THE ONE COMPONENT BOSE GAS

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The one component charged Bose gas (*Bosonic Jellium*):
N Bosons, unit charge , in box $\Lambda = [0, L]^3$.
Neutralizing background, density $\rho = \frac{N}{L^3}$.
Hamiltonian:

$$H = \sum_{i=1}^{N} -\tfrac{1}{2}\Delta_i - \rho \sum_{i=1}^{N} \int_{\Lambda} |x_i - y|^{-1} dy + \sum_{i<j} |x_i - x_j|^{-1} + \tfrac{1}{2}\rho^2 \iint_{\Lambda\times\Lambda} |x - y|^{-1} dx dy$$

Hilbert Space $\mathcal{H} = \bigotimes^{N} L^2(\Lambda)$ ($-\Delta$ Dirichlet or Neumann B.C.) or $\mathcal{H} = \bigotimes^{N} L^2(\mathbb{R}^3)$ (background still in box).

(Fermionic Jellium: $\mathcal{H} = \bigwedge^{N} L^2(\mathbb{R}^3; \mathbb{C}^2)$.)

Ground state energy: $E = \inf\ \mathrm{Spec}_{\mathcal{H}} H$

The two component Bose problem: N charged Bosons, charges $e_i = \pm 1$ (neutrality $\sum e_i = 0$).

$$H_2 = \sum_{i=1}^{N} -\tfrac{1}{2}\Delta_i + \sum_{i<j} \frac{e_i e_j}{|x_i - x_j|}$$

$\mathcal{H} = \bigotimes^N L^2(\mathbb{R}^3)$.

Two component ground state energy:

$$E_2 = \inf \ \mathrm{Spec}_{\mathcal{H}} H_2$$

THEOREM 1 (Lieb-Narnhofer 1973). *The thermodynamic limit exists:*

$$\lim_{\substack{L\to\infty \\ \frac{N}{L^3}=\rho}} \frac{E}{L^3} = e(\rho)$$

Foldy 1961: Using Bogolubov approximation gets

$$e(\rho) = -0.803(3/4\pi)^{1/4}\rho^{5/4}.$$

Should be good for large ρ. Using the variational principle Foldy's calculation rigorously establishes:

$$e(\rho) \leq -0.803(3/4\pi)^{1/4}\rho^{5/4} + o(\rho^{5/4}),$$

as $\rho \to \infty$

MAIN RESULT

THEOREM 2 (Lieb-Solovej 1999). *Foldy's calculation is correct:*

$$e(\rho) \geq -0.803(3/4\pi)^{1/4}\rho^{5/4} + o(\rho^{5/4})$$

as $\rho \to \infty$.

HISTORY

NON-RIGOROUS RESULTS:

Bogolubov 1947 Invents pairing approximation to explain superfluidity for Bosons.

Bardeen Cooper Schrieffer 1957: Explains superconductivity by similar approximation for fermions

Gell-Mann and Bruckner 1957: For fermionic jellium:

$$e(\rho) = C_{TF}\rho^{5/3} - C_D\rho^{4/3} + C_1\rho\log\rho + C_2\rho + \dots$$

(today used heavily in computational chemistry).

Bogolubov 1958: Explains BCS theory in terms of his approximation.

Foldy 1961: Bosonic Jellium

RIGOROUS RESULTS:

Lieb-Liniger 1963: Solves a 1-dim Bose gas exactly and verifies the Bogolubov approximation for the ground state energy in this case.

Dyson 1967: Motivated by Foldy, Dyson proves that the two component Bose gas is *NOT* stable: $E_2 \leq -CN^{7/5}$ (Stability requires energy $\geq -CN$). I will give the intuition behind Dyson's result in a moment.

Conlon-Lieb-Yau 1988: Dyson's bound has the right power

$$E_2 \geq -C'N^{7/5}.$$

As Corollary: Foldy's result has the right power

$$E \geq -cN\rho^{1/4} \qquad \text{(equivalent formulation).}$$

Graf-Solovej 1993: The first two terms for fermionic jellium are correct:

$$e(\rho) = C_{TF}\rho^{5/3} - C_D\rho^{4/3} + o(\rho^{4/3}).$$

The main theorem establishing Foldy's law for the charged bose gas is the first time that any aspect of the pairing approximation has been rigorously verified for 3-dimensional systems.

The heuristics behind Dyson's argument:

Construct trial state of N Bosons localized in ball of radius R. Density $\rho = \frac{N}{R^3}$.
Localization energy $= NR^{-2}$.
Foldy energy $= -N\rho^{1/4} = -N^{5/4}R^{-3/4}$.
Total energy:

$$NR^{-2} - N^{5/4}R^{-3/4}$$

optimal for $R = N^{-1/5}$ and gives energy $-N^{7/5}$.

