Problems in Markov chains

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This collection of problems was compiled for the course Statistik 1B. It contains the problems in Martin Jacobsen and Niels Keiding: Markovkæder (KUIMS 1990), and more.

September 1990

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In this edition a few misprints have been corrected

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1. Conditional independence

Problem 1.1 Suppose that there are functions (of sets) f_z and g_z such that for all sets A and B we have

$$P\{X \in A, Y \in B | Z = z\} = f_z(A) g_z(B)$$

for every z. Show that X and Y are conditionally independent given Z.

Problem 1.2 Use the result of the previous problem to show, or show directly, that if

$$P\{X = x, Y = y, Z = z\} = f(x, z) \cdot g(y, z)$$

for some functions f and g then X and Y are conditionally independent given Z.

Problem 1.3 Show that X and Y are conditionally independent given Z if and only if

$$P\{X \in A | Y = y, Z = z\} = P\{X \in A | Z = z\}$$

for every (measurable) set A and $((Y, Z)(P)$-almost) every (y, z).

Thus if X and Y are conditionally independent given Z, then X is independent of Y given Z.

Problem 1.4 Suppose that X, Y and Z are independent random variables. Show that

(a) X and Y are conditionally independent given Z

(b) X and $X + Y + Z$ are conditionally independent given $X + Y$

Problem 1.5 Let X_0, U_1 and U_2 be independent random variables and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function. Put

$$X_n = F(X_{n-1}, U_n) \quad n = 1, 2$$

Show that X_0 and X_2 are conditionally independent given X_1. May F depend on n?

Problem 1.6 Assuming that X is independent of Y and X is conditionally independent of Z given Y, show that X is independent of (Y, Z). (*Recall that independence of X and Y and of X and Z does not ensure independence of X and (Y, Z)*)

Problem 1.7 Suppose that the probability mass function of (X, Y, Z) is strictly positive.

(a) Show that if X and Y are conditionally independent given Z and X and Z are conditionally independent given Y then X is independent of (Y, Z).

(b) Show that the result in question (a) is not necessarily correct without the assumption on the probability mass function (consider the case where $X = Y = Z$ is non-degenerate).

Problem 1.8 Let $(X_i)_{i \in I}$ be a (at most countable) collection of random variables. Make a *graph* by writing down all the random variables X_i and connecting X_i and X_j if and only if the conditional distribution of X_i given $(X_k)_{k \in I \setminus \{i\}}$ depends on X_j. (The correct mathematical way of saying this is that there should be an edge (a connection) between X_i and X_j unless

$$P\{X_i \in A | X_k = x_k, k \in I \setminus \{i\}\}$$

may be chosen so that is does not depend x_j for any A.)

The resulting graph is called the *conditional independence graph*. The X_is are called the *vertices* and the connections between two vertices are called the edges. If we can "go" from X_i to X_j by following a sequence of edges, we say there is a *path* between X_i and X_j.

(a) Show that this is well-defined, i.e. that if the conditional distribution of X_i given $(X_k)_{k \in I \setminus \{i\}}$ depends on X_j, then the conditional distribution of X_j given $(X_k)_{k \in I \setminus \{j\}}$ depends on X_i.

(b) Sketch the conditional independence graph for a Markov chain.

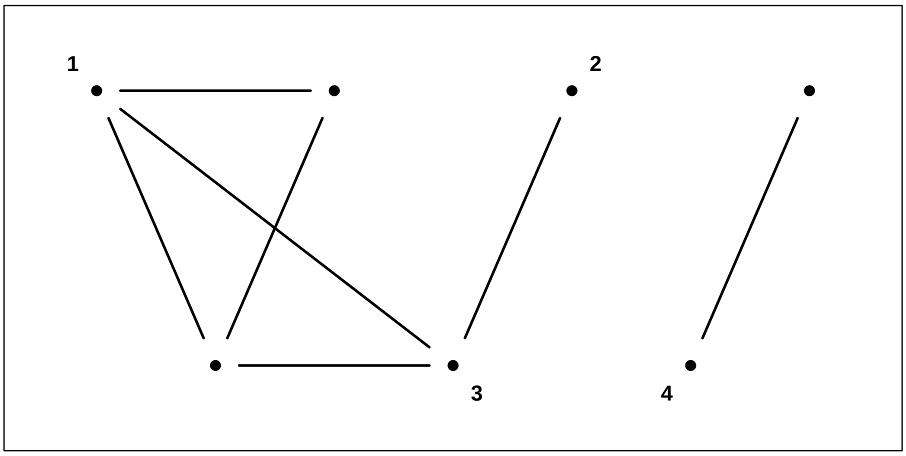
(c) Show that if there is no edge between X_i and X_j then they are conditionally independent given the rest.

(d) Define the neighbours of X_i to be the variables that are connected to X_i by an edge. Let

$$N_i^c = \{j \in I \setminus \{i\} : X_j \text{ is not a neighbour of } X_i\}$$

Show that X_i are conditionally independent of $(X_j)_{j \in N_i^c}$ given the neighbours of X_i

A conditional independence graph



In the graph above, the variables marked 1 and 2 are conditionally independent given the rest, and given the variable marked 3, the variable marked 2 is independent of the rest (3 is the only neighbour of 2).

