Kapitel 11

One-dimensional homogeneous diffusions

Suppose given a real-valued process $\boldsymbol{X}$, which is a solution to an SDE, ($\boldsymbol{B}$ is BM(1))

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t), \quad X_0 \equiv x_0. \tag{11.1}$$

Suppose also that it is known that

$$\mathrm{P}\bigcap_{t\geq 0}\big(X(t) \in \,]l, r[\big) = 1,$$

where $]l, r[\subset \mathbb{R}$ is an open interval that could be a genuine subinterval. In particular $l < x_0 < r$.

Our first aim is to discuss conditions on the functions b and σ, that ensure that $\boldsymbol{X}$ in fact stays away from the boundary points l and r, also in the case where $l = -\infty$ or $r = +\infty$.

Assume from now on that b, σ are continuous functions on $]l, r[$, with $\sigma > 0$ (but do not assume Lipschitz conditions as in Sætning 9.1). At the moment we are just given a $]l, r[$-valued solution to (11.1)).

We start by looking for a twice differentiable function $S :]l, r[\to \mathbb{R}$, such that $S(\boldsymbol{X}) \in \mathbf{c}\mathscr{M}_{\mathrm{loc}}$. By Itô's formula

$$\mathrm{d}S(X(t)) = AS(X(t))\,\mathrm{d}t + S'(X(t))\sigma(X(t))\,\mathrm{d}B(t) \tag{11.2}$$

where A is the second order differential operator

$$Af(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Thus $S(\boldsymbol{X}) \in \mathbf{c}\mathscr{M}_{\mathrm{loc}}$ if $AS \equiv 0$. This gives that S satisfies

$$S'(x) = c\exp\left(-\int_{x_0}^{x}\frac{2b(y)}{\sigma^2(y)}\,\mathrm{d}y\right)$$

for some c. If $c > 0$ we have $S' > 0$ so S is strictly increasing. S is called a *scale function* for the diffusion $\boldsymbol{X}$. If S is a scale function, all others are of the form $c_1 + c_2 S$ for some $c_1 \in \mathbb{R}$, $c_2 > 0$.

With $x_0 \in]l, r[$ given, fix $a < x_0 < b$, $a, b \in]l, r[$. Define

$$\tau_{a,b} = \inf\{t : X(t) = a \text{ or } X(t) = b\}.$$

Then $(S(\boldsymbol{X}))^{\tau_{a,b}}$ is a bounded local martingale, hence a true martingale, and so, for all t

$$\mathrm{E}\, S(X(\tau_{a,b} \wedge t)) = \mathrm{E}\, S(X(0)) = S(x_0)$$

and for $t \to \infty$, by dominated convergence

$$\mathrm{E}\, S(X(\tau_{a,b})) = S(x_0),$$

where

$$S(X(\tau_{a,b})) = \begin{cases} S(b) & \text{on } (\tau_b < \tau_a) \\ S(a) & \text{on } (\tau_a < \tau_b) \\ \lim\limits_{t\to\infty} S(X(t)) & \text{on } (\tau_{a,b} = +\infty) \end{cases}$$

exists by the martingale convergence theorem. It will be shown below that $P(\tau_{a,b} < \infty) = 1$, believing it for the moment we find

$$S(X(0)) = \mathrm{E}\, S(X(\tau_{a,b})) = S(b)\, \mathrm{P}(\tau_b < \tau_a) + S(a)\, \mathrm{P}(\tau_a < \tau_b),$$

i.e

$$\mathrm{P}(\tau_b < \tau_a) = 1 - \mathrm{P}(\tau_a < \tau_b) = \frac{S(x_0) - S(a)}{S(b) - S(a)},$$

the first basic formula.

Notation. $\tau_c = \inf\{t : X(t) = c\}$ for $c \in]l, r[$.

Note that since $\lim_{t\to\infty} S(X(t))$ exists almost surely on $(\tau_{a,b} = +\infty)$ and because S is strictly increasing and continuous, also $\lim_{t\to\infty} X(t)$ exists almost surely on $(\tau_{a,b} = +\infty)$ (this is what we can say at the moment - remember that we shall show shortly that $\mathrm{P}(\tau_{a,b} = +\infty) = 0$).

With $a < x_0 < b$ as before, let $\varphi : [a, b] \to \mathbb{R}$ be continuous and let f denote the unique solution to

$$Af(x) = -\varphi(x), \quad a \le x \le b, \quad f(a) = f(b) = 0.$$

Then, see (11.2),

$$\mathrm{d}f(X(t)) + \varphi(X(t))\, \mathrm{d}t = S'(X(t))\sigma(X(t))\, \mathrm{d}B(t)$$

so

$$M_t(f) \equiv f(X(t)) + \int_0^t \varphi(X(s))\, \mathrm{d}s$$

is a continuous, local martingale, hence so is $\boldsymbol{M}(f)^{\tau_{a,b}}$. But since

$$\sup_{s \le t} |M_{\tau_{a,b} \wedge s}(f)| \le \sup_{a \le x \le b} |f(x)| + t \sup_{a \le x \le b} |\varphi(x)| < \infty,$$

$\boldsymbol{M}(f)^{\tau_{a,b}}$ is a true martingale, in particular

$$\mathrm{E}\int_0^{t\wedge\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s = f(x_0) - \mathrm{E}\,f(X(t\wedge\tau_{a,b})).$$

