QUANTUM COHERENT OPERATORS:
A GENERALIZATION OF COHERENT STATES

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1. **Glauber states**, parametrized by $z \in \mathbf{C}$:

$$\psi_z(x) = \pi^{-1/2} \exp[zx - \tfrac{1}{2}(x^2 + |z|^2)]$$

$$\Pi(z) \equiv |\psi_z\rangle\langle\psi_z| = \text{ Projection onto } \psi_z$$

$$\int_{\mathbf{C}} \Pi(z) d^2 z = I_{\mathcal{H}} = \text{ identity}$$

$$\mathrm{Tr}_{\mathcal{H}} \Pi(z) = \langle\psi_z|\psi_z\rangle = 1$$

2. **Bloch spin states**: parametrized by $\Omega \in \mathcal{S}^2$:

$$|\Omega\rangle_J \in \mathcal{H}_J = \mathbf{C}^{2J+1} \qquad \Omega \cdot \mathbf{S}_J |\Omega\rangle_J = J|\Omega\rangle_J$$

$$\Pi_J(\Omega) = \text{ Projection on } |\Omega\rangle_J$$

$$\frac{1}{4\pi} \int_{\mathcal{S}^2} \Pi_J(\Omega) d\Omega = \frac{1}{2J+1} I_{\mathcal{H}_J}$$

$$\mathrm{Tr} \Pi_J(\Omega) = 1.$$

MAIN APPLICATION: HEISENBERG MODEL.

Heisenberg Hamiltonian (E_{ij} real numbers):

$$H_J^{\mathrm{class}} = -\sum_{1\le i<j\le n} E_{ij}(J\Omega^{(i)})\cdot(J\Omega^{(j)}), \quad \Omega^{(i)}\in\mathcal{S}^2$$

$$H_J^{\mathrm{quant}} = -\sum_{1\le i<j\le n} E_{ij}\mathbf{S}_J^{(i)}\cdot\mathbf{S}_J^{(j)}, \quad \mathbf{S}_J^{(i)} \text{ operator on } \mathcal{H}_J$$

Free energies:

$$\exp\{-\beta f^{\mathrm{class}}(J,\beta)\} = (4\pi)^{-n}\int \exp[-\beta H_J^{\mathrm{class}}]\prod_{i=1}^{n} d\Omega^{(i)}$$

$$\exp\{-\beta f(J,\beta)\} = (2J+1)^{-n}\mathrm{Tr}\exp[-\beta H_J^{\mathrm{quant}}]$$

Berezin-Lieb inequalities for Heisenberg model:

$$f^{\mathrm{class}}(J,\beta) \ge f(J,\beta) \ge f^{\mathrm{class}}(J+1,\beta).$$

Suppose, now that K and J are 2 different spin values classically we have exact scaling:

$$f^{\text{class}}(J,\beta) = f^{\text{class}}\left(K, \left(\frac{J}{K}\right)^2 \beta\right).$$

Using Berezin-Lieb we get for quantum model an approximate scaling

$$f\left(K, \left(\frac{J}{K+1}\right)^2 \beta\right) \geq f(J,\beta) \geq f\left(K, \left(\frac{J+1}{K}\right)^2 \beta\right).$$

Question: Can we do better by a direct comparison?

Answer: With **Quantum Coherent Operators** we have:

THEOREM: *If* $K \geq J$

$$f\left(K, \left(\frac{J}{K}\right)^2 \beta\right) \geq f(J,\beta) \geq f\left(K, \left(\frac{J+1}{K+1}\right)^2 \beta\right)$$

$$f\left(J, \left(\frac{K+1}{J+1}\right)^2 \beta\right) \geq f(K,\beta) \geq f\left(J, \left(\frac{K}{J}\right)^2 \beta\right).$$

