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LECTURE 2: Martingales and the Itô formula

Properties of the Itô integral:

1) $\int_0^T (af(s)+bg(s))dB_s = a\int_0^T f(s)\,dB_s + b\int_0^T g_s\,dB_s$;

where $f, g \in \mathcal{V}(0,T)$, a, b are constants

2) $E[\int_0^T f(s)dB_s] = 0$; $f \in \mathcal{V}(0,T)$

3) (The Itô isometry) $E[(\int_0^T f_s\,dB_s)^2] = E[\int_0^T f_s^2\,ds]$; $f \in \mathcal{V}(0,T)$

4) The process $I(t) := \int_0^t f(s)dB_s$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ (and P). (For $f \in \mathcal{V}(0,T)$)

A filtration is an increasing family of σ-algebras $\{\mathcal{N}_t\}_{t\geq 0}$ (if $s<t$ then $\mathcal{N}_s \subseteq \mathcal{N}_t$)

DEFINITION A stochastic process $Y(t)$ is a martingale with respect to a given filtration $\{\mathcal{N}_t\}_{t\geq 0}$ (and P) if the following holds:

1) $\{Y(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{N}_t\}_{t\geq 0}$, i.e. for each t, $Y(t)$ is $\mathcal{N}_t$-measurable

2) $E[|Y(t)|] < \infty$ for all t

3) $E[Y(s) | \mathcal{N}_t] = Y(t)$ for all $s > t$.

RECALL THE BASICS OF CONDITIONAL EXPECTATION:

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a r.v, $E[|X|] < \infty$. Let $\mathcal{H} \subset \mathcal{F}$ be another σ-algebra. Then the conditional expectation of X with respect to $\mathcal{H}$, $E[X|\mathcal{H}]$, is defined by the following properties:

(i) $E[X|\mathcal{H}]$ is $\mathcal{H}$-measurable

(ii) $\int_H E[X|\mathcal{H}]\,dP = \int_H X\,dP$ for all $H \in \mathcal{H}$.

Some properties:

1) $E[aX + bY | \mathcal{H}] = a E[X|\mathcal{H}] + b E[Y|\mathcal{H}]$

2) $E[X|\mathcal{H}] = X$ if X is $\mathcal{H}$-measurable

3) $E[YX|\mathcal{H}] = X E[Y|\mathcal{H}]$, if X is $\mathcal{H}$-measurable

4) $E[Y|\mathcal{H}] = E[Y]$ if Y is independent of $\mathcal{H}$

5) The tower property: If $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ are σ-algebras, then

$$E[X|\mathcal{H}] = E[E[X|\mathcal{G}]|\mathcal{H}]$$

EXAMPLE $Y(t) := B(t)$ is an $\mathcal{F}_t$-martingale.

Proof. We must show that

$$E[B(s) \mid \mathcal{F}_t] = B(t) \quad \text{if } s > t.$$

$$E[B(s) \mid \mathcal{F}_t] = E[B(s) - B(t) + B(t) \mid \mathcal{F}_t]$$

$$\overset{1)}{=} E[B(s) - B(t) \mid \mathcal{F}_t] + E[B(t) \mid \mathcal{F}_t]$$

$$\overset{2),\,4)}{=} E[B(s) - B(t)] + B(t) = B(t). \quad \text{OK}$$

EXTENSION OF THE ITO INTEGRAL

Let $\mathcal{H}_t$ be a filtration such that $\mathcal{F}_t \subseteq \mathcal{H}_t$ for all t. Let $\mathcal{W}(0,T)$ be the set of measurable processes $f(t,\omega)$ such that

(i)' $f(t,\omega)$ is $\mathcal{H}_t$-adapted | (i) $f(t,\omega)$ is $\mathcal{F}_t$-adapted

(ii)' $\int_0^T f^2(s)\,ds < \infty$ a.s. | (ii) $E[\int_0^T f^2(s)\,ds] < \infty$

Then if $B(t)$ is a martingale w.r.t. $\mathcal{H}_t$, we can define the Itô integral

$$\int_0^T f(s)\,dB(s) \quad ; \quad f \in \mathcal{W}(0,T)$$

in a similar ways as we did for $\mathcal{V}(0,T)$, replacing convergence in L^2 by convergence in probability.