Since f is continuous, by the remark on the previous page,

$$\lim_{t\to\infty} f(X(t\wedge\tau_{a,b})) = f(X(\tau_{a,b}))$$

exists almost surely, and by dominated convergence

$$\mathrm{E}\,f(X(\tau_{a,b})) = \lim_{t\to\infty} \mathrm{E}\,f(X(t\wedge\tau_{a,b})).$$

Further, by monotone convergence if $\varphi \geq 0$ or $\varphi \leq 0$,

$$\lim_{t\to\infty} \mathrm{E}\int_0^{t\wedge\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s = \mathrm{E}\int_0^{\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s$$

so that for such φ,

$$\mathrm{E}\int_0^{\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s = f(x_0) - \mathrm{E}\,f(X(\tau_{a,b})).$$

Taking $\varphi \equiv 1$ on $[a,b]$ this gives

$$\mathrm{E}\,\tau_{a,b} = f_0(x_0) - \mathrm{E}\,f_0(X(\tau_{a,b})),$$

where f_0 solves $Af_0 \equiv -1$, $f_0(a) = f_0(b) = 0$. Since the expression on the right hand side is finite $\mathrm{E}\,\tau_{a,b} < \infty$ follows. In particular $\mathrm{P}(\tau_{a,b} < \infty) = 1$, and we have shown the basic scale function formula on p. 114. Also, for general φ, since it is now clear that $\mathrm{E}\,f(X(\tau_{a,b})) = 0$ because $f(a) = f(b) = 0$,

$$\mathrm{E}\int_0^{\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s = f(x_0).$$

This is certainly true if $\varphi \geq 0$ or $\varphi \leq 0$. For general continuous φ, write $\varphi = \varphi^+ - \varphi^-$. Below, in Theorem 11.2, we give an identity which is true for alle bounded Borel functions $\varphi : [a,b] \to \mathbb{R}$.

Lemma 11.1. *Let S be an arbitrary scale function with*

$$S'(x) = c\exp\left(-\int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)}\,\mathrm{d}y\right)$$

for some $c > 0$, and define $k :]l,r[\to \mathbb{R}_+$ by

$$k(x) = \frac{2}{\sigma^2(x)S'(x)}.$$

Then the unique solution f to $Af \equiv -\varphi$ on $[a,b]$, $f(a) = f(b) = 0$, where φ is a given continuous function, is

$$f(x) = \int_a^b G_{a,b}(x,y)\varphi(y)k(y)\,\mathrm{d}y, \tag{11.3}$$

where $G_{a,b}$ is the Green function $G_{a,b}(x,y) = G_{a,b}(y,x)$,

$$G_{a,b}(x,y) = \frac{(S(x)-S(a))(S(b)-S(y))}{S(b)-S(a)}, \quad a \leq x \leq y \leq b$$

Proof. Since $G_{a,b}(a,y) = G_{a,b}(x,b) = 0$, clearly $f(a) = f(b) = 0$. If $x < z \in [a,b]$,

$$\begin{aligned} f(z) - f(x) = & \frac{S(x)-S(z)}{S(b)-S(a)} \int_a^x \big(S(y)-S(a)\big)\varphi(y)k(y)\,\mathrm{d}y \\ & + \frac{S(z)-S(x)}{S(b)-S(a)} \int_z^b \big(S(b)-S(y)\big)\varphi(y)k(y)\,\mathrm{d}y \\ & + \frac{1}{S(b)-S(a)} \int_x^z \Big(\big(S(y)-S(a)\big)\big(S(b)-S(z)\big) \\ & \qquad - \big(S(x)-S(a)\big)\big(S(b)-S(y)\big)\Big)k(y)\,\mathrm{d}y. \end{aligned} \tag{11.4}$$

Look at the last term in (11.4). We get that

$$\begin{aligned} \big(S(y)-S(a)\big)\big(S(b)-S(z)\big) & - \big(S(x)-S(a)\big)\big(S(b)-S(y)\big) \\ & \le \big(S(z)-S(a)\big)\big(S(b)-S(z)\big) - \big(S(x)-S(a)\big)\big(S(b)-S(z)\big) \\ & = \big(S(z)-S(x)\big)\big(S(b)-S(z)\big) \end{aligned}$$

and, by similar reasoning

$$\begin{aligned} \big(S(y)-S(a)\big)\big(S(b)-S(z)\big) & - \big(S(x)-S(a)\big)\big(S(b)-S(y)\big) \\ & \ge -\big(S(z)-S(x)\big)\big(S(x)-S(a)\big). \end{aligned}$$

For the integral itself we therefore get that

$$\begin{aligned} \int_x^z \Big(\big(S(y)-S(a)\big)\big(S(b)-S(z)\big) - \big(S(x)-S(a)\big)\big(S(b)-S(y)\big)\Big)k(y)\,\mathrm{d}y. \\ \le \big(S(z)-S(x)\big)\big(S(b)-S(z)\big) \int_x^z k(y)\,\mathrm{d}y, \end{aligned}$$

and

$$\begin{aligned} \int_x^z \Big(\big(S(y)-S(a)\big)\big(S(b)-S(z)\big) - \big(S(x)-S(a)\big)\big(S(b)-S(y)\big)\Big)k(y)\,\mathrm{d}y. \\ \ge -\big(S(z)-S(x)\big)\big(S(x)-S(a)\big) \int_x^z k(y)\,\mathrm{d}y. \end{aligned}$$

