Example: the Ornstein-Uhlenbeck process

$$dX_t = -\beta(X_t - \alpha)dt + \sigma dW_t$$

where $\beta > 0, \alpha \in \mathbb{R}, \sigma > 0$ and $X_0 = x_0$.

Solution:

$$X_t = \alpha + (x_0 - \alpha)e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-s)} dW_s$$

Note that this is a sum of deterministic terms and an integral of a deterministic function with respect to a Wiener process with normally distributed increments. The distribution is thus normal.

The conditional expectation is

$$\begin{aligned} E[X_t|X_0 = x_0] &= E\left[\alpha + (x_0 - \alpha)e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-s)} dW_s\right] \\ &= \alpha + (x_0 - \alpha)e^{-\beta t} \end{aligned}$$

The conditional variance is

$$\text{Var}[X_t|X_0 = x_0] = E\left[(\sigma \int_0^t e^{-\beta(t-s)} dW_s)^2\right]$$

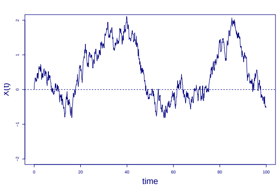
Use Ito's isometry to obtain

$$\text{Var}[X_t|X_0 = x_0] = \sigma^2 E\left[\int_0^t e^{-2\beta(t-s)} ds\right] = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)$$

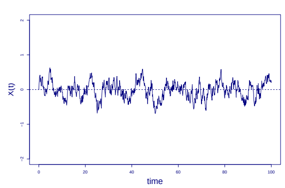
Thus $(X_t|X_0 = x_0) \sim N(\alpha + (x_0 - \alpha)e^{-\beta t}, \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right))$.

Asymptotically $X_t \sim N(\alpha, \frac{\sigma^2}{2\beta})$ (or always if $X_0 \sim N(\alpha, \frac{\sigma^2}{2\beta})$).

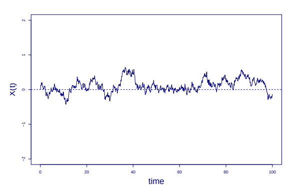
Parameter interpretation in the OU-process



$\beta = 0.01, \sigma = 1$



$\beta = 0.1, \sigma = 1$



$\beta = 0.01, \sigma = 0.5$

β	:	how "strongly" the system reacts to perturbations (the "decay-rate" or "growth-rate")
σ^2	:	the variation or the size of the noise.
α	:	the asymptotic mean

Example: population growth model

Also called the *geometric Brownian motion*

$$dN_t = aN_t dt + \sigma N_t dW_t$$

The Itô solution:

$$N_t = N_0 \exp\left\{(a - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}$$

The Stratonovich solution:

$$N_t = N_0 \exp\{at + \sigma W_t\}$$

Qualitative behavior of the Itô solution

$$N_t = N_0 \exp\left\{(a - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}$$

- If $a > \frac{1}{2}\sigma^2$ then $N_t \to \infty$ when $t \to \infty$, a.s.
- If $a < \frac{1}{2}\sigma^2$ then $N_t \to 0$ when $t \to \infty$, a.s.
- If $a = \frac{1}{2}\sigma^2$ then N_t will fluctuate between arbitrary large and arbitrary small values as $t \to \infty$, a.s.

Whereas for the Stratonovich solution we have

$$N_t = N_0 \exp\{at + \sigma W_t\}$$

- If $a > 0$ then $N_t \to \infty$ when $t \to \infty$, a.s.
- If $a < 0$ then $N_t \to 0$ when $t \to \infty$, a.s.

... just like in the deterministic case.

Apparently it makes a huge difference which interpretation we choose.