This extension is important for the Itô formula.

THE ITÔ FORMULA (1-DIMENSIONAL CASE)

An Itô process (1-dimensional) is a process X_t of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\,ds + \int_0^t v(s,\omega)\,dB(s) \quad ; \quad 0 \le t \le T$$

where $u(s)$ and $v(s)$ are $\mathcal{H}$-adapted, for some $\mathcal{H}$ s.t. $\tilde{B}$ is a martingale wrt. $\mathcal{H}$; and

$$\int_0^T |u(s)|\,ds < \infty , \quad \int_0^T v^2(s)\,ds < \infty \quad \text{a.s.}$$

In short hand notation,

$$dX_t = u_t\,dt + v_t\,dB_t \; ; \; X_0 = x \in \mathbb{R}$$

THEOREM (The Itô formula)

Let X_t be an Itô process as above.

Let $g \in C^{1,2}([0,\infty)\times\mathbb{R})$, i.e. g is cont. diff. wrt. the first variable and twice cont. diff. wrt. the second variable. Define

$$Y(t) = g(t, X(t))$$

Then $Y(t)$ is again an Itô process, and in differential form $Y(t)$ is given by

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\,dt + \frac{\partial g}{\partial x}(t, X(t))\,dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))\left(dX(t)\right)^2,$$

where $(dX(t))^2 = (u(t)dt + v(t)dB(t))^2$ $\quad v^2(t)\,dt$

$= (u(t)dt)^2 + 2u(t)v(t)\,dt\,dB(t) + (v(t)dB(t))^2,$

where multiplication of the differentials is carried out according to the following multiplication table

	dt	$dB(t)$
dt	0	0
$dB(t)$	0	dt

Recall that $E\left[(B_{t+h} - B_t)^2\right] = h$

<u>REMARK</u> Note that $\int_0^t f(s)\,dB(s)$ is <u>not</u> necessarily a martingale if we only assume $f \in \mathcal{W}(0,T)$. (But it is a <u>local</u> martingale)

<u>EXAMPLE</u> Consider $Y(t) = \frac{1}{2}B^2(t)$

$dY(t) = ?$

Choose $g(t,x) = \frac{1}{2}x^2$, $X(t) = B(t)$

Then by the Itô formula $\left(\frac{\partial g}{\partial x} = x,\ \frac{\partial^2 g}{\partial x^2} = 1\right)$

$$dY(t) = \underbrace{B(t)\,dB(t)}_{\frac{\partial g}{\partial x}(t,X(t))\,dX(t)} + \frac{1}{2}\cdot 1\cdot (dB(t))^2$$

So

$$dY(t) = \frac{1}{2}dt + B(t)\,dB(t)$$

i.e.

$$Y(t) = Y(0) + \int_0^t \frac{1}{2}ds + \int_0^t B(s)\,dB(s)$$

If $B(0)=0$, this gives

$$\frac{1}{2}B^2(t) = \frac{1}{2}t + \underbrace{\int_0^t B(s)\,dB(s)}$$

This implies that

$$\int_0^T B(s)\,dB(s) = \frac{1}{2}B^2(T) - \frac{1}{2}T.$$

<u>EXAMPLE</u> $Y(t) = \exp(\alpha t + \beta B(t))$;

α, β constants. $dY(t) = ?$

Choose $X(t) = \alpha t + \beta B(t)$, $g(x) = e^x$

Then

$$dY(t) = \underbrace{Y(t)}_{\frac{\partial g}{\partial x}(t,X(t))}\underbrace{(\alpha\,dt + \beta\,dB(t))}_{dX(t)} + \frac{1}{2}Y(t)\beta^2 dt$$

$$= Y(t)\left[\left(\alpha + \frac{1}{2}\beta^2\right)dt + \beta\,dB(t)\right]$$

Alternatively, we "could have chosen"

$$X(t) = B(t), \quad g(t,x) = \exp(\alpha t + \beta x)$$

Then we get the same final result, but with different computation.