It now follows, dividing in (11.4) by $S(z) - S(x)$ and taking limits, that

$$\begin{aligned} \frac{f'(x)}{S'(x)} = & -\frac{1}{S(b)-S(a)} \int_a^x (S(y)-S(a))\varphi(y)k(y)\,\mathrm{d}y \\ & + \frac{1}{S(b)-S(a)} \int_x^b (S(b)-S(y))\varphi(y)k(y)\,\mathrm{d}y \end{aligned}$$

and differentiating this after x gives

$$\begin{aligned} \left(\frac{f'}{S'}\right)'(x) & = -\frac{1}{S(b)-S(a)}\big((S(x)-S(a))\varphi(x)k(x) + (S(b)-S(x))\varphi(x)k(x)\big) \\ & = -\varphi(x)k(x). \end{aligned}$$

It remains only to check that

$$\left(\frac{f'}{S'}\right)' = \frac{1}{S'}(f'' - (\log S')'f') = \frac{k\sigma^2}{2}\left(f'' + \frac{2b}{\sigma^2}f'\right) = kAf. \qquad \square$$

The measure m on $]l,r[$ with density k, $m(\mathrm{d}x) = k(x)\,\mathrm{d}x$, is called the *speed measure* for the diffusion $\boldsymbol{X}$. Note that if the scale function S is replaced by $c_1 + c_2 S$ (where $c_1 \in \mathbb{R}$, $c_2 > 0$), k is replaced by $\frac{1}{c_2}k$.

We summarize the results obtained so far, writing $\mathrm{P}^{x_0}, \mathrm{E}^{x_0}$ instead of P, E to emphasize the initial value $X(0) \equiv x_0$, which is called x in the theorem.

Theorem 11.2. *With $\boldsymbol{X}$ given by (11.1) a diffusion with values in $]l,r[$, where b and $\sigma > 0$ are continuous, and where $X(0) \equiv x \in]l,r[$, it holds for $a < x < b$, $a, b \in]l,r[$, that $\mathrm{P}^x(\tau_{a,b} < \infty) = 1$,*

$$\mathrm{P}^x(\tau_b < \tau_a) = 1 - \mathrm{P}^x(\tau_a < \tau_b) = \frac{S(x) - S(a)}{S(b) - S(a)},$$

and for $\varphi : [a,b] \to \mathbb{R}$ bounded and measurable, that

$$\mathrm{E}^x \int_0^{\tau_{a,b}} \varphi(X(s))\,\mathrm{d}s = \int_a^b G_{a,b}(x,y)\varphi(y)k(y)\,\mathrm{d}y,$$

in particular

$$\mathrm{E}^x\,\tau_{a,b} = \int_a^b G_{a,b}(x,y)k(y)\,\mathrm{d}y.$$

In the formulas above, S, given by (apart from an additive constant),

$$S'(x) = \exp\left(-\int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)}\,\mathrm{d}y\right)$$

is an arbitrary scale function and

$$k(x) = \frac{2}{\sigma^2(x)S'(x)}$$

in the corresponding speed measure density.

Example 11.3. If $\boldsymbol{X}$ is a BM(1)-process, $\boldsymbol{X}$ is a martingale, so $S(x) = x$ is a scale function which corresponds to $k \equiv 2$, i.e the speed measure is two times the Lebesgue measure. Further

$$\mathrm{P}^x(\tau_b < \tau_a) = \frac{x-a}{b-a},$$
$$\mathrm{E}^x\,\tau_{a,b} = (x-a)(b-x)$$

for $a \le x \le b \in \mathbb{R}$.

If $\boldsymbol{X}$ is a Brownian motion with drift ξ, diffusion coefficient σ ($X(t) = X(0) + \xi t + \sigma B(t)$)

$$S'(x) = e^{-\frac{2\xi}{\sigma^2}x}, \quad k(x) = \frac{2}{\sigma^2} e^{\frac{2\xi}{\sigma^2}x}.$$ ◦

So far we have assumed that

$$\mathrm{P}^x \bigcap_{t \ge 0} \big(X(t) \in]l,r[\big) = 1,$$

i.e. that $\tau_r = \tau_l \equiv \infty$ P^x-almost surely. The next result will tell us what are the properties of S and k that prevents $\boldsymbol{X}$ from reaching either of the boundaries l and r. Throughout S is a given scale, k the matching density for the speed measure.

Theorem 11.4. *Define*

$$S(r) = \lim_{y \uparrow r} S(y) \le \infty, \quad S(l) = \lim_{y \downarrow l} S(y) \ge -\infty.$$

(i) *Either*

$$S(r) = \infty \quad \text{or} \quad \int_y^r (S(r) - S(z))k(z)\,\mathrm{d}z = \infty, \quad y \in \,]l, r[$$

and similarly, either

$$S(l) = -\infty \quad \text{or} \quad \int_l^y (S(z) - S(l))k(z)\,\mathrm{d}z = \infty, \quad y \in \,]l, r[.$$

(ii) *For* $a < x < b$*,*

$$\mathrm{P}^x(\tau_a < \infty) = \frac{S(r) - S(x)}{S(r) - S(a)}, \quad \mathrm{P}^x(\tau_b < \infty) = \frac{S(x) - S(l)}{S(b) - S(l)}.$$

In particular $\mathrm{P}^x(\tau_y < \infty) > 0$ *for all* $x, y \in \,]l, r[$, $\mathrm{P}^x(\tau_a < \infty) = 1$ *if and only if* $S(r) = \infty$ *and* $\mathrm{P}^x(\tau_b < \infty) = 1$ *if and only if* $S(l) = -\infty$.