We would like to have the following result: *Let I_1, I_2 and I_3 be disjoint subsets of I and suppose that every path form a variable in I_1 to a variable in I_2 passes through I_3, then $(X_i)_{i\in I_1}$ and $(X_i)_{i\in I_2}$ are conditionally independent given $(X_i)_{i\in I_3}$.* (If $I_3 = \emptyset$ then read independent for conditionally independent). For instance, this would imply that 1 and 2 are conditionally independent given 3 and that 2 and 4 are independent. This is true under additional assumptions, for instance if $(X_i)_{i\in I}$ has a strictly positive joint density wrt a product measure.

Conditional independence graphs are important in a class of statistical models known as *graphical models.*

2. Discrete time homogeneous Markov chains.

Problem 2.1 *(Random Walks).* Let $Y_0, Y_1, \dots$ be a sequence of independent, identically distributed random variables on $\mathbb{Z}$. Let

$$X_n = \sum_{j=0}^{n} Y_j \qquad n = 0, 1, \dots$$

Show that $\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain.

Problem 2.2 Let $Y_0, Y_1, \dots$ be a sequence of independent, identically distributed random variables on $\mathbb{N}_0$. Let $X_0 = Y_0$ and

$$X_n = \begin{cases} X_{n-1} - Y_n & \text{if } X_{n-1} > 0 \\ X_{n-1} + Y_n & \text{if } X_{n-1} \leq 0 \end{cases} \qquad n = 0, 1, \dots$$

Show that $\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain.

Problem 2.3 *(Branching processes).* Let $U_{i,j}, i = 0, 1, \dots, j = 1, 2, \dots$ be a sequence of independent, identically distributed random variables on $\mathbb{N}_0$, and let X_0 be a random variable independent of the $U_{i,j}$s. Let

$$X_n = \begin{cases} \displaystyle\sum_{j=1}^{X_{n-1}} U_{n-1,j} & \text{if } X_{n-1} > 0 \\ 0 & \text{if } X_{n-1} = 0 \end{cases} \qquad n = 1, 2, \dots$$

Show that $\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain.

Problem 2.4 Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with countable state space S and transition probabilities $p_{ij}, i, j \in S$. Let N be a random variable independent of $\{X_n\}_{n\geq 0}$ with values in $\mathbb{N}_0$. Let

$$N_n = N + n$$
$$Y_n = (X_n, N_n)$$

for all $n \in \mathbb{N}_0$.

(a) Show that $\{Y_n\}_{n\geq 0}$ is a homogeneous Markov chain, and determine the transition probabilities.

(b) Instead of assuming that N is independent of $\{X_n\}_{n\geq 0}$, it is now only assumed that N is conditional independent of $\{X_n\}_{n\geq 0}$ given X_0 i.e.

$$\begin{aligned} &P((X_1,\dots,X_n)=(i_1,\dots,i_n),N=j \mid X_0=i_0) \\ =&P((X_1,\dots,X_n)=(i_1,\dots,i_n) \mid X_0=i_0)\cdot P(N_0=j \mid X_0=i_0) \end{aligned}$$

for all $i_1,\dots,i_n\in S, n\in\mathbb{N}, j\in\mathbb{N}_0$, and all $i_0\in S$ with $P(X_0=i_0)>0$. Show that $\{Y_n\}_{n\geq 0}$ is a homogeneous Markov chain and determine the transition probabilities.

Problem 2.5 Let $\{X_n\}_{n\geq 0}$ be a stochastic process on a countable state space S. Suppose that there exists a $k\in\mathbb{N}_0$ such that

$$\begin{aligned} &P(X_n=j \mid X_1=i_1,\dots,X_{n-1}=i_{n-1}) \\ =&P(X_n=j \mid X_{n-k}=i_{n-k},\dots,X_{n-1}=i_{n-1}) \end{aligned}$$

for all $n\geq k$ and all $i_0,\dots,i_{n-1},j\in S$ for which

$$P(X_0=i_0,\dots,X_{n-1}=i_{n-1})>0$$

Such a process is called a *k-dependent chain*. The theory for these processes can be handled within the theory for Markov chains by the following construction:
Let

$$Y_n=(X_n,\dots,X_{n+k-1}) \qquad n\in\mathbb{N}_0.$$

Then $\{Y_n\}_{n\geq 0}$ is a stochastic process with countable state space S^k, sometimes refered to as the *snake chain*. Show that $\{Y_n\}_{n\geq 0}$ is a homogeneous Markov chain.