THE MULTI-DIMENSIONAL CASE

Suppose we have n Itô processes

$$\begin{cases} dX_1(t) = u_1(t)\,dt + V_{11}(t)\,dB_1(t) + \cdots + V_{1m}(t)\,dB_m(t) \\ dX_2(t) = u_2(t)\,dt + V_{21}(t)\,dB_1(t) + \cdots + V_{2m}(t)\,dB_m(t) \\ \vdots \\ dX_n(t) = u_n(t)\,dt + V_{n1}(t)\,dB_1(t) + \cdots + V_{nm}(t)\,dB_m(t) \end{cases}$$

Introduce

$$dX(t) = \begin{bmatrix} dX_1(t) \\ \vdots \\ dX_n(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \quad dB(t) = \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{bmatrix}$$

$$V(t) = \begin{bmatrix} V_{11}(t) & \cdots & V_{1m}(t) \\ \vdots & & \vdots \\ V_{n1}(t) & \cdots & V_{nm}(t) \end{bmatrix} \in \mathbb{R}^{n \times m}$$

Then the system gets the form

$$dX(t) = u(t)\,dt + V(t)\,dB(t)$$

THE MULTI-DIMENSIONAL ITÔ FORMULA

Let $X(t)$ be as above and let

$g = g(t,x) = g(t, x_1, \dots, x_n) \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$

Define

$$Y(t) = g(t, X(t)).$$

Then

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\,dt + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(t, X(t))\,dX_i(t)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(t, X(t))\,dX_i(t)\,dX_j(t),$$

where $dX_i(t)\,dX_j(t)$ is computed by using the multiplication rules

$$dt \cdot dt = dt \cdot dB_i(t) = 0$$

$$dB_i(t)\,dB_j(t) = \begin{cases} 0 & \text{if } i \neq j \\ dt & \text{if } i = j \end{cases}$$

EXAMPLE (Integration by parts)

Let $dX_i(t) = u_i(t)\,dt + \sum_{j=1}^{m} v_{ij}(t)\,dB_j(t)$; $i = 1, 2.$

Then

$$d(X_1(t) X_2(t)) = X_1(t)\,dX_2(t) + X_2(t)\,dX_1(t) + dX_1(t)\,dX_2(t)$$

Proof. Apply the Itô formula with

$$g(x_1, x_2) = x_1 x_2 .$$

THE MARTINGALE REPRESENTATION THEOREM

We have seen that if $f \in \mathcal{V}(0,T)$ then

$$I(t) := \int_0^t f(s)\,dB(s) \;; \quad 0 \le t \le T$$

is an $\mathcal{F}_t$-martingale (and continuous).

The martingale representation theorem states that the converse is also true:

THEOREM Let $M(t)$ be a continuous $\mathcal{F}_t$-martingale with $E[M^2(t)] < \infty$ for all $t \in [0,T]$. Then there exists $f \in \mathcal{V}(0,T)$ such that

$$M(t) = M(0) + \int_0^t f(s)\,dB(s)$$

A proof of this can be given by using the following related result:

THEOREM (The Itô representation theorem)

Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable.

Then there exists $f \in \mathcal{V}(0,T)$ such that

$$F = E[F] + \int_0^T f(s)\,dB(s)$$

The classical proof gives the existence and uniqueness of f, but no information about how to find it. But using Malliavin calculus one can show that f can be given the representation

$$f(s) = E[D_s F \mid \mathcal{F}_s]$$ (the Clark-Ocone formula)

where $D_s F$ is the Malliavin derivative of F at s.