(iii) *If* $S(r) < \infty$*, then* $\lim_{t\to\infty} X(t) = r$ P^x*-almost surely on* A_-*, where*

$$A_- = \bigcup_{a: a < x} (\tau_a = \infty), \quad \text{and} \quad \mathrm{P}^x(A_-) = \frac{S(x) - S(l)}{S(r) - S(l)},$$

and if $S(l) > -\infty$*, then* $\lim_{t\to\infty} X(t) = l$ P^x*-almost surely on* A_+*, where*

$$A_+ = \bigcup_{b: b > x} (\tau_b = \infty), \quad \text{and} \quad \mathrm{P}^x(A_+) = \frac{S(r) - S(x)}{S(r) - S(l)}.$$

(iv)

$$\mathrm{P}^x(\lim_{t\to\infty} X(t) = r) = 1, \quad \text{if } S(r) < \infty,\ S(l) = -\infty,$$
$$\mathrm{P}^x(\lim_{t\to\infty} X(t) = r) = 1, \quad \text{if } S(r) = \infty,\ S(l) > -\infty.$$

(v) *If* $S(r) < \infty$ *and* $S(l) > -\infty$ *then*

$$\mathrm{P}^x(\lim_{t\to\infty} X(t) = r) = 1 - \mathrm{P}^x(\lim_{t\to\infty} X(t) = l) = \frac{S(x) - S(l)}{S(r) - S(l)}.$$

(vi) *If* $S(r) = \infty$ *and* $S(l) = -\infty$ *then* $\boldsymbol{X}$ *is recurrent in the sense that*

$$\mathrm{P}^x \bigcap_{y \in]l,r[} \bigcap_{t>0} \bigcup_{s>t} (X(s) = y) = 1,$$

i.e. $\boldsymbol{X}$ *hits any level infinitely often in any interval* $[t, \infty[$, $t \ge 0$.

Proof. Let $l < a < x < b < r$. For $b \uparrow r$, $\tau_b \uparrow \tau_r \equiv \infty$ (by the assumption that $\boldsymbol{X}$ never hits r), so $1_{(\tau_b < \tau_a)} \to 1_{(\tau_a = \infty)}$, hence

$$\mathrm{P}^x(\tau_a = \infty) = \lim_{b \to r} \frac{S(x) - S(a)}{S(b) - S(a)} = \frac{S(x) - S(a)}{S(r) - S(a)}$$

proving (ii).

If $S(r) < \infty$, $S(\boldsymbol{X})^{\tau_a}$ is a bounded local martingale, hence a true martingale, so the random variable

$$S(X(\tau_a)) = \begin{cases} S(a) & \text{on } (\tau_a < \infty) \\ \lim_{t \to \infty} S(X(t)) & \text{on } (\tau_a = \infty) \end{cases}$$

is well defined P^x-almost surely and satisfies

$$\mathrm{E}^x\, S(X(\tau_a)) = S(x).$$

On the other hand

$$S(x) = \mathrm{E}^x\, S(X(\tau_a)) = S(a) \frac{S(r) - S(x)}{S(r) - S(a)} + \mathrm{E}^x \left(S(X(\tau_a)) 1_{(\tau_a = \infty)} \right).$$

implying that

$$\mathrm{E}^x \left(S(X(\tau_a)) 1_{(\tau_a = \infty)} \right) = S(r) \frac{S(x) - S(a)}{S(r) - S(a)} = S(r)\, \mathrm{P}^x(\tau_a = \infty).$$

Since $S(X(\tau_a)) \le S(r)$, it follows that $S(X(\tau_a)) = S(r)$ P^x-almost surely on $(\tau_a = \infty)$, i.e. $\lim_{t \to \infty} X(t) = r$ P^x-almost surely on $(\tau_a = \infty)$ and (iii) follows since $(\tau_a = \infty) \uparrow A_-$ as $a \downarrow l$ so

$$\mathrm{P}^x(A_-) = \lim_{a \downarrow l} \mathrm{P}^x(\tau_a = \infty) = \frac{S(x) - S(l)}{S(r) - S(l)}.$$

Now we can prove (i): if $S(r) < \infty$, $\lim_{b \uparrow r} \tau_{a,b} = \tau_a$ P^x-almost surely and so by monotone convergence

$$\mathrm{E}^x\, \tau_a = \lim_{b \uparrow r} \int_a^b G_{a,b}(x, y) k(y)\, \mathrm{d}y.$$

But since by (ii), $\mathrm{P}^x(\tau_a = \infty) > 0$, the left hand side equals ∞. The right hand side equals

$$\begin{aligned} &\lim_{b \uparrow r} \Bigg(\int_x^b \frac{(S(x) - S(a))(S(b) - S(y))}{S(b) - S(a)} k(y)\, \mathrm{d}y \\ &\qquad + \int_a^x \frac{(S(y) - S(a))(S(b) - S(x))}{S(b) - S(a)} k(y)\, \mathrm{d}y \Bigg) \\ &= \frac{S(x) - S(a)}{S(r) - S(a)} \int_x^r (S(r) - S(y)) k(y)\, \mathrm{d}y \\ &\qquad + \frac{S(r) - S(x)}{S(r) - S(a)} \int_a^x (S(y) - S(a)) k(y)\, \mathrm{d}y \end{aligned}$$

with the last term finite, hence the first integral equals $+\infty$ and (i) is proved.