Problem 2.6 An urn holds b black and r red marbles, $b,r\in\mathbb{N}$. Consider the experiment of successively drawing one marble at random from the urn and replacing it with $c+1$ marbles of the same colour, $c\in\mathbb{N}$. Define the stochastic process $\{X_n\}_{n\geq 1}$ by

$$X_n=\begin{cases} 1 & \text{if the } n\text{'th marble drawn is black} \\ 0 & \text{if the } n\text{'th marble drawn is red} \end{cases} \qquad n=1,2,\dots$$

Show that $\{X_n\}_{n\geq 1}$ is not a homogeneous Markov chain.

Problem 2.7 Let $Y_0, Y_1, \ldots$ be a sequence of independent, identically distributed random variables on $\mathbb{Z}$ such that

$$P(Y_n = 1) = P(Y_n = -1) = 1/2 \qquad n = 0, 1, \ldots$$

Consider the stochastic process $\{X_n\}_{n \geq 0}$ given by

$$X_n = \frac{Y_n + Y_{n+1}}{2} \qquad n = 0, 1, \ldots$$

(a) Find the transition probabilities

$$p_{jk}(m, n) = P(X_n = k \,|\, X_m = j)$$

for $m < n$ and $j, k = -1, 0, 1$.

(b) Show that the Chapman-Kolmogorov equations are not satisfied, and that consequently $\{X_n\}_{n \geq 0}$ is not a homogeneous Markov chain.

3. Transient and recurrent states.

Problem 3.1 Below a series of transition matrices for homogeneous Markov chains is given. Draw (or sketch) the transition graphs and examine whether the chains are irreducible. Classify the states.

(a)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

(e)
$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(f)
$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-p & p & 0 & 0 & \cdots \\ 0 & 1-p & p & 0 & \cdots \\ 0 & 0 & 1-p & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(h)

$$\begin{pmatrix} 1-p & 0 & p & 0 & 0 & \cdots \\ 1-p & 0 & 0 & p & 0 & \cdots \\ 1-p & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Problem 3.2 Let there be given r empty urns, $r \in \mathbb{N}$, and consider a sequence of independent trials, each consisting of placing a marble in an urn chosen at random. Let X_n be the number of empty urns after n trials, $n \in \mathbb{N}$. Show that $\{X_n\}_{n \geq 1}$ is a homogeneous Markov chain, find the transition matrix and classify the states.

Problem 3.3 Consider a homogeneous Markov chain with state space $\mathbb{N}_0$ and transition probabilities $p_{ij}, i, j \in \mathbb{N}_0$ given by

$$p_{ij} = \begin{cases} 1 & i = j = 0 \\ p & i = j > 0 \\ q & i - j = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $p + q = 1, p, q > 0$. Find $f_{j0}^{(n)} = P_j\{T_0 = n\}$ for $j \in \mathbb{N}$, and show that $E_j(T_0) = \frac{j}{q}$, where T_0 is the first return time to state 0.

Problem 3.4 *(Random Walks).* Let $Y_0, Y_1, \ldots$ be a sequence of independent, identically distributed random variables on $\mathbb{Z}$. Let

$$X_n = \sum_{j=0}^{n} Y_j \qquad n = 0, 1, \ldots$$

and let $p_{ij}, i, j, \in \mathbb{Z}$ be the transition probabilities for the Markov chain $\{X_n\}_{n \geq 0}$. Define further for all $i, j \in \mathbb{Z}$

$$G_{ij}^{(n)} = \sum_{k=0}^{n} p_{ij}^{(k)} \qquad n = 1, 2, \ldots$$

(a) Show that $G_{ij}^{(n)} \leq G_{00}^{(n)}$ for all $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(b) Establish for all $m \in \mathbb{N}$ the inequalities

$$(2m+1)G_{00}^{(n)} \geq \sum_{\substack{j: \\ |j| \leq m}} G_{0j}^{(n)} \geq \sum_{k=0}^{n} \sum_{\substack{j: \\ |\frac{j}{k}| \leq \frac{m}{n}}} p_{0j}^{(k)}$$

Assume now that $E(Y) = 0$.

(c) Use the Law of Large Numbers to show that

$$\forall a > 0 : \lim_{n \to \infty} \sum_{\substack{j: \\ |j| < na}} p_{0j}^{(n)} = 1$$

(d) Show that $\{X_n\}_{n \geq 0}$ is recurrent if it is irreducible
(Hint: Use (b) with $m = an, a > 0$)

It can be shown that if $\mu \neq 0$ then $\{X_n\}_{n \geq 0}$ is transient. Furthermore, it can be shown that a recurrent RW cannot be positive - see e.g. Karlin(1966): *A first course in stochastic processes.*

4. Positive states and invariant distributions

Problem 4.1 Consider the Markov chains in problem 3.1 (g)+(h). Decide whether there exists positive respectively null recurrent states. Find the invariant distributions for the positive classes.