Note though:

If W_t is independent of N_t we would expect that

$$E[N_t] = E[N_0]e^{at}$$

i.e. the same as when there is no noise in a_t. Let us check:

Let

$$Y_t = e^{\sigma W_t}$$

and apply Ito's formula

$$dY_t = \frac{1}{2}\sigma^2 e^{\sigma W_t} dt + \sigma e^{\sigma W_t} dW_t$$

i.e.

$$Y_t = Y_0 + \frac{1}{2}\sigma^2 \int_0^t e^{\sigma W_s} ds + \sigma \int_0^t e^{\sigma W_s} dW_s$$

Thus

$$E[Y_t] = \underbrace{E[Y_0]}_{=1} + \frac{1}{2}\sigma^2 \int_0^t \underbrace{E[e^{\sigma W_s}]}_{=E[Y_s]} ds + \sigma \underbrace{E\left[\int_0^t e^{\sigma W_s} dW_s\right]}_{=0}$$

We obtain the differential equation for $E[Y_t]$:

$$\frac{d}{dt}E[Y_t] = \frac{1}{2}\sigma^2 E[Y_t] \quad ; \quad E[Y_0] = 1$$

so that

$$E[Y_t] = E[e^{\sigma W_t}] = e^{\sigma^2 t/2}$$

Finally

$$\begin{aligned} E[N_t] &= E\left[N_0 \exp\left\{(a-\frac{1}{2}\sigma^2)t+\sigma W_t\right\}\right] \\ &= E[N_0]\exp\left\{(a-\frac{1}{2}\sigma^2)t\right\} E\left[\exp\left\{\sigma W_t\right\}\right] \\ &= E[N_0]\exp\left\{(a-\frac{1}{2}\sigma^2)t\right\}\exp\left\{\frac{1}{2}\sigma^2 t\right\} \\ &= E[N_0]e^{at} \end{aligned}$$

exactly as we expected! However, for the Stratonovich solution, the same calculations give

$$E[N_t] = E[N_0]e^{(\tilde{a}+\sigma^2/2)t}$$

where $\tilde{a}$ is seen to be a *different* parameter from a.

An existence and uniqueness result

Linear growth and local Lipschitz conditions:

For each $N \in \mathbb{N}$ there exists a constant K_N such that

$$|b(x,t)| + |\sigma(x,t)| \leq K_N(1+|x|)$$

and

$$|b(x,t)-b(y,t)| + |\sigma(x,t)-\sigma(y,t)| \leq K_N(x-y)$$

for all $t \in [0, \mathbb{N}]$ and for all x, where $|\sigma|^2 = \text{tr } \sigma\sigma^T$.

Then

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = U, \quad U \perp \{W_t\}_{t\geq 0}$$

has a unique t-continuous solution X_t.

Examples from ODEs

The equation

$$\frac{dx_t}{dt} = x_t^2 \quad , \quad x_0 = 1$$

does not satisfy the linear growth condition. It has the unique solution

$$x_t = \frac{1}{1-t} \quad ; \quad 0 \leq t < 1$$

but no global solution (defined for all t).

The linear growth condition ensures that the solution X_t does not *explode*, i.e. $|X_t|$ does not tend to ∞ in finite time.

Examples from ODEs

The equation

$$\frac{dx_t}{dt} = 3x_t^{2/3} \quad , \quad x_0 = 0$$

does not satisfy the Lipschitz condition at $x = 0$. It has more than one solution:

$$x_t = \begin{cases} 0 & \text{for} \quad t \leq a \\ (t-a)^3 & \text{for} \quad t > a \end{cases}$$

for any $a > 0$.

The Lipschitz condition ensures that a solution X_t is unique: If $X_t^{(1)}$ and $X_t^{(2)}$ are two t-continuous processes satisfying the conditions then

$$X_t^{(1)} = X_t^{(2)} \quad \text{for all} \quad t \leq T, \text{ a.s.}$$

The solution X_t where drift and diffusion coefficients fulfill the growth and Lipschitz conditions is a *strong* solution:

- the version of W_t is given in advance
- The solution X_t is $\mathcal{F}_t^U$-adapted

$\mathcal{F}_t^U$ is the filtration generated by the initial U and $W_s, s \leq t$.

If only $b(\cdot)$ and $\sigma(\cdot)$ are given, and we ask for a pair of processes $(\tilde{X}_t, \tilde{W}_t)$ then the solution is called a *weak* solution.

Strong uniqueness means *pathwise* uniqueness, weak uniqueness means that any two solutions are identical in law, i.e. have the same finite dimensional distributions.

Sufficient condition for the *existence of a unique weak solution*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = U$$

$a(x) = \sigma(x)\sigma(x)^T$

- a continuous
- $a(x)$ strictly positive definite for all x
- There exists a constant K such that

$$\begin{aligned} |a_{ij}(x)| &\leq K(1 + |x|^2) \\ |b_i(x)| &\leq K(1 + |x|) \end{aligned}$$

for all $i, j = 1, \cdots, d$ and x.

