Dyson's conjecture for the energy of a charged Bose gas[a]

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[a] Joint work with E.H. Lieb

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The oldest problem in Quantum Mechanics

The energy of a charged gas of N particles, with charges $e_i \in \{1, -1\}$:

$$H_N = \sum_{i=1}^{N} -\tfrac{1}{2}\Delta_i + \sum_{1\le i<j\le N} \frac{e_i e_j}{|x_i - x_j|}, \quad \text{on } \bigotimes^{N} L^2(\mathbb{R}^3) = L^2(\mathbb{R}^{3N})$$

$$\begin{aligned} E(N) &= \min\{\inf \mathrm{spec} H_N \ : \ e_i = \pm 1, \ i = 1, 2, \ldots, N\} \\ &= \inf \mathrm{spec}_{L^2((\mathbb{R}^3\times\{1,-1\})^N)} H_N \end{aligned}$$

On $L^2((\mathbb{R}^3 \times \{1,-1\})^N) = \bigotimes^N L^2(\mathbb{R}^3 \times \{1,-1\})$ we may restrict to fully symmetric functions. Therefore this is a *charged Bose gas*.

Note that $E(1) = 0$, $E(2) = -1/4$ (Schrödinger 1926-27 (*hydrogen*)).

Dyson's conjecture (1967):

$$\lim_{N\to\infty} \frac{E(N)}{N^{7/5}} = \inf\left\{ \tfrac{1}{2}\int (\nabla\Phi)^2 - J\int \Phi^{5/2} \ : \ \int \Phi^2 = 1, \ \Phi \ge 0 \right\}$$

$$J = (2/\pi)^{3/4} \int 1 + x^4 - x^2(x^4+2)^{1/2} dx.$$

History (after 1927)

THEOREM 1 (Dyson 1967 Instability of charged bose gas).

$$E(N) \leq -CN^{7/5}$$

THEOREM 2 (Dyson-Lenard 1967). $E(N) \geq -CN^{5/3}$

THEOREM 3 (Conlon-Lieb-Yau 1988). $E(N) \geq -CN^{7/5}$

THEOREM 4 (Lieb-Solovej 2003).

$$\liminf_{N\to\infty} \frac{E(N)}{N^{7/5}} \geq \inf\left\{ \int (\nabla\Phi)^2 - J\int \Phi^{5/2} \ : \ \int \Phi^2 = 1,\ \Phi \geq 0 \right\}$$

Remark on proof of Thm. 1: If we use a product trial function $\prod_{i=1}^N \phi(x_i)$, with $e_i = (-1)^i$ we only get $E(N) \leq -CN$ (not instability). Dyson uses a BCS type trial function (for $2N$ particles):

$$\prod_{i=1}^{2N} \phi(x_i) \sum_{\text{Perm. }\sigma} \prod_{j=1}^{N} \left[1 - e_{\sigma(2j)} e_{\sigma(2j-1)} \sum_\alpha \psi_\alpha(x_{\sigma(2j)}) \psi_\alpha(x_{\sigma(2j-1)}) \right]$$

Sketching the proof of Dyson's conjecture

Physics: Global length scale (of ϕ) is $N^{-1/5}$, $\phi^2 \sim N^{8/5}$. Local length scales (of ψ_α) is $N^{-2/5} \ll N^{-1/5}$.

Step 1: The local (short scale) energy : Consider gas confined to box of size ℓ, with $N^{-2/5} \ll \ell \ll N^{-1/5}$ and with particle number $\nu \sim N^{8/5}\ell^3$. I.e., $\boxed{\ell^{-1} \ll \nu \ll \ell^{-5}}$. Use second quantization

$$H_{\text{Box}} = \sum_{p,e} \epsilon(p) a^*_{p,e} a_{p,e} + \frac{1}{2} \sum_{p,q,\mu,\nu,e,e'} ee' \omega_{pq;\mu\nu} a^*_{p,e} a^*_{q,e'} a_{\nu,e'} a_{\mu,e}$$

Relevant part is $\omega_{pq;00} = \omega_{00;pq} = \omega_{p0;0-q} = \omega_{0p;-q0} \sim g(p)\delta(p+q)$. Conclude energy $\sim -J\nu(\nu/\ell^3)^{1/4}$.