Problem 4.2 *(Ehrenfest's diffusion model).* Let two urns A and B contain r marbles in total. Consider a sequence of trials, each consisting of choosing a marble at random amongst the r marples and transferring it to the other urn. Let X_n denote the number of marbles in A after n trials, $n \in \mathbb{N}$. Find the transition matrix for the homogeneous Markov chain $\{X_n\}_{n\geq 1}$. Show that the chain is irreducible and positive, and that the stationary initial distribution $(a_j)_{j=0}^r$ is given by

$$a_j = \binom{r}{j}\frac{1}{2^r} \qquad j = 0, \ldots, r.$$

Problem 4.3 *(Bernoulli-Laplace's diffusion model).* Let two urns A and B consist of r red respectively r white marbles. Consider a sequence of trials each consisting in drawing one marble from each urn and switching them. Let X_n be the number of red marbles in urn A after n trials, $n \in \mathbb{N}$. Find the transition matrix for the homogeneous Markov chain $\{X_n\}_{n\geq 1}$, and classify the states. Find the stationary initial distribution (it is a hypergeometric distribution).

Problem 4.4 Consider a homogeneous Markov chain with transition matrix

$$\begin{pmatrix} q_1 & p_1 & 0 & 0 & 0 & \cdots \\ q_2 & 0 & p_2 & 0 & 0 & \cdots \\ q_3 & 0 & 0 & p_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $p_i = 1 - q_i$ and $p_i, q_i \geq 0, i \in \mathbb{N}$. The chain is irreducible if $0 < p_i < 1$ for all $i \in \mathbb{N}$. Find the necessary and sufficient conditions for transience, positive recurrence, null recurrence respectively.

Problem 4.5 Consider a homogeneous Markov chain with transition matrix

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\sum p_i = 1$ and $p_i > 0$ for all $i \in \mathbb{N}$. Show that the chain is irreducible and recurrent. Find a necessary and sufficient condition for the chain to be positive. Find the stationary initial distribution when it exists.

5. Absorption probabilities

Problem 5.1 Consider the Markov chains in problem 3.1. Find the absorption probabilities

$$\alpha_j(C) = P_j\{\exists n \in \mathbb{N}_0 : X_n \in C\},$$

if there exists transient states, j, and recurrent subclasses, C

Problem 5.2 Let there be given two individuals with genotype Aa. Consider a sequence of trials, each consisting of drawing two individuals at random from the offsprings of the previous generation. Let X_n state the genotypes for the individuals drawn in the n'th trial, $n \in \mathbb{N}$. Thus X_n can take 6 different values

$$\begin{array}{lll} E_1 = \{AA, AA\} & E_2 = \{AA, Aa\} & E_3 = \{Aa, Aa\} \\ E_4 = \{Aa, aa\} & E_5 = \{aa, aa\} & E_6 = \{AA, aa\} \end{array}$$

Assume that the probability for A respectively a is $1/2$. Find the transition matrix and classify the states for this homogeneous Markov chain. Determin the absorption probabilities $\alpha_j(C)$ for all transient states j and $C = \{E_1\}$ respectively $C = \{E_5\}$.

Problem 5.3 Consider a game of tennis between two players A and B. Let us assume that A wins the points with probability p, and that points are won independent. In a game there is essentially 17 different states: *0-0, 15-0, 30-0, 40-0, 15-15, 30-15, 40-15, 0-15, 0-30, 0-40, 15-30, 15-40, advantage A, advantage B, game A, game B, deuce* since *30-30* and *deuce*, respectively *30-40* and *advantage B*, respectively *40-30* and *advantage A* may be considered to be the same state.

Show that the probability for A winning the game, p_A, is

$$p_A = p^4 + 4p^4q + \frac{10p^4q^2}{1-2pq} = \begin{cases} \dfrac{p^4(1-16q^4)}{p^4-q^4} & p \neq q \\ \dfrac{1}{2} & p = q \end{cases}$$

where $q = 1 - p$.
(Hint: It is sufficient to look at the Markov chain consisting of the 5 states: advantage A, advantage B, game A, game B, deuce).

Problem 5.4 *(Martingales)*. Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space $S = \{0, \dots, N\}$ and transition probabilities $p_{ij}, i, j \in S$.

Assume that

$$E_j(X_1) = \sum_{k=0}^{N} k p_{jk} = j \qquad j = 0, \dots, N$$

Thus, in average the chain will neither increase nor decrease. Then $\{X_n\}_{n\geq 0}$ is a *martingale*. It follows immediately that $p_{00} = p_{NN} = 1$, i.e. 0 and N are absorbing states. We assume that the other states are all transient.

(a) Show that $E_j(X_n) = j$ for all $n \in \mathbb{N}$.

(b) Show that the probability for absorption in N is given by

$$\alpha_j(\{N\}) = \frac{j}{N} \qquad j = 0, \dots, N.$$

Problem 5.5 *(Waiting times to absorption).* Consider a homogeneous Markov chain with state space S, and let $C' \subseteq S$ denote the set of transient states. Let T be the first time a recurrent state is visited and let

$$d_j = \sum_{k=0}^{\infty} k P_j(T = k) \qquad j \in C'$$

Assume that $P_j(T = \infty) = 0$ for all $j \in S$.

