Note Title 06/08/2008

# LECTURE 3: Stochastic differential equations.

Return to the population growth equation

$$dN_t = (r + \alpha \text{ "noise"}) N_t \, dt$$

Formally, replace "noise" by "$\frac{dB}{dt}$" and multiply by $dt$:

$$dN_t = r N_t \, dt + \alpha N_t \, dB_t$$

How do we solve this stochastic differential equation?

Define $\quad Y_t = \ln N_t$

Then by the Itô formula

$$dY_t = \frac{1}{N_t} dN_t + \frac{1}{2}\left(-\frac{1}{N_t^2}\right)(dN_t)^2$$

$$(dN_t)^2 = \alpha^2 N_t^2 \, dt \,, \quad \text{Hence}$$

$$dY_t = r\,dt + \alpha\, dB_t - \frac{1}{2}\alpha^2\, dt$$

So
$$Y_t = \ln N_t = Y_0 + \int_0^t \left(r - \frac{1}{2}\alpha^2\right) ds + \int_0^t \alpha\, dB_s$$

Therefore

$$N_t = N_0 \exp\left( \left(r - \frac{1}{2}\alpha^2\right) t + \alpha B_t \right)$$

Some consequences of the solution formula:

The law of the iterated logarithm for $B_t$ states that

$$\overline{\lim_{t \to \infty}} \frac{B_t}{\sqrt{2t \ln(\ln t)}} = 1 \quad \text{a.s.}$$

Therefore

(i) If $r - \frac{1}{2}\alpha^2 > 0$ then $N_t \to \infty$ as $t \to \infty$

(ii) If $r - \frac{1}{2}\alpha^2 < 0$ then $N_t \to 0$ as $t \to \infty$

(iii) If $r - \frac{1}{2}\alpha^2 = 0$ then $N_t$ fluctuates between arbitrary large and arbitrary small values.

Observe that

$$E[N_t] = N_0 \, E\left[\exp\left(\left(r - \tfrac{1}{2}\alpha^2\right)t + \alpha B_t\right)\right]$$

$$= N_0 \exp\left(\left(r - \tfrac{1}{2}\alpha^2\right)t\right) E\left[\exp(\alpha B_t)\right] = N_0 e^{rt},$$

which is the solution in case there is no noise.

Alternatively, one could model population growth in a random environment as follows:

$$\frac{dN_t}{dt} = rN_t + \alpha \text{"noise"}$$

i.e.

$$dN_t = rN_t\,dt + \alpha\,dB_t \quad \text{(Ornstein-Uhlenbeck)}$$

To solve this we proceed as follows:

$$dN_t - rN_t\,dt = \alpha\,dB_t$$

Multiply by $e^{-rt}$:

$$e^{-rt}dN_t - re^{-rt}N_t\,dt = \alpha e^{-rt}dB_t$$

$$d\left(e^{-rt}N_t\right) = \alpha e^{-rt}\, dB_t$$

$$e^{-rt}N_t = N_0 + \int_0^t \alpha e^{-rs}\, dB_s$$

Hence

$$N_t = N_0 e^{rt} + e^{rt}\int_0^t \alpha e^{-rs}\, dB_s$$

Note that $E[N_t] = N_0 e^{rt}$,

as in the previous example.

Now consider a general stochastic differential equation of the form

$$\textcircled{*} \quad \begin{aligned} dX_t &= b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t; \quad 0 \le t \le T \\ X_0 &\text{ independent of } \{B_s\}. \end{aligned}$$

Here $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are given functions.

<u>THEOREM</u> Suppose there exist constants $C, D$ such that

(1) (Lipschitz) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x-y|$

for all $x, y \in \mathbb{R}^n$, $t \in [0,T]$

(2) (Linear growth) $|b(t,x)| + |\sigma(t,x)| \leq D(1+|x|)$

for all $x \in \mathbb{R}^n$, $t \in [0,T]$

Suppose $E[|X_0|^2] < \infty$. Then there exists a unique adapted solution $X_t$ of the equation (*) such that $E[|X_t|^2] < \infty$ for all $t \in [0,T]$.