Step 2: The global energy :

$$\sum_{\text{n.n. boxes i,j}} \tfrac{1}{2}\ell^{-2} \left(\sqrt{\nu(i)} - \sqrt{\nu(j)}\right)^2 - \sum_{\text{boxes i}} J\nu(i) \left(\nu(i)/\ell^3\right)^{1/4}.$$

A discrete approximation to $\frac{1}{2}\int (\nabla\Phi)^2 - J\int \Phi^{5/2}$, $\int \Phi^2 = N$.

Local energy calculation

For $p \neq 0$ let $b^*_{p,e} = a^*_{p,e} a_{0,e} / \sqrt{\nu_e}$. Then

$$b^*_{p,e} b_{p,e} \leq a^*_{p,e} a_{p,e}, \qquad [b^*_{p',e'}, b^*_{p,e}] = 0, \qquad [b_{p,e}, b^*_{p,e}] \leq 1 \tag{1}$$

and H_{Box} "$\geq$" $\sum_{p \neq 0} h_p$ where h_p is

$$\frac{\epsilon(p)}{2} \sum_{e=\pm 1} (b^*_{p,e} b_{p,e} + b^*_{-p,e} b_{-p,e})$$
$$+ g(p) \sum_{e,e'=\pm 1} \sqrt{\nu_e \nu_{e'}} e e' (b^*_{p,e} b_{p,e'} + b^*_{-p,e} b_{-p,e'} + b^*_{p,e} b^*_{-p,e'} + b_{p,e} b_{-p,e'})$$

It follows from (1) (by completing squares) that

$$h_p \geq -(\epsilon(p)/2 + \nu g(p)) + \sqrt{(\epsilon(p)/2 + \nu g(p))^2 - (\nu g(p))^2}$$

After replacing sums by integrals this gives

$$H_{\mathrm{Box}} \text{"} \geq \text{"} - J\nu(\nu/\ell^3)^{1/4}$$

The localization scheme

THEOREM 5 (The sliding method (Conlon-Lieb-Yau 1988)). $\chi_z =$ *"smooth characteristic" function of* ℓ*-cube centered at* $z \in \mathbb{R}^3$*. Then*

$$\sum_{1\leq i<j\leq N} \frac{e_i e_j}{|x_i - x_j|} \geq \int \sum_{1\leq i<j\leq N} \chi_z(x_i) \frac{e_i e_j}{|x_i - x_j|} \chi_z(x_j) dz - C\frac{N}{\ell}$$

THEOREM 6 (A many body kinetic energy bound ($\ell = 1$)). $\chi_z =$ *"smooth characteristic" function of unit cube centered at* $z \in \mathbb{R}^3$*.* $a^*(z)$ *creation operator of constant in cube.* $\mathcal{P}_z =$*projection orthogonal to constants in cube.* $\Omega \subset \mathbb{R}^3$*.* e_1, e_2, e_3 *standard basis. For all* $0 < s < 1$

$$(1 + \varepsilon(\chi, s)) \sum_{i=1}^{N} -\Delta_i \geq \int_{\Omega} \Big[\sum_{i=1}^{N} \mathcal{P}_z^{(i)} \chi_z^{(i)} \frac{(-\Delta_i)^2}{-\Delta_i + s^{-2}} \chi_z^{(i)} \mathcal{P}_z^{(i)}$$

$$+ \sum_{j=1}^{3} \left(\sqrt{a_0^*(z + e_j) a_0(z + e_j) + 1/2} - \sqrt{a_0^*(z) a_0(z) + 1/2} \right)^2 \Big] dz$$

$$-3\mathrm{vol}(\Omega), \qquad \varepsilon(\chi, s) \to 0 \text{ as } s \to 0.$$