The Matter of Instability[a]

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STABILITY MATTERS

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[a]Joint work with Elliott H. Lieb

List of Slides

The charged gas in Quantum Mechanics

The Hamiltonian of a gas of charged particles:

$$H_N = \sum_{i=1}^{N} -\frac{1}{2}\Delta_i + \sum_{1\leq i<j\leq N} \frac{e_i e_j}{|x_i - x_j|}$$

We consider (for simplicity) the charges $e_i = \pm 1$, $i = 1, \ldots, N$ as variables. Thus the Hilbert space is $\mathcal{H} = L^2\left((\mathbb{R}^3 \times \{-1, 1\})^N\right)$. If

$$\mathcal{H}_{\mathrm{B}} = \bigotimes_{\mathrm{sym}}^{N} L^2\left(\mathbb{R}^3 \times \{-1, 1\}\right),$$

then

$$E(N) := \inf \operatorname{spec}_{\mathcal{H}} H_N = \inf \operatorname{spec}_{\mathcal{H}_{\mathrm{B}}} H_N$$

Stability of Matter (i.e., that H_N obeys a lower bound linear in N) holds on the subspace of $\mathcal{H}$, where either the positively or negatively charged particles (or both) are fermions.

The Instability of the charged Bose Gas

THEOREM 1 (Instability of the charged (Bose) gas. *Dyson '67***).** *There is a constant $C_+ > 0$ such that*

$$E(N) \leq -C_+ N^{7/5}.$$

INSTABILITY: $7/5 > 1$.

The trial state: The Dyson trial state is a complicated *Bogolubov pair function*. Stablity cannot be proved with simple *product state*:

$\Psi(x_1, e_1, \ldots, x_N, e_N) = \prod_{i=1}^{N} \phi(x_i)$ (and $= 0$ if $\sum_{i=1}^{N} e_i \neq 0$) ($N$ even):

$$\langle \Psi, H_N \Psi \rangle = \underbrace{CNR^{-2}}_{\text{kinetic energy}} - \underbrace{CNR^{-1}}_{\text{potential=self-energy}} = -CN$$

where R is the extent of the support of ϕ.

The $N^{7/5}$ and $N^{5/3}$ laws for Bosons

THEOREM 2 (The $N^{7/5}$ law. *Conlon-Lieb-Yau '88***).** *There is a constant $C_- > 0$ such that $E(N) \geq -C_- N^{7/5}$.*

THEOREM 3 (The $N^{5/3}$ law. *Dyson '67, Lieb '78***).** *If the positive or negative bosons are inifinitely heavy there are constant $C_\pm > 0$ such that $-C_- N^{5/3} \underbrace{\leq}_{\text{Dyson}} E(N) \underbrace{\leq}_{\text{Lieb}} -C_+ N^{5/3}$.*

Proof of lower bound.

Electrostatic inequality:

$$\sum_{i<j} \frac{e_i e_j}{|x_i - x_j|} \text{“} \geq \text{”} \sum_{i=1}^{N} - \max_{j \neq i} |x_i - x_j|^{-1}$$

Sobolev's inequality :

$$-\Delta - \max_j |x - x_j|^{-1} \geq \sup_R \{ \underbrace{-NR^{1/2}}_{\text{Sobolev}} \underbrace{-R^{-1}}_{\text{distant part}} \} = N^{2/3}. \qquad \square$$

Stability of matter can be proved similarly except that one should use the *Lieb-Thirring inequality* instead of the Sobolev inequality.

Foldy's law and Dyson's conjecture

THEOREM 4 (Foldy's law. *Lieb-Solovej '01***).** *The thermodynamic energy per particle $e(\rho)$ of positively charged bosons in a constant negative background of density ρ satisfies*

$$\lim_{\rho\to\infty}\frac{e(\rho)}{\rho^{1/4}}=J,\qquad J=(2/\pi)^{3/4}\int_0^\infty 1+x^4-x^2\left(x^4+2\right)^{1/2}\,dx.$$

Foldy calculated this in '61 using the method of Bogolubov. This is what motivated Dyson in constructing his trial function for the upper bound in the two-component gas.

DYSON'S CONJECTURE ('67): For the two component gas we have

$$\lim_{N\to\infty}\frac{E(N)}{N^{7/5}}=\inf\left\{\tfrac{1}{2}\int|\nabla\phi|^2-J\int\phi^{5/2}\ :\ \phi\geq 0,\ \int\phi^2=1\right\}$$

THEOREM 5 (Dysons's conjecture. *Lieb-Solovej in prep.***).** *Dyson's conjecture is correct as a lower bound.*

The Foldy-Bogolubov method (in a box)