**REMARK** Conditions (1) and (2) are sufficient to guarantee uniquess and existence.

**Example 1** Consider the equation

$$\frac{dX_t}{dt} = X_t^2 \; ; \; X_0 = 1$$

Here the linear growth condition fails.
The solution of the equation is

$$X_t = \frac{1}{1-t} \; ; \quad 0 \le t < 1$$

which explodes at time $t = 1$.

**Example 2** Consider the equation

$$\frac{dX_t}{dt} = 3 X_t^{2/3} \; ; \quad X_0 = 0$$

For any $a>0$ the function

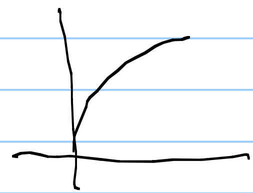
$$X_t = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$$

solves the equation. So the solution is not unique.

The function

$$b(x) = 3x^{2/3}$$

is not Lipschitz conti. at 0.



Nevertheless, there are many examples of equations with unique solutions where conditions (1) or (2) do not hold.

__Example 1__ A stochastic logistic equation for population growth could be.

$$dX_t = rX_t(K - X_t)\,dt + \beta X_t\,dB_t \; ; \; X_0 = x > 0$$

The solution exists and is unique (even though we do not have linear growth as required in the theorem), and its given by

$$X_t = \frac{\exp\left((rK - \tfrac{1}{2}\beta^2)t + \beta B_t\right)}{x^{-1} + r\int_0^t \exp\{(rK - \tfrac{1}{2}\beta^2)s + \beta B_s\}\,ds} \qquad (\text{Gard})$$

Example 2   A related logistic model is

$$dY_t = rY_t(K - Y_t)\,dt + \beta Y_t(K - Y_t)\,dB_t \; ; \; Y_0 > 0$$

For $r > 0$, $K > 0$ there is a unique solution of this equation (but no solution formula seems to be known) (Lungu & Ø.)

Not all SDEs have solutions:

Example (the Tanaka equation)

$$dX_t = \text{sign}(X_t)\, dB_t \;; \qquad X_0 = 0$$

where

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

It can be proved that this equation has no (strong) solution. (Lipschitz continuity fails)

An interesting example:

The Brownian bridge

Let $a, b$ be given real numbers. Consider the equation

$$dY_t = \frac{b - Y_t}{1-t}\,dt + dB_t\,; \quad 0 \leq t < 1$$

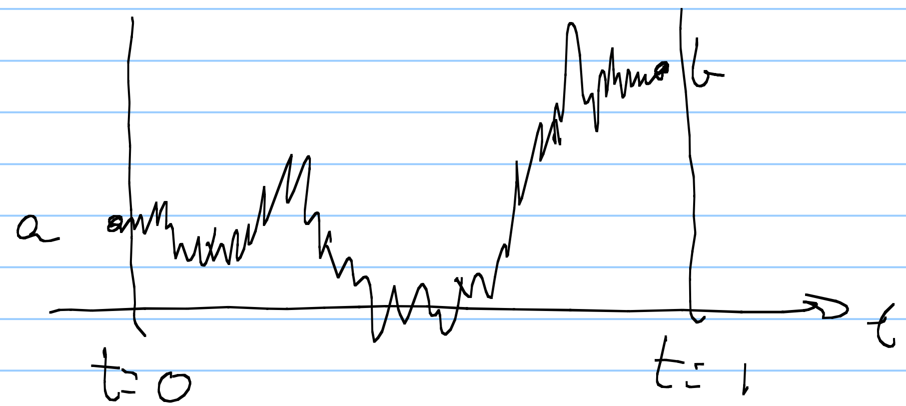
$$Y_0 = a$$

The solution is

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}\,; \quad 0 \leq t < 1$$

One can show that

$$\lim_{t \to 1^-} Y_t = b$$



# AN OPTIMAL HARVESTING PROBLEM

(N. Chr. Stenseth, K.A. Brekke, B. Ø.) PNAS (2007)

Consider the following equation

$$dX(t) = X(t)\left[\mu(t)dt + \sigma(t)dB(t)\right] - c(t)dt \quad ; t \geq 0$$
$$X(0) = x > 0$$

This models a population with size $X(t)$ at time $t$, being harvested at the rate $c(t)$.