(a) Show that $(d_j)_{j\in C'}$ satisfies the system of equations

$$d_j = 1 + \sum_{i\in C'} p_{ji} d_i \qquad j \in C' \qquad\qquad (*)$$

(b) Show that $(d_j)_{j\in C'}$ is the smallest non-negative solution to $(*)$, i.e. if $(z_j)_{j\in C'}$ is a solution to $(*)$ with $z_j \in [0, \infty], j \in C'$, then

$$z_j \geq d_j \qquad j \in C'.$$

(c) Assume that S is finite. Show that $(d_j)_{j\in C'}$ is the only solution to $(*)$.

6. Convergence of transition probabilities.

Problem 6.1 A transition matrix $P = (p_{ij})_{i,j \in S}$ for a homogeneous Markov chain with state space S, is called *doubly stochastic* if it is stochastic and

$$\sum_{i \in S} p_{ij} = 1 \qquad j \in S$$

(a) Assume that $S = \{0, \dots, n\}, n \in \mathbb{N}$. Show that if P is irreducible and doubly stochastic, then the Markov chain is positive. Find the stationary initial distribution.

(b) Assume that $S = \mathbb{N}$ and that P is irreducible, aperiodic and doubly stochastic. Show that the Markov chain is not positive.
(Hint: Use that the equation

$$\sum_{j=1}^{\infty} p_{jk}^{(n)} \geq \sum_{j=1}^{N} p_{jk}^{(n)}$$

is valid for all n, $N \in \mathbb{N}$*).*

Problem 6.2 Let $P = (p_{ij})_{i,j \in S}$ be an irreducible transition matrix for a homogeneous Markov chain with state space S. Suppose that P is *idempotent*, i.e.

$$P^n = P \qquad n = 2, 3, \dots$$

(a) Show that all states are recurrent and aperiodic.

(b) Show that the chain is positive

(c) Show that

$$\forall i, j \in S : \qquad p_{ij} = p_{jj}$$

i.e. the rows in P are identical.

Problem 6.3 Let $\{X_n\}_{n \geq 0}$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{ij})_{i,j \in S}$. Let $C \subseteq S$ be a non-empty, aperiodic recurrent subclass.

(a) Show that
$$\pi_j = \lim_{n\to\infty} p_{jj}^{(n)} < \infty \qquad j \in C.$$

Let $i \in S \setminus C$ and $k \in C$ and let

$$\begin{aligned}
\alpha_{ik}(C,n) =& P_i(X_1 \notin C, \dots, X_{n-1} \notin C, X_n = k)\\
\alpha_i(C,n) =& P_i(X_1 \notin C, \dots, X_{n-1} \notin C, X_n \in C)\\
\Pi_i(C) =& P_i(\exists n \in \mathbb{N} \forall k > n : X_k \in C)
\end{aligned}$$

(b) Show that for any $\epsilon > 0$ there exists a finite subset $\tilde{C} \subseteq C$ and an $N_\epsilon \in \mathbb{N}$ such that

$$\forall n > N_\epsilon : |\alpha_i(C,n) - \sum_{\nu=1}^{n} \sum_{k\in\tilde{C}} \alpha_{ik}(C,\nu)| < \epsilon$$

(c) Let $j \in C$. Show that

$$p_{ij}^{(n)} = \sum_{\nu=1}^{n} \sum_{k\in C} p_{kj}^{(n-\nu)} \alpha_{ik}(C,\nu)$$

(d) Show that

$$\begin{aligned}
|p_{ij}^{(n)} - \pi_j \sum_{\nu=1}^{n} \sum_{k\in\tilde{C}} \alpha_{ik}(C,\nu)| \leq & |\sum_{\nu=1}^{n} \sum_{k\in\tilde{C}} \alpha_{ik}(C,\nu)(p_{kj}^{(n-\nu)} - \pi_j)|\\
&+ |\sum_{\nu=N+1}^{n} \sum_{k\in\tilde{C}} \alpha_{ik}(C,\nu)(p_{kj}^{(n-\nu)} - \pi_j)|\\
&+ \sum_{\nu=1}^{n} \sum_{k\in C\setminus\tilde{C}} p_{kj}^{(n-\nu)} \alpha_{ik}(C,\nu)
\end{aligned}$$

(e) Now show that
$$\lim_{n\to\infty} p_{ij}^{(n)} = \Pi_i(C)\pi_j$$
(Hint: Use (b) to bound the terms in (d)).

7. Markov chains with finite state space

Problem 7.1 Consider the Markov chains in problem 3.1 (a)-(f). Decide whether or not there exists positive states. If so, find the stationary initial distributions for the positive classes.

Problem 7.2 Consider a homogeneous Markov chain with state space $S = \{1, \ldots, n\}$ and transition matrix

$$\begin{pmatrix} q & p & 0 & 0 & 0 & \cdots & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & q & 0 & p \\ 0 & \cdots & 0 & 0 & 0 & q & p \end{pmatrix}$$

Show that the chain is positive and find the stationary initial distribution.