It remains to establish (vi). From (ii) we know that

$$\mathrm{P}^x(\tau_a < \infty) = \mathrm{P}^x(\tau_b < \infty) = 1$$

for all $a < x$, $b > x$. Let $a_n \downarrow l$, $b_n \uparrow r$, then $\tau_{a_n} \uparrow \infty$, $\tau_{b_n} \uparrow \infty$ P^x-almost surely and between τ_{a_n} and τ_{b_n} $\boldsymbol{X}$ passes through all levels $y \in [a_n, b_n]$ since it is continuous. (vi) follows easily from this. □

Instead of starting with a solution to the SDE (11.1), assume given an open interval $]l, r[$ and continuous functions $b :]l, r[\to \mathbb{R}$, $\sigma :]l, r[\to \mathbb{R}_+ =]0, \infty[$ that satisfy the condition from Theorem 11.4 (i):

$$S(r) = +\infty, \quad \text{or} \quad \int_y^r (S(r) - S(z))k(z)\,\mathrm{d}z = +\infty, \quad y \in]l, r[$$
$$S(l) = -\infty, \quad \text{or} \quad \int_l^y (S(r) - S(z))k(z)\,\mathrm{d}z = +\infty, \quad y \in]l, r[,$$

where for some $x_0 \in]l, r[$,

$$S'(x) = \exp\Big(- \int_{x_0}^x \frac{2b(y)}{\sigma^2(y)}\,\mathrm{d}y \Big), \quad k(x) = \frac{2}{\sigma^2(x) S'(x)}.$$

Theorem 11.5. *Let $]l, r[$, b, σ be as above, let $\boldsymbol{B}$ be a* BM(1)*-process on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ and let $U \in \mathcal{F}_0$ be a given random variable with values in $]l, r[$. Then the SDE*

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t), \quad X_0 \equiv U,$$

has a unique solution, which is a diffusion.

If $U \equiv x_0$, the distribution Π^{x_0} of $\boldsymbol{X}$, viewed as a random variable with values in $C_{\mathbb{R}_0}(]l, r[)$, the space of continuous paths $w : \mathbb{R}_0 \to]l, r[$, does not depend on the choice of $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ and $\boldsymbol{B}$ (uniqness in law), and with an arbitrary boundary condition $U \in \mathcal{F}_0$, the distibution of $\boldsymbol{X}$ is the mixture $\int_{]l,r[} \Pi^x\, \mathrm{P}(U \in \mathrm{d}x)$.

This very important result we cannot prove/don't have the time to prove. At best we could give a proof when b, σ are Lipschitz on any interval $]\lambda, \rho[$ where $l < \lambda < \rho < r$. Some of the ideas in a proof is contained in the following.

Example 11.6. Let $\boldsymbol{B}$ be a BM(d)-process where $d \geq 2$, let $a > 0$ and define

$$X(t) = \|\tilde{B}(t)\| = \Big(\sum_{j=1}^d \big(\tilde{B}^{(i)}(t)\big)^2 \Big)^{\frac{1}{2}}$$

where $\tilde{B}^{(j)}(t) = \tilde{B}^{(1)}(t) + a$ for $j = 1$, $\tilde{B}^{(j)}(t) = B^{(j)}(t)$ for $j \geq 2$.

$\boldsymbol{X}$ is a *d-dimensional Bessel process* (BES(d)) starting at $a > 0$. We shall first study the properties of $\boldsymbol{X}$ using Itô's formula. However, $x \mapsto \|x\|$ is C^2 only on $\mathbb{R}^d \setminus 0$, so it is necessary to stop $\boldsymbol{X}$ before it hits 0; let $0 < r < a$ and define

$$\tau = \inf\{t : X(t) = r\}.$$

Then $\boldsymbol{X}^\tau = \|\tilde{\boldsymbol{B}}^\tau\|$, and by Itô's formula, using that

$$\mathrm{D}_i \|x\| = \frac{x_i}{\|x\|}, \quad \mathrm{D}^2_{ij} \|x\| = \frac{\delta_{ij}}{\|x\|} - \frac{x_i x_j}{\|x\|^3}$$

we get

$$\mathrm{d}X^\tau(t) = \frac{1}{X^\tau(t)} \sum_{j=1}^{d} \big(\tilde{B}^{(j)}(t)\big)^\tau \,\mathrm{d}\big(\tilde{B}^{(j)}(t)\big) + \frac{1}{2}\frac{d-1}{X^\tau(t)}\,\mathrm{d}(t \wedge \tau),$$

in particular

$$[X^\tau](t) = \left(\frac{1}{(X^\tau)^2} \sum_{j=1}^{d} \big((\tilde{B}^{(j)})^\tau\big)^2 \cdot \big[(\tilde{B}^{(j)})^\tau\big]\right)(t) = \tau \wedge t.$$

Next, let $f_d : \mathbb{R}_0 \to \mathbb{R}$ solve $\frac{d-1}{2x} f_d'(x) = -\frac{1}{2} f_d''(x)$ i.e.

