family $\{\phi_j\}_{j=1}^N$ we have using Bessel’s inequality that for all $x \in \mathbb{R}^d$

$$\sum_{j=1}^N |\phi_j^{e,-}(x)|^2 = \sum_{j=1}^N |(2\pi)^{-d/2} \int e^{ipx} 1_{(0, e)}(p^2) \hat{\phi}_j(p) \, dp|^2 \leq (2\pi)^{-d} \int |e^{ipx} 1_{(0, e)}(p^2)|^2 \, dp = (2\pi)^{-d} \kappa_d e^{d/2}. \quad (4)$$
Step 5: Combining (2–4) we obtain for any orthonormal family \( \{ \phi_j \}_{j=1}^N \)
\[
\sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 \, dx \geq \int \int_0^\infty \left[ \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{1/2} - (2\pi)^{-d/2} \kappa_d^{1/2} e^{d/4} \right]^2 \, dx \, dedx
\]
\[
\geq \frac{(2\pi)^2 d^2 \kappa_d^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{\frac{d+2}{d}} \, dx.
\]

Step 6: Using Step 5 and Hölder’s inequality we arrive at the final result as follows
\[
\sum_{j=1}^N \langle \phi_j, H \phi_j \rangle = \sum_{j=1}^N \int |\nabla \phi_j(x)|^2 + V(x)|\phi_j(x)|^2 \, dx
\]
\[
\geq \sum_{j=1}^N \int |\nabla \phi_j(x)|^2 - |V_-(x)||\phi_j(x)|^2 \, dx
\]
\[
\geq \frac{(2\pi)^2 d^2 \kappa_d^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{\frac{d+2}{d}} \, dx
\]
\[
- \left( \sqrt{\int |V_-(x)|^{\frac{d+2}{d}} \, dx} \right)^{\frac{d}{d+2}} \left( \int \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{\frac{d+2}{d}} \, dx \right)^{\frac{d}{d+2}}
\]
\[
\geq - \frac{2(2\pi)^{-d} \kappa_d}{d+2} \left( \frac{d}{d+2} \right)^{d/2} \int |V_-(x)|^{\frac{d+2}{d}} \, dx,
\]
where we have used the fact that the function \( \mathbb{R}_+ \ni t \mapsto At - Bt^{\frac{d}{d+2}} \) for \( A, B > 0 \) has the minimal value \(-\frac{2}{d+2} \left( \frac{d}{d+2} \right)^{d/2} A^{-d/2} B^{\frac{d+2}{d+2}} \).

The estimate in the Lieb-Thirring inequality should be compared to the classical phase space integral
\[
(2\pi)^{-d} \int [p^2 + V(x)]_- \, dp \, dx = -\frac{2(2\pi)^{-d} \kappa_d}{d+2} \int |V_-(x)|^{\frac{d+2}{d}} \, dx.
\]
The celebrated Lieb-Thirring conjecture states that the inequality holds with the classical constant above, i.e.,
\[
\sum_{n=1}^{\infty} [\mu_n]_- \geq (2\pi)^{-d} \int [p^2 + V(x)]_- \, dp \, dx.
\]
As we saw the Lieb-Thirring inequality implies that $\Sigma(H) \geq 0$. If $V$ tends to zero at infinity we have $\Sigma(H) \leq 0$.

**Theorem 2.** If $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ is such that $\{x \mid V(x) > \varepsilon\}$ is a bounded set (except for a set of measure zero) for all $\varepsilon > 0$ then $\Sigma(H) \leq 0$.

**Proof.** Given $\varepsilon > 0$ we will show that there exists a subspace $M$ of $C^2_0(\mathbb{R}^d)$ such that $\Sigma(H) \leq 0$.

Given an integer $N$ we may define $\phi_j(x) = R^{-d/2}g((x - u_j)/R)$, $j = 1, \ldots, N$ where the $u_j \in \mathbb{R}^d$ are chosen in such a way that all $\phi_j$ have disjoint support and are supported away from the set $\{x \mid V(x) > \varepsilon/2\}$. Then the $\phi_j$ form an orthonormal family. If $\phi = \sum_{j=1}^N \alpha_j \phi_j$ then

$$\langle \phi, H\phi \rangle \leq \sum_{j=1}^N |\alpha_j|^2 \left( \int R^{-2} |\nabla g|^2 + \varepsilon/2 \int |g|^2 \right) \leq \varepsilon \sum_{j=1}^N |\alpha_j|^2.$$ 

Hence the space $M = \text{span}\{\phi_1, \ldots, \phi_N\}$ has the desired property. \hfill $\square$

## 2 The CLR bound

We will now prove a bound on the number of negative eigenvalues in dimension $d \geq 3$. The argument is due to Rupert Frank.

**STEP 1:**

We proceed as in the proof of the Lieb-Thirring inequality, but instead of assuming that $\{\phi_j\}_{j=1}^N$ is an orthonormal family, we assume that $\{\sqrt{-\Delta} \phi_j\}_{j=1}^N$ is an orthonormal family. The calculation of the kinetic energy is unchanged, but Bessel’s inequality now yields (under the assumption $d \geq 3$),

$$\sum_{j=1}^N |\phi_j^{e^{-}(x)}|^2 = \sum_{j=1}^N \left| (2\pi)^{-d/2} \int \frac{e^{ipx}1_{(0,e)}(p^2)}{|p|} (|p|\hat{\phi}_j(p)) \, dp \right|^2 \leq (2\pi)^{-d} \int \frac{|e^{ipx}1_{(0,e)}(p^2)|^2}{|p|^2} \, dp = (2\pi)^{-d} k_d e^{(d-2)/2}. \quad (5)$$
STEP 2:
Upon inserting (5) in the old Step 5, we get (with $\rho(x) = \sum_{j=1}^{N} |\phi_j(x)|^2$)

$$\sum_{j=1}^{N} \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 dx \geq \int \int_0^\infty \left[ \rho(x)^{1/2} - (2\pi)^{-d/2} \kappa_d^{1/2} e^{(d-2)/4} \right]^2 dx \geq C_d \int \rho(x)^{1+\frac{2}{d}} dx.$$ 

STEP 3:
Let now $\{\psi_1, \ldots, \psi_N\}$ be linearly independent eigenvectors of $H$ with eigenvalue below 0 (or more generally satisfy $1_{(-\infty,0]}(H)\psi_j = \psi_j$). We can now apply Gram-Schmidt to obtain a collection $\{\phi_1, \ldots, \phi_N\}$ with $\langle \sqrt{-\Delta} \phi_j, \sqrt{-\Delta} \phi_k \rangle = \delta_{j,k}$.

Notice that the number $N$ is unchanged because if $\sqrt{-\Delta} \psi = 0$, then $|p|^\psi(p) = 0$ in $L^2$ which implies that $\hat{\psi} = 0$ almost everywhere.

We now get that (using Hölder and STEP 2)

$$0 \geq \sum_{j=1}^{N} \langle \phi_j, (-\Delta - V) \phi_j \rangle = N - \int V(x) \rho(x) \, dx \geq N - (\int V^{d/2})^{2/d} (\int \rho^{d/(d-2)} (d-2)/d) \geq N - C (\int V^{d/2})^{2/d} N^{(d-2)/d}. \quad (6)$$

A Hölder inequality now gives that

$$N \leq \tilde{C} \int V^{d/2}. \quad (7)$$