Foldy's calculation and pairing theory:

Foldy uses periodic boundary conditions for $-\Delta$. Problem is on torus.
Replaces $|x-y|^{-1}$ by

$$\sum_{p \neq 0} L^{-3}|p|^{-2} \exp(ip(x-y)).$$

sum is over 'periodic momenta' (note $p \neq 0$ so average is 0).

Hamiltonian

$$H' = \sum_{i=1}^{N} -\tfrac{1}{2}\Delta_i + \sum_{i<j} \sum_{p\neq 0} L^{-3}|p|^{-2} \exp(ip(x_i - x_j))$$

$p \neq 0$ supposed to make up for missing background!!??

2nd quantization formulation

$$H' = \sum_p |p|^2 a_p^* a_p + \sum_{p\neq 0} L^{-3}|p|^{-2} \sum_{k,q} a_{k+p}^* a_{q-p}^* a_q a_k$$

Observation: since $p \neq 0$ no terms with 3 or 4 $a_0^\sharp$.

Bogolubov approximation: Motivation is Bose condensation: Almost all particles are in the state of momentum $p = 0$ created by a_0^*. Thus:

Step 1: Keep only quartic terms with precisely two $a_0^\sharp$ (ignore terms with one or no $a_0^\sharp$).

$$\begin{aligned} H'' \;=\; & \sum_p |p|^2 a_p^* a_p + \sum_{p\neq 0} L^{-3}|p|^{-2} [a_p^* a_0^* a_p a_0 \\ & + a_0^* a_{-p}^* a_0 a_{-p} + a_p^* a_{-p}^* a_0 a_0 + a_0^* a_0^* a_p a_{-p}] \end{aligned}$$

Step 2 in Bogolubov appr.: Replace the *OPERATORS* $a_0^\sharp$ by the number $\sqrt{N}$:

$$H''' = \sum_{p\neq 0} |p|^2 a_p^* a_p + \rho |p|^{-2}[a_p^* a_p + a_{-p}^* a_{-p} + a_p^* a_{-p}^* + a_p a_{-p}]$$

Complete the square:

$$\begin{aligned} H''' &= \sum_p A_p(a_p^* + \beta_p a_{-p})(a_p + \beta_p a_{-p}^*) \\ &\quad + A_p(a_{-p}^* + \beta_p a_p)(a_{-p} + \beta_p a_p^*) \\ &\quad -2\sum_{p\neq 0} A_p \beta_p^2 \end{aligned}$$

Last term due to $[a_p, a_q^*] = \delta_{pq}$.

$$\begin{aligned} A_p(1+\beta_p^2) &= \tfrac{1}{2}|p|^2 + \rho|p|^{-2} \\ 2A_p\beta_p &= \rho|p|^{-2} \end{aligned}$$

The ground state energy is given by the last term above.

$$e = \lim_{L\to\infty} -\frac{2}{L^3}\sum_{p\neq 0} A_p\beta_p^2 = \int A_p\beta_p^2 = C_F\rho^{5/4}.$$

Ground state wave function ψ satisfies

$$(a_p + \beta_p a^*_{-p})\psi = 0,$$

for all $p \neq 0$.

In the original language (a_0 an operator) this corresponds to function of the form

$$\begin{aligned} \psi &= 1 + \sum_{i<j} f(x_i - x_j) \\ &\quad + c \sum_{\substack{i,j,l,k \\ \text{different}}} f(x_i - x_j) f(x_l - x_k) + \ldots \end{aligned}$$

where $\widehat{f}(p) = \beta_p$. In fact, $\widehat{f}(p) = G(|p|^4/\rho)$, G independent of ρ.
Thus f varies on a length scale $\rho^{-1/4}$ (the typical interpair distance).

Ideas in rigorous proof: No need to prove Bose condensation globally enough to do it on short scale.

- Localize by Neumann bracketing in "small" boxes of size ℓ. Constant function (condensate not affected). Function f "not affected" if $\ell \gg \rho^{-1/4}$. We choose ℓ close to $\rho^{-1/4}$.
- Control electrostatics between boxes using an averaging method of Conlon-Lieb-Yau. Error $= N/\ell \ll N\rho^{1/4}$.
- Establish condensation on scale ℓ: First non-zero Neumann eigenvalue $\sim \ell^{-2}$. The expected number N_+ of particles not in condensate in the "small box". Their energy: $N_+\ell^{-2} \sim N_+\rho^{1/2}$. if consistent with total energy $-N\rho^{1/4}$ we should expect $N_+ \ll N\rho^{-1/4}$, i.e., local condensation.

One establishes this through a bootstrapping procedure. Having established local condensation one starts the hard work of establishing the Bogolubov approximation. Difficulty: We cannot use periodic b.c.