$$f_d(x) = \begin{cases} \log x & if\ d = 2 \\ -\frac{1}{d-2} x^{-(d-2)} & if\ d \geq 3. \end{cases}$$

Then

$$\mathrm{d}f_d(\boldsymbol{X}^\tau) = f_d'(\boldsymbol{X}^\tau)\,\mathrm{d}\boldsymbol{X}^\tau + \frac{1}{2} f_d''(\boldsymbol{X}^\tau)\,\mathrm{d}[\boldsymbol{X}^\tau] = \frac{1}{(\boldsymbol{X}^\tau)^d} \sum_{j=1}^{d} \big(\boldsymbol{B}^{(j)}\big)^\tau \,\mathrm{d}\big(\boldsymbol{B}^{(i)}\big)^\tau,$$

i.e. $f_d(\boldsymbol{X}^\tau)$ is a continuous local martingale. It follows that for any $n \in \mathbb{N}$, $N \in \mathbb{N}$, $f_d(\boldsymbol{X}^{\tau_{n,N}})$ is a true martingale (being a bounded local martingale), where

$$\tau_{n,N} = \inf\Big\{t : X(t) = \frac{1}{n} \text{ or } X(t) = N\Big\},$$

assuming that $\frac{1}{n} < a < N$. Clearly $\mathrm{P}(\tau_{n,N} < \infty) = 1$ (because $\tau_{n,N} \leq \inf\{t : |B_t^{(2)}| = N\} < \infty$ almost surely) so

$$f_d(X^{\tau_{n,N}}(\infty)) = f_d(X(\tau_{n,N})) = f_d\Big(\frac{1}{n}\Big) \quad \text{or} \quad f_d(N). \tag{11.5}$$

Using optional sampling on the uniformly integrabel (since bounded) martingale $f_d(\boldsymbol{X}^{\tau_{n,N}})$ we get

$$\mathrm{E}(f_d(X(\tau_{n,N})) \mid \mathcal{F}_{\tau_{n-1,N}}) = f_d(X(\tau_{n-1,N})),$$

i.e. for $N > a$ fixed, $(f_d(X(\tau_{n,N})))_n$ is a discrete time martingale, bounded above by the constant $f_d(N)$. Hence

$$\lim_{n\to\infty} f_d(X(\tau_{n,N})) = Z_N$$

exists almost surely as a finite limit, but since (11.5) holds and $f_d(\frac{1}{n}) \to -\infty$, necessarily $X(\tau_{n,N}) = N$ for n sufficiently large i.e. $\boldsymbol{X}$ hits any given high level N before it hits levels sufficiently close to 0. *It follows that with probability 1,* $\boldsymbol{X}$ *will never hit 0.* But then we may use Itô's formula directly and deduce that

$$\mathrm{d}X(t) = \frac{1}{2}\frac{d-1}{X(t)}\,\mathrm{d}t + \frac{1}{X(t)} \sum_{j=1}^{d} \tilde{B}^{(j)}(t)\,\mathrm{d}\tilde{B}^{(j)}(t).$$

The last term corresponds to a continuous local martingale with $Y_0 \equiv 0$ and

$$[Y](t) = \sum_{j=1}^{d} \int_0^t \frac{\big(\tilde{B}^{(j)}(s)\big)^2}{X^2(s)} \,\mathrm{d}s = t,$$

hence by Lévys characterisation of Brownian motion (Eksempel 8.5) $\boldsymbol{Y}$ is a BM(1)-process $\boldsymbol{B}^*$, and $\boldsymbol{X}$ solves

$$\mathrm{d}X(t) = \frac{1}{2}\frac{d-1}{X(t)}\,\mathrm{d}t + \mathrm{d}B^*(t), \quad X(0) \equiv a.$$

We have now shown that BES(d) is a diffusion with values in $]0, \infty[$, with scale function f_d and speed measure density

$$k_d(x) = \frac{2}{f_d'(x)} = 2x^{-(d-1)}.$$

By Theorem 11.4, $\boldsymbol{X}$ is recurrent for $d = 2$ (in particular it gets arbitrarily close to 0 without ever hitting), while for $d \geq 3$,

$$\mathrm{P}\big(\lim_{t\to\infty} X(t) = \infty\big) = 1,$$

$$\mathrm{P}(\tau_r < \infty) = \Big(\frac{r}{a}\Big)^{d-2}, \quad r \leq a.$$

Note that if you are good at integration, you should be able to prove that for $d = 2$ the local martingale $f_d(\boldsymbol{X}) = \log \boldsymbol{X}$ is *not* a true martingale, simply by shoving that

$$\mathrm{E}\log X(1) = \log a + \int_a^\infty \frac{1}{r} e^{-\frac{1}{2}r^2}\,\mathrm{d}r > \log a. \qquad \circ$$

Let now again $\boldsymbol{X}$ be a diffusion on $]l, r[$,

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t), \quad X(0) \equiv U \in \mathcal{F}_0$$

with scale function S, speed measure density k, satisfying the critical condition from Theorem 11.4, repeated on p. 120. As usual b, σ are continuous with $\sigma > 0$.

The problem we shall now study is that of investigating whether there exists a probability μ on $]l, r[$, such that if U has distribution μ, $\boldsymbol{X}$ is *stationary* for all t, $X(t)$ has distribution μ. μ is also called an *invariant probability* for U. If it exists, μ is uniquely determined and typically, for all x, $p_t(x, \cdot) \xrightarrow{w} \mu$ (weak convergence) as $t \to \infty$ ($p_t(x, \cdot)$ is the transition probability from x).