Example: $K = 1,\ J = 1/2$

$$f\left(1, \frac{1}{16}\beta\right) \geq f\left(\frac{1}{2}, \beta\right) \geq f\left(1, \frac{9}{4}\beta\right) \qquad \text{Ratio} = 36 \qquad \text{OLD}$$

$$f\left(1, \frac{1}{4}\beta\right) \geq f\left(\frac{1}{2}, \beta\right) \geq f\left(1, \frac{9}{16}\beta\right) \qquad \text{Ratio} = 2.25 \qquad \text{NEW}$$

DEFINITION: 2 Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$. A *quantum coherent operator* Γ is a positive semidefinite operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$\overline{\mathrm{Tr}}_{\mathcal{H}_1} \Gamma = I_{\mathcal{H}_2} \qquad (*)$$

$$\overline{\mathrm{Tr}}_{\mathcal{H}_2} \Gamma = I_{\mathcal{H}_1} \qquad (**)$$

$$\overline{\mathrm{Tr}}_{\mathcal{H}} = \begin{cases} \frac{1}{\dim \mathcal{H}} \mathrm{Tr}\mathcal{H} & \text{finite dimension} \\ \mathrm{Tr}_{\mathcal{H}} & \infty \text{ dimension} \end{cases}$$

"Analogy" with previous case: $\mathcal{H}_2 = \mathcal{H}$ and $\mathcal{H}_1 =$ "classical phase space $\mathbf{C}$ or $\mathcal{S}^2$" ($\overline{\mathrm{Tr}}_{\mathcal{H}_1}$ becomes an integral)

Γ is $(2J+1)\Pi_J(\Omega)$

$$(*) \qquad \Leftrightarrow \qquad \frac{1}{4\pi} \int_{\mathcal{S}^2} (2J+1)\Pi_J(\Omega) d\Omega = I_{\mathcal{H}_J}$$

$$(**) \qquad \Leftrightarrow \qquad \frac{1}{(2J+1)} \mathrm{Tr}_{\mathcal{H}_J} \left[(2J+1)\Pi_J(\Omega)\right] = 1.$$

If A is an operator on $\mathcal{H}_1$ we define $\check{A}$ on $\mathcal{H}_2$ by

$$\check{A} = \overline{\mathrm{Tr}}_{\mathcal{H}_1}(\Gamma A)$$

$\check{A}$ is the **lower symbol** for A.

A is *an* **upper symbol** for $\check{A}$. If A always exists (it may not) we say Γ is **operator complete**.

THEOREM: *(Generalization of Berezin-Lieb):* If A is a Hermitean operator on $\mathcal{H}_1$ and g is a convex function (e.g., $g(x) = e^{-\beta x}$) then

$$\overline{\mathrm{Tr}}_{\mathcal{H}_1} g(A) \geq \overline{\mathrm{Tr}}_{\mathcal{H}_2} g(\check{A}).$$

Proof:

$\{w_j\}$ othonormal basis (for $\mathcal{H}_1$) of eigenvectors of A.

$\{v_i\}$ othonormal basis (for $\mathcal{H}_2$) of eigenvectors of $\check{A}$.

$$\begin{aligned}\overline{\mathrm{Tr}}_{\mathcal{H}_1} g(A) &= \overline{\mathrm{Tr}}_{\mathcal{H}_1}\left[\overline{\mathrm{Tr}}_{\mathcal{H}_2}[\Gamma] A\right] \\ &= \frac{1}{\dim \mathcal{H}_1} \frac{1}{\dim \mathcal{H}_2} \sum_{i,j} \left\langle v_i \otimes w_j \middle| \Gamma \middle| v_i \otimes w_j \right\rangle g\Big(\langle w_j | A | w_j \rangle\Big).\end{aligned}$$

But $\quad \frac{1}{\dim \mathcal{H}_1} \sum_j \left\langle v_i \otimes w_j \middle| \Gamma \middle| v_i \otimes w_j \right\rangle = \left\langle v_i \middle| \overline{\mathrm{Tr}}_{\mathcal{H}_2}[\Gamma] \middle| v_i \right\rangle = 1.$