The solution is a strong Markov process.

A strong solution is also a weak solution.

There are SDEs with no strong solution, but still a unique weak solution.

Remark: Note that the above conditions are *sufficient* conditions, not *necessary* conditions.

Transition densities:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

$$y \mapsto p(t, x, y)$$

Conditional density of X_t given $X_0 = x$;
also conditional density of X_{t+s} given $X_s = x$.

Data: $X_{t_1}, \cdots, X_{t_n}, \quad t_1 < \cdots < t_n$.

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} p(t_i - t_{i-1}, X_{t_{i-1}}, X_{t_i}; \theta)$$

- Chapman-Kolmogorov equation:

$$p(t+s,x,y) = \int p(t,z,y)p(s,x,z)dz$$

- Kolmogorov's backward equation:

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2} + b(x)\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t}$$

with the initial condition

$$p(t,x,y) \to \delta_x \quad \text{as } t \to 0.$$

δ_x is the Dirac measure at x.

- Kolmogorov's forward equation:

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma(y)^2 p] - \frac{\partial}{\partial y}[b(y)p] = \frac{\partial p}{\partial t}$$

(Fokker-Planck equation)

Examples:

- Ornstein-Uhlenbeck

$$dX_t = -\beta(X_t - \alpha)dt + \sigma dW_t$$

Remember that
$(X_t|X_0 = x_0) \sim N(\alpha + (x-\alpha)e^{-\beta t}, \sigma^2(1-e^{-2\beta t})/2\beta)$.

$$p(t,x,y) = \frac{1}{\sqrt{\pi\sigma^2(1-e^{-2\beta t})/\beta}} \exp\left[-\frac{(y-\alpha-(x-\alpha)e^{-\beta t})^2}{\sigma^2(1-e^{-2\beta t})/\beta}\right]$$

- Cox-Ingersoll-Ross

$$dX_t = -\theta(X_t - \alpha)dt + \sigma\sqrt{X_t}dW_t$$

$\theta > 0$, $\alpha > 0$, $\sigma > 0$.

$$\begin{aligned} p(t,x,y) &= \frac{\beta(y/x)^{\frac{1}{2}\nu}\exp(\frac{1}{2}\theta\nu t - \beta y)}{\Gamma(\beta\alpha)(1-\exp(-\theta t))} \\ &\times \exp\left[\frac{-\beta(x+y)}{\exp(\theta t)-1}\right] \mathrm{I}_\nu\left(\frac{\beta\sqrt{xy}}{\sinh(\frac{1}{2}\theta t)}\right), \end{aligned}$$

where $\beta = 2\theta\sigma^{-2}$ and $\nu = \beta\alpha - 1$.

I_ν is a modified Bessel function with index ν.

The transition density is a non-central χ^2-distribution.

- Radial Ornstein-Uhlenbeck

$$dX_t = (\theta X_t^{-1} - X_t)dt + dW_t,$$

$\theta > 0$.

$$\begin{aligned} p(t,x,y) &= \frac{(y/x)^\theta\sqrt{xy}\exp(-y^2 + (\theta + \frac{1}{2})t)}{\sinh(t)} \\ &\times \exp\left[\frac{-(x^2+y^2)}{\exp(2t)-1}\right] \mathrm{I}_{\theta-\frac{1}{2}}\left(\frac{xy}{\sinh(t)}\right) \end{aligned}$$

I_ν is a modified Bessel function with index ν.

Taylor expansions

Review of deterministic expansions:

Consider

$$\frac{d}{dt}x_t = a(x_t)$$

with initial value x_{t_0} for $t \in [t_0, T]$, and $a(\cdot)$ is sufficiently smooth. We can write

$$x_t = x_{t_0} + \int_{t_0}^{T} a(x_s)ds$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. By the chain rule

$$\frac{d}{dt}f(x_t) = a(x_t)f'(x_t)$$

Define the operator

$$Lf = af'$$

where ' denotes differentiation with respect to x. Express the above equation for $f(x)$ in integral form