Theorem 11.7. *$\boldsymbol{X}$ has an invariant probability μ if and only if*

$$K \equiv \int_l^r k(x)\,\mathrm{d}x < \infty,$$

and in that case

$$\mu(\mathrm{d}x) = \frac{1}{K} k(x)\,\mathrm{d}x.$$

In particular, in order for the invariant probability to exist, it is necessary that $\boldsymbol{X}$ be recurrent, $S(r) = \infty, S(l) = -\infty$.

Proof (partial). Suppose first that the invariant probability measure μ exists. Let $\mathcal{K}$ denote the class of $f \in C^2(]l,r[)$ such that for some $l' < r'$ it holds that f is constant on $]l,l'[$ and constant on $]r',r[$ (the constant values may be different). Then, since

$$f(X(t)) = f(X(0)) + \int_0^t Af(X(s))\,\mathrm{d}s + \int_0^t f'(X(s))\sigma(X(s))\,\mathrm{d}B(s), \qquad (11.6)$$

and since f is bounded, Af, f' have compact support so that also Af, f' are bounded, taking expectations and using $X(t) \overset{D}{=} X(0)$, we obtain

$$\mu(Af) = 0, \quad f \in \mathcal{K}.$$

Thus

$$\int_l^r \Big(bf' + \frac{1}{2}\sigma^2 f''\Big)\mu(\mathrm{d}) = 0, \quad f \in \mathcal{K}.$$

Assuming now that $\mu(\mathrm{d}x) = u(x)\,\mathrm{d}x$, this gives (use partial integration and that f' has compact support)

$$\int_l^r \Big(bu - \frac{1}{2}(\sigma^2 u)'\Big)f'\,\mathrm{d}x = 0.$$

But as f' we can obtain any $g \in C^1$ with compact support (since $\int_{x_0}^x g(y)\,\mathrm{d}y$ is constant close to l and r respectively since g vanishes close to l and r), and the class of such g is dense in L^2 (Lebesgue (l,r)). Deduce that $bu - \frac{1}{2}(\sigma^2 u)' \equiv 0$ on (l,r) and the desired expression for the invariant density follows.

We still need that the existence of μ implies $\int_l^r k(x)\,\mathrm{d}x < \infty$ and uniqueness of μ. We claim that If $\int_l^r k(x)\,\mathrm{d}x < \infty$, then $u = \frac{1}{K}k$ is the density for the invariant measure, but the proof of this requires Markov process theory. We now know that $\mu(Af) = 0$ for $f \in \mathcal{K}$ where $\mu(\mathrm{d}x) = u(x)\,\mathrm{d}x$. We also have

$$\mathrm{P}_t\, f(x) = f(x) + \int_0^t \mathrm{P}_s(Af)(x)\,\mathrm{d}s$$

from (11.6). Also $\mu(\mathrm{P}_t\, f) = \mu(f) + \int_0^t \mu(\mathrm{P}_s(Af)\,\mathrm{d}s)$. One must now argue that $\mathrm{P}_s\, f \in \mathscr{D}(A)$ (standard definiton of the domain; $f \in C^2$ bounded, Af bounded), that $\mathrm{P}_s(Af) = A(\mathrm{P}_s\, f)$ and finally that $\mu(A(\mathrm{P}_s\, f)) = 0$ (or $\mu(A(g)) = 0$ for all $g \in \mathscr{D}(A)$). Then

$$\mu(\mathrm{P}_t\, f) = \mu(f) + \int_0^t \mu(A(\mathrm{P}_s\, f))\,\mathrm{d}s = \mu(f).$$

It remains to verify that if $\int_l^r k(x)\,\mathrm{d}x < \infty$, then $S(r) = \infty$, $S(l) = -\infty$. But for an arbitrary $x \in]l,r[$

$$\infty = \int_x^r (S(r) - S(y))k(y)\,\mathrm{d}y \le \int_x^r k(y)\,\mathrm{d}y(S(r) - S(x))$$

so $\int_x^r k(y)\,\mathrm{d}y < \infty$ forces $S(r) = \infty$. $\square$

Example 11.8. The *Cox-Ingersoll-Ross process* solves the SDE

$$\mathrm{d}X(t) = (a + bX(t))\,\mathrm{d}t + \sigma\sqrt{X(t)}\,\mathrm{d}B(t)$$

with parameters $a, b \in \mathbb{R}$, $\sigma > 0$. One is interested in a solution which is strictly positive and finite i.e. $]l, r[\, = \,]0, \infty[$.

We use Theorem 11.4 to decide for what values of a, b, $\sigma > 0$ a strictly positive and finite solution exists. By computation

$$S'(x) = \exp\Big(-\int_l^x \frac{2(a+by)}{\sigma^2 y}\,\mathrm{d}y\Big) = x^{-\frac{2a}{\sigma^2}} \exp\Big(-\frac{2b}{\sigma^2}(x-1)\Big),$$

and

$$k(x) = \frac{2}{\sigma^2} x^{\frac{2a}{\sigma^2}-1} \exp\Big(\frac{2b}{\sigma^2}(x-1)\Big).$$

It follows that

$$S(0) = -\infty \iff \frac{2a}{\sigma^2} \geq 1,$$
$$S(\infty) = \infty \iff b < 0 \text{ or } b = 0,\ \frac{2a}{\sigma^2} \leq 1.$$

In the case where $S(\infty) < \infty$ we next compute

$$I = \int_x^\infty (S(\infty) - S(y))k(y)\,\mathrm{d}y$$

for large x.