By Jensen's inequality

$$\overline{\mathrm{Tr}}_{\mathcal{H}_1} g(A) \geq \frac{1}{\dim \mathcal{H}_2} \sum_i g\Big(\left\langle v_i \middle| \overline{\mathrm{Tr}}_{\mathcal{H}_1}[\Gamma A] \middle| v_i \right\rangle\Big) = \overline{\mathrm{Tr}}_{\mathcal{H}_2} g(\check{A}).$$

$\mathcal{QED}$

MAIN EXAMPLE: $\mathcal{H}_1 = \mathcal{H}_J = \mathbf{C}^{2J+1}$, $\mathcal{H}_2 = \mathcal{H}_K = \mathbf{C}^{2K+1}, K \geq J$.

Let $K - J \leq L \leq K + J$. Let $P_L =$ Projection onto spin L in $\mathcal{H}_J \otimes \mathcal{H}_K$.

$$\Gamma_L = \frac{(2K+1)(2J+1)}{2L+1} P_L$$

LEMMA: *Γ_L is a quantum coherent operator.*

LEMMA: *Let $\mathbf{S}_K$ be the 3 spin operators in $\mathcal{H}_K$*

$$\check{\mathbf{S}}_K = \overline{Tr}\Gamma_L \mathbf{S}_K = \frac{L(L+1) - K(K+1) - J(J+1)}{2J(J+1)} \mathbf{S}_J$$

and the reverse with $J \leftrightarrow K$.

$$\begin{array}{ll}
\Gamma_{\min} \leftrightarrow L = K - J & \Gamma_{\max} \leftrightarrow L = K + J \\
\check{\mathbf{S}}_K = -\dfrac{K+1}{J+1}\mathbf{S}_J & \check{\mathbf{S}}_K \quad \leftrightarrow \dfrac{K}{J+1}\mathbf{S}_J \\
\check{\mathbf{S}}_J = -\dfrac{J}{K}\mathbf{S}_K & \check{\mathbf{S}}_J \quad \leftrightarrow \dfrac{J}{K+1}\mathbf{S}_K
\end{array}$$

THEOREM (not easy): *$\Gamma_{\min}$ and $\Gamma_{\max}$ are operator complete. In general, Γ_L is* not *operator complete.*

Example of non completeness: If $K = J = 2$ and $L = 3$ then $\check{\mathbf{S}}_K = 0$.

HEISENBERG MODEL:

$$\check{H}_J^{\text{quant}} = \overline{\text{Tr}}[\overbrace{(\Gamma_{\min} \otimes \ldots \otimes \Gamma_{\min})}^{n \text{ times}} H_J^{\text{quant}}] = \left(\frac{J}{K}\right)^2 H_K^{\text{quant}}.$$

Therefore

$$(2J{+}1)^{-n}\text{Tr}\exp[-\beta H_J^{\text{quant}}] \geq (2K{+}1)^{-n}\text{Tr}\exp\left[-\left(\frac{J}{K}\right)^2 \beta H_K^{\text{quant}}\right]$$

Note: $\Gamma_{\min}$ (*not* $\Gamma_{\max}$) is the interesting operator to use with the generalized Berezin-Lieb inequality to get the bounds given before. $\Gamma_{\max}$ would give old inequalities. This is strange!

Finally, there are inequalities relating the *ferromagnetic* $(+)$ and *antiferromagnetic* $(-)$ free energies on a *bipartite* lattice (with E_{ij} non-vanishing if and only if i and j belong to different sublattices). Use as quantum coherent operator the product of $\Gamma_{\min}$ on one sublattice and $\Gamma_{\max}$ on the other sublattice.

$$\check{H}_J^{\text{quant}} = -\left(\frac{J}{J+1}\right) H_J^{\text{quant}}$$

$$f_{\pm}\left(J, \frac{J}{J+1}\beta\right) \leq f_{\mp}(J,\beta) \leq f_{\pm}\left(J, \frac{J+1}{J}\beta\right).$$