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Malliavin Calculus - a Crash Course

Note Title 8/5/2008

INTRODUCTION. THE WIENER-ITÔ CHAOS EXPANSION THEOREM.

Let $f(t_1, \dots, t_n)$ be a deterministic function on $[0,T]^n$. Then the iterated Itô integral of f is defined by

$$J_n(f) = \int_0^T \left(\int_0^{t_n} \left(\int_0^{t_{n-1}} \cdots \left(\int_0^{t_2} f(t_1, \dots, t_n)\, dB(t_1) \right) dB(t_2) \cdots \right) dB(t_n) \right)$$

This integral converges if $f \in L^2([0,T]^n)$.

Similarly, if $f \in \hat{L}^2([0,T]^n)$, i.e. f is symmetric and in $L^2([0,T]^n)$, then we define

$$I_n(f) := n!\, J_n(f) = \text{"} \int_{[0,T]^n} f(t_1, \dots, t_n)\, dB^{\otimes n} \text{"}$$

(Note that if we put

$$G_n = \{ (t_1, \dots, t_n) ;\ 0 \le t_1 \le t_2 \le \cdots \le t_n \le T \}$$

then the volum of G_n is $\frac{1}{n!} \cdot \text{Volum}\, [0,T]^n$)

THEOREM (Chaos expansion)

Let $F \in L^2(P)$ be $\mathcal{F}_T$ - measurable

Then there exist a sequence $f_n \in \hat{L}^2([0,T]^n)$

such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{(convergence in } L^2(P))$$

Moreover, $E[F^2] = \sum_{n=0}^{\infty} n! \, \|f_n\|^2_{L^2([0,T]^n)}$

<u>DEFINITION</u> (Malliavin derivative)

Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$ be $\mathcal{F}_T$-measurable.

Assume that $F \in \mathbb{D}_{1,2}$, i.e.

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=0}^{\infty} n \, n! \, \|f_n\|^2_{L^2([0,T]^n)} < \infty$$

Then we define the Malliavin derivative of F at t, $D_t F$, by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

<u>EXAMPLE</u> Suppose $F = \int_0^T f(t)\,dB(t)$, $f \in L^2[0,T]$ det.

Then $F = I_1(f)$, so

$$D_t F = f(t).$$

<u>APPLICATION</u>:

The Clark-Ocone theorem:

Let $F \in \mathbb{D}_{1,2}$. Then $F = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t]\,dB(t)$

Sketch of proof. Consider

$$\int_0^T E[D_t F \mid \mathcal{F}_t]\, dB(t) = \int_0^T E\left[\sum n I_{n-1}(f_n(\cdot,t)) \mid \mathcal{F}_t\right] dB(t)$$

$$= \sum \int_0^T n E[I_{n-1}(f_n(\cdot,t)) \mid \mathcal{F}_t]\, dB(t)$$

$$= \sum \int_0^T n \quad I_{n-1}(f_n(\cdot,t)\cdot \chi_{[0,t]}(\cdot)^{\otimes n})\, dB(t)$$

$$= \sum \int_0^T n(n-1)!\, J_{n-1}(f_n(\cdot,t)\chi_{(0,t]}(\cdot)^{\otimes n})\, dB(t)$$

$$= \sum_{n=1}^{\infty} n!\, J_n(f_n) = \sum_{n=1}^{\infty} I_n(f_n) = F - E[F]. \quad \square$$

SOME BASIC PROPERTIES

(I) If X is $\mathcal{F}_t$-measurable, then

$$D_s X = 0 \quad \text{if } s > t$$

(II) (Chain rule)

$$D_t(\varphi(F)) = \varphi'(F) D_t F$$

(III) (Fundamental theorem)

$$D_t\left(\int_0^T g(s,\omega)\, dB(s)\right) = g(t,\omega) + \int_0^T D_t g(s,\omega)\, dB(s)$$

(IV) (Duality formula)

$$E\left[G \int_0^T u(s,\omega)\, dB(s) \right] = E\left[\int_0^T u(s,\omega)\, D_s G\, ds \right]$$

APPLICATION: Efficient computation of the "greeks".
(Fournié et al, Finance & Stochastic 1999)

Consider the following market

(risk free asset) $dS_0(t) = \rho\, S_0(t)\, dt$; $S_0(0) = 1$

(risky asset) $dS_1(t) = S_1(t)\left[\mu(t)\, dt + \sigma(t)\, dB(t)\right]$; $S_1(0) > 0$

Assume that $\mu(t) = \mu(S_1(t))$, $\sigma(t) = \sigma(S_1(t))$ (Markovian system), ρ constant.