8. Examples of Markov chains

Problem 8.1 Consider a usual p-q Random Walk, starting in 0, and let $a, b \in \mathbb{N}$

(a) Show that the probability $\alpha(a)$, that a is reached before $-b$ is

$$\alpha(a) = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^{b} - 1}{\left(\frac{q}{p}\right)^{a+b} - 1} & p \neq q \\ \dfrac{b}{a+b} & p = q \end{cases}$$

(b) Let $a = b = n, n \in \mathbb{N}$, and let $p > q$. Find

$$\lim_{n\to\infty} \alpha(n)$$

Problem 8.2 A gambler is playing the roulette, placing a bet of 1 counter on *black* in each game. When she wins, she receives a total of two counters. The ongoing capital size is thus a usual p-q Random Walk with state space $\mathbb{N}_0$ and with 0 as an absorbing barrier. Find the probability of the gambler having a capital of $n+k$ counters, $k \in \mathbb{N}$, at some time given that her initial capital is n counters.

Problem 8.3 Consider a p-q Random Walk with absorbing barriers 0 and $N, N \in \mathbb{N}$. Find the expected waiting time d_j until absorption in 0 or N, $j = 1, \ldots, N-1$.
(Hint: Use the result from problem 5.5 and that the complete solution to the inhomogeneous system of equations is equal to a partial solution plus the complete solution for the homogeneous system of equations).

Problem 8.4 Consider a p-q Random Walk with state space $\mathbb{N}_0$ and with 0 as an absorbing wall. Find the mean waiting time d_j until absorption in 0, $j \in \mathbb{N}$.
(Hint: Use the result from problem 5.5(b))

Problem 8.5 Let $\{X_n\}_{n\geq 0}$ be a usual p-q Random Walk starting in 0.

(a) Find the probability

$$\alpha(0) = P(X_n = 0 \text{ for at least one } n \in \mathbb{N})$$

of the event that $\{X_n\}_{n\geq 0}$ visits 0.

Let Q_0 denote the number of times $\{X_n\}_{n\geq 0}$ visits 0.

(b) Find the distribution of Q_0.
(Hint: Use that

$$P(Q_0 = k) = \sum_{n=1}^{\infty} P(X_1 \neq 0, \dots, X_{n-1} \neq 0, X_n = 0, \sum_{j=n+1}^{\infty} 1_{\{X_j=0\}} = k-1) \quad)$$

Problem 8.6 Consider a branching process $\{X_n\}_{n\geq 0}$ with $X_0 = 1$ and offspring distribution given by

$$p_0(k) = \begin{cases} bc^{k-1} & k \geq 1 \\ 1 - \frac{b}{1-c} & k = 0 \end{cases}$$

where $b, c > 0$ and $b + c \leq 1$. This is known as a modified geometric distribution.

(a) Find the generating function for the offspring distribution.

(b) Find the extinction probability and the mean of the offspring distribution.

(c) Find the generating function for X_n and use this to find the distribution of X_n. Note that the distribution is of the same type (modified geometric distribution) as the actual offspring distribution.
(Hint: Start with $n = 2$ and $n = 3$ and try to guess the general expression).

Problem 8.7 Let N be a random variable, which is Poisson distributed with parameter $\lambda \in \mathbb{R}_+$. Consider N independent Markov chains with state space $\mathbb{Z}$, starting in 0, and all with the same transition matrix $P = (p_{ij})_{i,j\in\mathbb{Z}}$. Let $Z_k^{(n)}$ be the number of Markov chains that after n steps are in state k, $n \in \mathbb{N}, k \in \mathbb{Z}$. Show that $Z_k^{(n)}$ is Poisson distributed with parameter $\lambda p_{0k}^{(n)}, \lambda \in \mathbb{R}, k \in \mathbb{Z}$.

9. Definition of homogeneous Markov chains in continuous time.

Problem 9.1 Let T be exponentially distributed with mean λ. Define the stochastic process $\{X(t)\}_{t\geq 0}$ by $X(t) = 1_{\{T\leq t\}}, t \geq 0$. Show that $\{X(t)\}_{t\geq 0}$ is a homogeneous Markov process and find $P(t), t \geq 0$.

Problem 9.2 Let T be a non-negative continuous random variable. Consider the stochastic process $\{X(t)\}_{t\geq 0}$ given by $X(t) = 1_{\{T\leq t\}}, t \geq 0$. Show that unless T is exponentially distributed, $\{X(t)\}_{t\geq 0}$ cannot be a continuous time homogeneous Markov chain.

Problem 9.3 Let $\{P(t)\}_{t\geq 0}$ be substochastic on a countable state space S, i.e.

$$p_{ij}(t) \geq 0 \text{ and } S_i(t) = \sum_{j\in S} p_{ij}(t) \leq 1,$$

such that $\{P(t)\}_{t\geq 0}$ satisfies the Chapman-Kolmogorov equations and $P(0) = I$. Assume that $p_{ik}(t) > 0$ for all $i, k \in S, t > 0$. Show that either is $S_i(t) = 1, \forall i \in S, \forall t > 0$ or $S_i(t) < 1, \forall i \in S, \forall t > 0$.