$$f(x_t) = f(x_{t_0}) + \int_{t_0}^{t} Lf(x_s)ds$$

Note that if $f(x) = x$ then $Lf = a, L^2 f = La$ and

$$x_t = x_{t_0} + \int_{t_0}^{t} a(x_s)ds$$

If $f(x) = a(x)$ then $La = aa'$ and

$$a(x_s) = a(x_{t_0}) + \int_{t_0}^{s} La(x_z)dz$$

Apply this to the equation for x_t

$$\begin{aligned} x_t &= x_{t_0} + \int_{t_0}^{t} \left(a(x_{t_0}) + \int_{t_0}^{s} La(x_z)dz \right) ds \\ &= x_{t_0} + a(x_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} La(x_z)dz\, ds \\ &= x_{t_0} + a(x_{t_0})(t - t_0) + R_1 \end{aligned}$$

which is the simplest non-trivial expansion for x_t.

Apply again to the function $f = La$ to obtain

$$\begin{aligned} x_t &= x_{t_0} + a(x_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} La(x_z)dzds \\ &= x_{t_0} + a(x_{t_0}) \int_{t_0}^{t} ds + La(x_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz\, ds + R_2 \\ &= x_{t_0} + a(x_{t_0})(t - t_0) + La(x_{t_0})\frac{1}{2}(t - t_0)^2 + R_2 \end{aligned}$$

where

$$R_2 = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^2 a(x_u)du\, dz\, ds$$

For a general $r+1$ times continuously differentiable function f we obtain the classical Taylor formula in integral form

$$f(x_t) = f(x_{t_0}) + \sum_{l=1}^{r} \frac{(t-t_0)^l}{l!} L^l f(x_{t_0}) + \int_{t_0}^{t} \cdots \int_{t_0}^{s_r} L^{r+1} f(x_{s_1}) ds_1 \dots ds_{r+1}$$

The Ito-Taylor expansion

Iterated application of Ito's formula!

Consider

$$X_t = X_{t_0} + \int_{t_0}^{t} b(X_s)ds + \int_{t_0}^{t} \sigma(X_s) dWs$$

We introduce the operators

$$L^0 f = bf' + \frac{1}{2}\sigma^2 f''$$
$$L^1 f = \sigma f'$$

For f twice continuously differentiable, Ito's formula yields

$$\begin{aligned} f(X_t) &= f(X_{t_0}) + \int_{t_0}^{t} \left(b(X_s) f'(X_s) + \frac{1}{2}\sigma^2(X_s) f''(X_s) \right) ds \\ &\quad + \int_{t_0}^{t} \sigma(X_s) f'(X_s) dWs \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f'(X_s) dWs \end{aligned}$$

Note that for $f(x) = x$ we have $L^0 f = b$ and $L^1 f = \sigma$, and the original equation for X_t is obtained

$$X_t = X_{t_0} + \int_{t_0}^{t} b(X_s)ds + \int_{t_0}^{t} \sigma(X_s) dWs$$

Like in the deterministic expansions, we apply Ito's formula to the functions $f = b$ and $f = \sigma$ and obtain

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z) dz + \int_{t_0}^{s} L^1 b'(X_z) dWz \right) ds \\ &\quad + \int_{t_0}^{t} \left(\sigma(X_{t_0}) + \int_{t_0}^{s} L^0 \sigma(X_z) dz + \int_{t_0}^{s} L^1 \sigma'(X_z) dWz \right) dWs \\ &= X_{t_0} + b(X_{t_0}) \int_{t_0}^{t} ds + \sigma(X_{t_0}) \int_{t_0}^{t} dWs + R \\ &= X_{t_0} + b(X_{t_0})(t-t_0) + \sigma(X_{t_0})(W_t - W_{t_0}) + R \end{aligned}$$

This is the simplest non-trivial Ito-Taylor expansion of X_t involving single integrals with respect to both time and the Wiener process. The remainder contains multiple integrals with respect to both.

In the previous expansion we had

$$R = \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z) dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z) dW_z\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 \sigma(X_z) dz\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 \sigma(X_z) dW_z\, dW_s$$

Note that $dz\, ds, dW_z\, ds$ and $dz\, dW_s$ "scales like 0", whereas $dW_z\, dW_s$ scales like dt, comparable to the terms in the simplest expansion with two single integrals.

We therefore continue the expansion by applying the Ito formula to $f = L^1\sigma$.