The coefficients $\mu(t)$, $\sigma(t)$ are $\mathcal{F}_t$-adapted processes.

Consider the following performance criterion

$$J(c) = E\left[\int_0^\infty e^{-\rho t} \log c(t)\,dt\right]$$

Interpretation: This is the expected total discounted logarithmic utility of the harvesting.
($\rho > 0$ is a given constant)

Introduce $\lambda(t) = \dfrac{c(t)}{X(t)}$

Then the system becomes

$$dX(t) = X(t)\left[(\mu(t) - \lambda(t))dt + \sigma(t)dB(t)\right]$$

By the Itô formula the solution is ($X(0)=1$)

$$X(t) = \exp\left(\int_0^t \{\mu(s) - \lambda(s) - \tfrac{1}{2}\sigma^2(s)\}ds + \int_0^t \sigma(s)dB(s)\right)$$

The performance criterion gets the form

$$J(c) = E\left[\int_0^\infty e^{-\rho t} \log c(t)\, dt\right]$$

$$= E\left[\int_0^\infty e^{-\rho t} \left(\log \lambda(t) + \log X(t)\right) dt\right]$$

$$= E\left[\int_0^\infty e^{-\rho t} \left(\log \lambda(t) + \int_0^t \{\mu(s) - \lambda(s) - \tfrac{1}{2}\sigma^2(s)\}\, ds + \int_0^t \sigma(s)\, dB(s)\right) dt\right]$$

$$= E\left[\int_0^\infty e^{-\rho t} \left(\log \lambda(t) - \int_0^t \lambda(s)\, ds\right) dt\right] + C,$$

where

$$C = E\left[\int_0^\infty e^{-\rho t} \left(\int_0^t \{\mu(s) - \tfrac{1}{2}\sigma^2(s)\}\, ds + \int_0^t \sigma(s)\, dB(s)\right) dt\right]$$

does not depend on $\lambda$.

So it suffices to maximize

$$① \quad J_0(\lambda) = E\left[\int_0^\infty e^{-\rho t}\left(\log \lambda(t) - \int_0^t \lambda(s)\,ds\right)dt\right]$$

over all $\lambda \geq 0$.

Note that, by the Fubini theorem

$$\int_0^\infty e^{-\rho t}\left(\int_0^t \lambda(s)\,ds\right)dt = \int_0^\infty \Big( \underbrace{\int_s^\infty e^{-\rho t}\,dt}_{\frac{1}{\rho}e^{-\rho s}} \Big)\lambda(s)\,ds$$

$$= \int_0^\infty \frac{1}{\rho} e^{-\rho s} \lambda(s)\,ds$$

$$= \int_0^\infty \frac{1}{\rho} e^{-\rho t} \lambda(t)\,dt$$

Substituting this into ① we get

$$J_0(\lambda) = E\left[\int_0^\infty e^{-\rho t}\left(\log \lambda(t) - \frac{1}{\rho}\lambda(t)\right)dt\right]$$

"We can maximize this pointwise, for each $t$:

The function

$$h(\lambda) := \log \lambda - \frac{1}{\rho}\lambda$$

is maximal when

$$\frac{1}{\lambda} - \frac{1}{\rho} = 0$$

i.e. $\lambda = \lambda^*(t) = \rho$

We have proved:

<u>THEOREM</u> The optimal harvesting rate for the performance $J(c) = E\left[\int_0^\infty e^{-\rho t} \log c(t)\, dt\right]$

is

$$c(t) = c^*(t) = \rho X(t)$$

POSSIBLE FURTHER STUDIES:

1) Stochastic control theory
(dynamic programming, HJB equations)

2) Optimal stopping
(Variational inequalities)

3) Filtering theory
a) Linear filter (Kalman filter)
b) Non-linear case (→ stochastic partial diff. eq.)

## 4) Extensions to Lévy processes

Any Lévy process $\eta(t)$ with $E[\eta^2(t)] < \infty$ can be written

$$\eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz).$$