Problem 9.4 Let T and U be two independent exponentially distributed random variables with parameter α respectively β. Consider the following process: At time 0 an individual is in state 0. Afterwards it can move to in either state 1 or 2, and then remain in that state. The individual moves to state 1 at time T if $T < U$, and to state 2 at time U if $U < T$. Show that $X_t =$ "State at time t", $t \geq 0$ is a homogeneous Markov process and find the transition probabilities.

10. Construction of homogeneous Markov chains

Problem 10.1 Let $\{X(t)\}_{t\geq 0}$ be a regular jump process on the state space $S = \{1, \ldots, N\}$ with intensity matrix $Q = (q_{ij})_{i,j\in S}$, that satisfies $q_{ij} = q_{ji}, i, j \in S$. Define

$$E(t) = -\sum_{k=1}^{N} p_{ik}(t)\log(p_{ik}(t))$$

with $x \log x = 0$ for $x = 0$.

(a) Show that $p'_{ik}(t) = \sum_{j=1}^{N} q_{kj}(p_{ij}(t) - p_{ik}(t))$

(b) Show that

$$E'(t) = \frac{1}{2}\sum_{j,k=1}^{N} q_{kj}(p_{ij}(t) - p_{ik}(t))(\log p_{ij}(t) - \log p_{ik}(t))$$

(c) Finally show that $E(t)$ is a nondecreasing function of $t \geq 0$.

Problem 10.2 Let $\{X(t)\}_{t\geq 0}$ be a homogeneous Markov chain with state space $\mathbb{Z}$, 0 as the initial value and parameters (λ, Π), where $\lambda_i = \lambda > 0$ for all i, and π_{ij} only depends on $j - i$ for all $i, j \in \mathbb{Z}$.

(a) The increase of the process over an interval $[t, t + h[, t \geq 0, h > 0$, is defined as the random variable $X(t+h)-X(t)$. Show that the increases corresponding to disjoint intervals are independent.

(b) Find the characteristic function for $X(t)$'s distribution, expressed by the characteristic function of the distribution F, which is determined by the probability masses $(\pi_{0j})_{j\in\mathbb{Z}}$.

(c) Show that $E(X(t))^k$ exists if F has k'th moments, and find the first and second order moments of $X(t)$ expressed by first and second order moments of F.

(d) Examine in particular the case where $\{X(t)\}_{t\geq 0}$ is a Poisson process.

(e) Examine in particular the case where Π is the transition matrix for a Random Walk.

Problem 10.3 Consider the following infinitesimal generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \qquad \alpha, \beta > 0$$

(a) Find the corresponding jump matrix, Π, and the intensities of waiting times between jumps, λ

(b) Show that a HMC with infinitesimal generator Q is uniformisable and construct the corresponding uniform Markov chain

(c) Find the transition probabilities.

Hint: Q is diagonalisable:

$$Q = V \Lambda U^\top$$

with

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -(\alpha+\beta) \end{bmatrix} \quad V = \frac{1}{\alpha+\beta} \begin{bmatrix} \alpha & \alpha \\ \alpha & -\beta \end{bmatrix} \quad \textit{and} \quad U^\top = \begin{bmatrix} \beta/\alpha & 1 \\ 1 & -1 \end{bmatrix}$$

(d) Write down the forward and backward differential equations and use them to verify your solution from question 3.

(e) Find $\lim_{t\to\infty} P(t)$

Problem 10.4 Consider the matrix

$$Q = \begin{bmatrix} -3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Verify that Q is an infinitesimal generator.

Let $(X_t)_{t\geq 0}$ be a Markov chain on $\{1, 2, 3\}$ with infinitesimal generator Q and initial distribution $\mu = (1, 0, 0)^\top$.

(b) Find the jump matrix Π.

(c) 3 of the transition probaibilties $p_{i,j}(t)$ are 0 for all $t \geq 0$. Which ones?

(d) Find the remaining transition proababilities:

1. Find the forward differential equation for $p_{3,3}(t)$ and solve it.
2. Find the forward differential equation for $p_{2,2}(t)$ and solve it.
3. Find $p_{2,3}(t)$.
4. Find the forward differential equation for $p_{1,1}(t)$ and solve it.
5. Find the forward differential equation for $p_{1,2}(t)$ and solve it.
6. Find $p_{1,3}(t)$.

(e) Find $P\{X_t = 3\}$.

(f) Give a suggestion of how to find $P\{X_t = 3\}$ if the initial distribution is $\mu = (1/2, 1/2, 0)^\top$.

Problem 10.5 Consider a Markov chain $(X_t)_{t\geq 0}$ on $\{1, 2, 3\}$ with infinitesimal generator

$$Q = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and initial distribution $\mu = (1, 0, 0)^\top$.

(a) Find $P\{\tau_1 > t\}$ where τ_1 denotes the first transition time of the chain.

(b) Let τ_2 be the second transition time of the chain. Write an integral expression for $P\{\tau_2 \leq t\}$

(c) Find the invariant distribution of the embedded Markov chain.