(i) If $b = 0$, $\frac{2a}{\sigma^2} > 1$,

$$I = K \int_x^\infty \int_y^\infty z^{-\frac{2a}{\sigma^2}}\,\mathrm{d}z\, y^{\frac{2a}{\sigma^2}-1}\,\mathrm{d}y = +\infty.$$

(ii) If $b > 0$,

$$I = K \int_x^\infty \int_y^\infty z^{-\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2}z}\,\mathrm{d}z\, y^{\frac{2a}{\sigma^2}-1} e^{\frac{2b}{\sigma^2}y}\,\mathrm{d}y.$$

Rewrite the inner integral as

$$y^{-\frac{2a}{\sigma^2}} \int_y^\infty \Big(\frac{z}{y}\Big)^{-\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2}z}\,\mathrm{d}z$$

and use that $(\frac{z}{y})^{-\frac{2a}{\sigma^2}}$ stays bounded when $y \leq z \leq y + c$ for arbitrary $c > 0$, to deduce that

$$I \sim \tilde{K} \int_x^\infty y^{-\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2}y} y^{\frac{2a}{\sigma^2}-1} e^{\frac{2b}{\sigma^2}y}\,\mathrm{d}y = +\infty.$$

Thus the conditions on S, k p. 120 are always satisfied for $r = \infty$.

Finally we evaluate $J = \int_0^x (S(y) - S(0))k(y)\,\mathrm{d}y$ for small $x > 0$ when $S(0) > -\infty$, i.e. when $\frac{2a}{\sigma^2} < 1$. But since $e^{\pm\frac{2b}{\sigma^2}z}$ is close to 1 for small z,

$$J \sim K \int_0^x \int_0^y z^{-\frac{2a}{\sigma^2}}\,\mathrm{d}z\, y^{\frac{2a}{\sigma^2}-1}\,\mathrm{d}y < \infty.$$

so when $\frac{2a}{\sigma^2} < 1$, the conditions on S, k p. 120 are *not* satisfied for $l = 0$.

The conclusion is that the Cox-Ingersoll-Ross SDE has a strictly positive and finite solution if and only if

$$\frac{2a}{\sigma^2} \geq 1.$$

The solution is recurrent ($S(\infty) = \infty$, $S(0) = -\infty$) if and only if either $\frac{2a}{\sigma^2} \geq 1$, $b < 0$ or $\frac{2a}{\sigma^2} = 1$, $b = 0$.

In the recurrent case, the process has an invariant probability if and only if $\frac{2a}{\sigma^2} \geq 1$, $b < 0$. The invariant probability is then a Γ-distribution. ∘

We conclude this chapter with a discussion of the expected time for a diffusion to hit a given level.

Proposition 11.9. *Let* $\boldsymbol{X}$ *solve*

$$\mathrm{d}\boldsymbol{X} = b(\boldsymbol{X})\,\mathrm{d}t + \sigma(\boldsymbol{X})\,\mathrm{d}\boldsymbol{B}$$

on $]l, r[$, *with scale* S, *speed density* k. *Then if* $l < a < x$,

(i) *if* $S(r) < \infty$ *then* $\mathrm{P}^x(\tau_a = \infty) > 0$ *and* $\mathrm{E}^x\,\tau_a = \infty$,

(ii) *if* $S(r) = \infty$ *then* $\mathrm{P}^x(\tau_a < \infty) = 1$ *and* $\mathrm{E}^x\,\tau_a < \infty$ *if and only if*

$$\int_x^r k(y)\,\mathrm{d}y < \infty.$$

(iii) *if* $S(r) = \infty$ *and* $S(l) = -\infty$,*we have* $\mathrm{E}^x\,\tau_a < \infty$, $\mathrm{E}^x\,\tau_b < \infty$ *for all* $a \in]l, x[$, *all* $b \in]x, r[$ *if and only if* $\boldsymbol{X}$ *has an invariant measure;* $\int_x^r k(y)\,\mathrm{d}y < \infty$.

Proof. (i): Follows from Theorem 11.4 (ii).

(ii): If $S(r) = \infty$, we also know that $\mathrm{P}^x(\tau_a < \infty) = 1$ from Theorem 11.4 (ii). And by monotone convergence, as $b \uparrow r$,

$$\begin{aligned}
\mathrm{E}^x\,\tau_a &= \lim_{b \uparrow r} \mathrm{E}^x\,\tau_{a,b} \\
&= \lim \Big(\int_x^b \frac{(S(x) - S(a))(S(b) - S(y))}{S(b) - S(a)} k(y)\,\mathrm{d}y \\
&\qquad + \int_a^x \frac{(S(y) - S(a))(S(b) - S(x))}{S(b) - S(a)} k(y)\,\mathrm{d}y \Big) \\
&= (S(x) - S(a)) \int_x^r k(y)\,\mathrm{d}y + \int_a^x (S(y) - S(a))k(y)\,\mathrm{d}y
\end{aligned}$$

and (ii) follows.

(iii): Is a direct consequense of (ii) (applying also the version of (ii) with $S(l) = -\infty$). □