$a^*_{p\pm}$: creation operators of momentum p states charge ± 1.
$\nu_\pm$ =number of particles of charge ± 1. $\nu = \nu_+ + \nu_-$.
$b^*_{p\pm} = (\nu_\pm)^{-1/2} a^*_{p\pm} a_{0\pm}$. $d^*_p = (\nu_+ + \nu_-)^{-1/2}(\nu_+^{1/2} b^*_{p+} - \nu_-^{1/2} b^*_{p-})$.
Condensation: most particles have momentum 0: $\nu_\pm \approx a^*_{0\pm} a_{0\pm}$.
Bogolubov approximation: The important part of the Hamiltonian can be written in terms of $b^*_{p\pm}$ or rather d^*_p: The Foldy-Bogolubov Hamiltonian:

$$\sum_{p\neq 0} \tfrac{1}{2}|p|^2 (d^*_p d_p + d^*_{-p} d_{-p}) + \frac{\nu}{\text{vol}} |p|^{-2} \left(d^*_p d_p + d^*_{-p} d_{-p} + d^*_p d^*_{-p} + d_p d_p \right)$$

$$= \sum_{p\neq 0} D(d^*_p + \alpha d_{-p})(d^*_p + \alpha d_{-p})^* + D(d^*_{-p} + \alpha d_p)(d^*_{-p} + \alpha d_p)^*$$

$$-D\alpha^2([d_p, d^*_p] + [d_{-p}, d^*_{-p}]), \qquad \text{(Note: } [d_p, d^*_p] \leq 1)$$

For specific D and α. In particular,
$2(2\pi)^{-3} \int D\alpha^2 dp = J\nu(\nu/\text{vol})^{1/4}$.

Length scales

- Size R of gas: $N(N/R^3)^{1/4} = NR^{-2} \Rightarrow R = N^{-1/5}$.
- Energy: $N(N/R^3)^{1/4} = NR^{-2} = N^{7/5}$.
- Momentum scale of the excited pairs:

$$p^2 = (N/R^3)|p|^{-2} \Rightarrow |p| = (N/R^3)^{1/4} = N^{2/5}$$

- Separation of scales: $|p| = N^{2/5} \gg R^{-1} = N^{1/5}$.

Steps in the rigorous proof

- Dirichlet localize gas into region of size $R = N^{-1/5}$.
- Neumann localize into boxes of size $|p|^{-1} = N^{-2/5}$.
- Electrostatic energy between regions is controlled by the method of sliding using the positivity of the Coulomb kernel.
- Control all terms in the Hamiltonian except the Foldy-Bogolubov part.
- Control condensation
- DIFFICULTY with kinetic energy localization: A pure Neumann localization is too crude. It ignores variation on scale $N^{1/5}$. One must use Neumann only for high momentum ($N^{2/5}$) and keep full energy for low momentum ($N^{1/5}$).
- DIFFICULTY with controlling condensation: It is not enough to know the *expectation* value of the condensation.

Kinetic energy bound

THEOREM 6 (A many body kinetic energy bound).
$\chi_z =$ *"smooth characteristic" function of unit cube centered at* $z \in \mathbb{R}^3$*.* $\mathcal{P}_z =$*projection orthogonal to constants in unit cube.* $\Omega \subset \mathbb{R}^3$*.* e_1, e_2, e_3 *standard basis.*
For all $0 < s < t < 1$

$$(1+\varepsilon(\chi, s)) \sum_{i=1}^{N} -\Delta_i \geq \int_\Omega \Big[\sum_{i=1}^{N} \mathcal{P}_z^{(i)} \chi_z^{(i)} \frac{(-\Delta_i)^2}{-\Delta_i + s^{-2}} \chi_z^{(i)} \mathcal{P}_z^{(i)}$$
$$+ \sum_{j=1}^{3} \left(\sqrt{a_0^*(z+e_j) a_0(z+e_j) + 1/2} - \sqrt{a_0^*(z) a_0(z) + 1/2} \right)^2 \Big] dz$$
$$-3 vol(\Omega).$$

Controlling condensation: Localizing large matrices

THEOREM 7 (Localizing large matrices). *Suppose that $\mathcal{A}$ is an $N \times N$ Hermitean matrix and let $\mathcal{A}^k$, with $k = 0, 1, ..., N-1$, denote the matrix consisting of the k^{th} supra- and infra-diagonal of $\mathcal{A}$. Let $\psi \in \mathbf{C}^N$ be a normalized vector and set $d_k = (\psi, \mathcal{A}^k \psi)$ and $\lambda = (\psi, \mathcal{A}\psi) = \sum_{k=0}^{N-1} d_k$. ($\psi$ need not be an eigenvector of $\mathcal{A}$.)*

Choose some positive integer $M \leq N$. Then, with M fixed, there is some $n \in [0, N-M]$ and some normalized vector $\phi \in \mathbf{C}^N$ with the property that $\phi_j = 0$ unless $n + 1 \leq j \leq n + M$ **(i.e., ϕ has length M)** *and such that*

$$(\phi, \mathcal{A}\phi) \leq \lambda + \frac{C}{M^2} \sum_{k=1}^{M-1} k^2 |d_k| + C \sum_{k=M}^{N-1} |d_k| \ , \tag{1}$$

where $C > 0$ is a universal constant. (Note that the first sum starts with $k = 1$.)