We want to replicate a payoff at time T given by
$$F = \varphi(S_1(T)).$$

Let $\theta(t) = (\theta_0(t), \theta_1(t))$ be a portfolio, giving the number of units held at time t of the two assets, respectively. The corresponding wealth process is given by

(1) $$V(t) = \theta_0(t) S_0(t) + \theta_1(t) S_1(t)$$

The portfolio is assumed to be self-financing, in the sense that

② $$dV(t) = \underline{\theta_0(t)}\, dS_0(t) + \theta_1(t)\, dS_1(t)$$

Combining ① and ② we get

$$dV(t) = \frac{V(t) - \theta_1(t) S_1(t)}{S_0(t)}\, dS_0(t) + \theta_1(t)\, dS_1(t)$$

or

$$dV(t) = V(t)\rho\, dt + \theta_1(t) S_1(t)\left[(\mu-\rho)\,dt + \sigma\, dB(t)\right]$$

or

$$dV(t) - \rho V(t)\,dt = \sigma(t)\,\theta_1(t) S_1(t)\Big[\underbrace{\frac{\mu-\rho}{\sigma}\,dt + dB(t)}_{d\tilde{B}(t)}\Big]$$

or

$$e^{-\rho T}\underbrace{V(T)}_{F} = V(0) + \int_0^T e^{-\rho t}\sigma(t)\,\theta_1(t) S_1(t)\, d\tilde{B}(t)$$

$\tilde{B}(t)$ is a B.m. with respect to another probability measure, say Q (the Girsanov theorem).

Alternatively, one can show that

$$\theta_1(t) = e^{-\rho(T-t)}\frac{d}{dx}E\left[\varphi\left(X^x(T-t)\right)\right],$$

where

$$dX^x(t) = X^x(t)\left[\rho\, dt + \sigma(X(t))\,dB(t)\right];\ X^x(0) = x$$

PROBLEM How do we compute $\theta_1(t)$ if φ is not smooth?

We generalize the problem to the following:

Suppose

$$dX(t) = dX^x(t) = b(X(t))\,dt + \sigma(X(t))\,dB(t),$$

for some smooth functions b, $\sigma \neq 0$ $(X^x(0) = x)$

Suppose φ is a bounded function.

Compute $\frac{d}{dx} E[\varphi(X^x(T))]$!

THEOREM (Fournié et al)

Let $Y(t)$ be the derivative process, i.e.

$$dY(t) = b'(X^x(t))Y(t)\,dt + \sigma'(X^x(t))Y(t)\,dB(t)$$

$$Y(0) = 1.$$

Then

$$\frac{d}{dx} E[\varphi(X^x(T))] = E[\varphi(X^x(T))\Lambda_a(\omega)]$$

where

$$\Lambda_a = \int_0^T a(t)\sigma^{-1}(X^x(t))Y(t)\,dB(t).$$

Here $a(t)$ is an arbitrary continuous deterministic function s.t

$$\int_0^T a(t)\,dt = 1.$$

<u>Sketch of proof.</u>

<u>LEMMA 1</u> $\quad D_s X(t) = Y(t)Y^{-1}(s)\,\sigma(X(s))\,\chi_{[0,t]}(s)$

$s < t$

<u>Proof.</u>

$$D_s X(t) = D_s\left(\int_0^t b(X(u))\,du + \int_0^t \sigma(X(u))\,dB(u)\right)$$

$$= \int_s^t b'(X(u))\,D_s X(u)\,du + \int_s^t \sigma'(X(u))\,D_s X(u)\,dB(u) + \sigma(X(s))$$

If we put $Z(t) = D_s X(t)$ (s fixed)

then

$$dZ(t) = b'(X(t))Z(t)\,dt + \sigma'(X(t))Z(t)\,dB(t);\ t > s$$

$$Z(s) = \sigma(X(s))$$

Itô formula gives the solution

$$Z(t) = D_s X(t) = \sigma(X(s))\exp\left(\int_s^t \sigma'(X(u))\,dB(u) + \int_s^t \left(b'(X(u)) - \tfrac{1}{2}\sigma'^2(X(u))\right)du\right)$$

$$= Y(t)Y(s)^{-1}\sigma(X(s)).$$

$\square$

LEMMA 2 Let $a(t)$ be a continuous, determ. function such that $\int_0^T a(t)\,dt = 1$.

Then
$$Y(T) = \int_0^T D_s X(T)\, a(s)\, \sigma^{-1}(X(s))\, Y(s)\, ds.$$

Proof. Chose $t = T$ in Lemma 1, multiply by $a(s)$ and integrate. □

Proof of theorem:

We may assume that φ is smooth.

Then
$$\frac{d}{dx} E[\varphi(X^x(T))] = E[\varphi'(X^x(T))Y(T)]$$

Lemma 2
$$= E[\varphi'(X^x(T)) \int_0^T D_s X^x(T)\, a(s)\, \sigma^{-1}(X^x(s))\, Y(s)\, ds]$$

$$= E[\int_0^T \underbrace{\varphi'(X^x(T))\, D_s X^x}_{D_s(\varphi(X^x(T)))}(T)\, a(s)\, \sigma^{-1}(X^x(s))\, Y(s)\, ds]$$

duality
$$= E[\varphi(X^x(T)) \underbrace{\int_0^T a(s)\, \sigma^{-1}(X^x(s))\, Y(s)\, dB(s)}_{\Lambda}]. \qquad \square$$