The next Ito-Taylor expansion becomes

$$\begin{aligned} X_t &= X_{t_0} + b(X_{t_0})\int_{t_0}^{t} ds + \sigma(X_{t_0})\int_{t_0}^{t} dWs + L^1\sigma(X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s + \bar{R} \\ &= X_{t_0} + b(X_{t_0})\Delta t + \sigma(X_{t_0})\Delta W_t + \sigma(X_{t_0})\sigma'(X_{t_0})\frac{1}{2}(\Delta W_t^2 - \Delta t) + \bar{R} \end{aligned}$$

with remainder

$$\begin{aligned} \bar{R} &= \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z) dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z) dW_z\, ds \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} L^0 \sigma(X_z) dz\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 \sigma(X_u) du\, dW_z\, dW_s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 \sigma(X_u) dW_u\, dW_z\, dW_s \end{aligned}$$

Numeric solutions

When no explicit solution is available we can approximate different characteristics of the process by simulation. (Realizations, moments, qualitative behavior etc). We use the approximations from the Ito-Taylor expansions.

- Different schemes (Euler, Milstein, higher order schemes...)
- Rate of convergence (Weak and strong)

Consider the Itô stochastic differential equation

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t$$

and a time discretization

$$0 = t_0 < t_1 < \cdots < t_j < \cdots < t_N = T$$

Put

$$\begin{aligned} \Delta_j &= t_{j+1} - t_j \\ \Delta W_j &= W_{t_{j+1}} - W_{t_j} \end{aligned}$$

Then

$$\Delta W_j \sim N(0, \Delta_j)$$

The Euler-Maruyama scheme

We approximate the process X_t given by

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \; ; \; X(0) = x_0$$

at the discrete time-points $t_j, 1 \le j \le N$ by

$$Y_{t_{j+1}} = Y_{t_j} + b(Y_{t_j})\Delta_j + \sigma(Y_{t_j})\Delta W_j \; ; \; Y_{t_0} = x_0$$

where $\Delta W_j = \sqrt{\Delta_j} \cdot Z_j$, with $Z_j \sim N(0,1)$ for all j.

The Euler-Maruyama scheme

Let us consider the expectation of the absolute error at the final time instant T:

There exist constants $K > 0$ and $\delta_0 > 0$ such that

$$E(|X_T - Y_{t_N}|) \le K\delta^{0.5}$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$.

We say that the approximating process Y *converges in the strong sense* with order 0.5.

(Compare with the Euler scheme for an ODE which has order 1).

The Euler-Maruyama scheme

Sometimes we do not need a close *pathwise* approximation, but only some function of the value at a given final time T (e.g. $E(X_T)$, $E(X_T^2)$ or generally $E(g(X_T))$):

There exist constants $K > 0$ and $\delta_0 > 0$ such that for any polynomial g

$$|E(g(X_T) - E(g(Y_{t_N}))| \le K\delta$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$.

We say that the approximating process Y *converges in the weak sense* with order 1.

The Milstein scheme

We can even do better!

We approximate X_t by

$$\begin{aligned} Y_{t_{j+1}} &= Y_{t_j} + b(Y_{t_j})\Delta_j + \sigma(Y_{t_j})\Delta W_j \\ &\quad + \frac{1}{2}\sigma(Y_{t_j})\sigma'(Y_{t_j})\{(\Delta W_j)^2 - \Delta_j\} \quad \text{(now Milstein...)} \end{aligned}$$

where the prime $'$ denotes the derivative.

The Milstein scheme

The Milstein scheme converges in the strong sense with order 1:

$$E(|X_T - Y_{t_N}|) \quad \leq \quad K\delta$$

We could regard the Milstein scheme as the proper generalization of the deterministic Euler-scheme.

If $b(X_t)$ does not depend on X_t the Euler-Maruyama and the Milstein scheme coincide.

Multi-dimensional diffusions:

Euler scheme: Similar.

Milstein scheme: Involves multiple Wiener integrals.

$$\int_{n\delta}^{(n+1)\delta} \int_{n\delta}^{s} dW_u^{(1)} dW_s^{(2)}$$

Simulation schemes are based on stochastic Ito-Taylor expansions that are formally obtained by iterated use of Ito's formula.

Kloeden and Platen (1992)