(d) Suggest an invariant distribution for the Markov chain $(X_t)_{t\geq 0}$.

11. The Poisson process

Problem 11.1 Let $\{X(t)\}_{t\geq 0}$ be a Poisson process starting in 0 and with parameter $\lambda > 0$. Assume that each jump "is recorded" with probability $p \in]0,1[$ independent of the other jumps. Let $\{Y(t)\}_{t\geq 0}$ be the recording process. Show that $\{Y(t)\}_{t\geq 0}$ is a Poisson process with parameter λp.

Problem 11.2 Consider a Poisson process, starting in 0 and with parameter $\lambda > 0$. Given that n jumps has occurred, $n \in \mathbb{N}_0$, at time $t > 0$, show that the density for the time of the r'th jump ($r < n$) is the following

$$f(s) = \begin{cases} \dfrac{n!s^{r-1}}{(r-1)!(n-r)!t^r}\left(1-\dfrac{s}{t}\right)^{n-r} & 0 < s < t \\ 0 & s \geq t \end{cases}$$

Problem 11.3 Consider two independent Poisson processes $\{X(t)\}_{t\geq 0}$ and $\{Y(t)\}_{t\geq 0}$, both starting in 0 and where $E(X(t)) = \lambda t$ and $E(Y(t)) = \mu t$ with $\lambda, \mu > 0$. Let T and T', $T' > T$, be two successive jumps of the $X(t)_{t\geq 0}$ process such that $X(t) = X(T)$ for $T \leq t < T'$ and $X(T') = X(T) + 1$. Let $N = Y(T') - Y(T)$, i.e. the number of jumps the process $\{Y(t)\}_{t\geq 0}$ makes in the time interval $]T, T'[$. Show that N is geometrically distributed with parameter $\frac{\lambda}{\lambda+\mu}$.

Problem 11.4 Consider a Poisson process $\{X(t)\}_{t\geq 0}$, starting in 0 and with parameter $\lambda > 0$. Let T be the time until the first jump and let $N(\frac{T}{k})$ be the number of jumps in the next $\frac{T}{k}$ time units. Find the mean and variance of $T \cdot N(\frac{T}{k})$.

Problem 11.5 Consider a detector measuring electric shocks. The shocks are all of size 1 (measured on a suitable scale) and arrive at random times, such that the number of shocks, seen at time t, is given by the Poisson process $\{N(t)\}_{t\geq 0}$, starting in 0 and with parameter $\lambda > 0$, i.e. the waiting times between the shocks are independent exponential distributed with mean λ. The output of the detector at time t for a shock, arriving at the random time S_i is

$$[\exp\{-\beta(t - S_i)\}]_+ = \begin{cases} 0 & t < S_i \\ \exp\{-\beta(t - S_i)\} & t \geq S_i \end{cases}$$

where $\beta > 0$, i.e. the effect from a shock is exponentially decreasing. We now assume that the detector is linear, so that the total output at time t is given by:

$$\alpha(t) = \sum_{i=1}^{N(t)} [\exp\{-\beta(t - S_i)\}]_+$$

We wish to find the characteristic function $s \to \phi_t(s)$ for the process $\alpha(t)$.

(a) Show that given $N(t) = n$, i.e. there has been n shocks in the interval $]0, t]$, the arrival $S_1, \ldots, S_n$ of the shocks, are distributed as the ordered values of n independent uniformly distributed random variables $X_1, \ldots, X_n$ on $[0, t]$.

Let $Y_t(i) = [\exp\{-\beta(t - X_i)\}]_+, i = 1, \ldots, n$. Note that given $N(t) = n$, $Y_t(i), i = 1, \ldots, n$ are independent and identically distributed.

(b) Find the characteristic function $s \to \theta_t(s)$ of $Y_t(i)$

(c) Determine now the characteristic function of $\alpha(t)$ expressed by $\theta_t(s)$

(d) Use e.g. (c) to find the mean and variance for $\alpha(t)$

Problem 11.6 Arrivals of the Number 1 bus form a Poisson process with rate 1 bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate seven buses per hour.

(a) What is the probability that exactly three buses pass by in one hour?

(b) What is the probability that exactly three Number 7 buses pass by while I am waiting for a Number 1 bus?

(c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What, then, is the probability that I wait for 30 minutes without seeing a single bus?

Problem 11.7 A radioactive source emits particles in a Poisson process of rate λ. The particles are each emitted in an independent random direction. A Geiger counter placed near the source records a fraction p of the particles emitted. What is the distribution of the number of particles recorded in time t?

12. Birth and death processes

Problem 12.1 Consider a birth process with $\lambda_0 = a > 0, \lambda_n = b > 0$ for $n \in \mathbb{N}$ where $a < b$. Find the transition probabilities.

Problem 12.2 Consider a birth process where $\lambda_i > 0$ for all $i \in S$

(a) Show for an arbitrary fixed $n \in \mathbb{N}$ the function

$$t \to p_{i,i+n}(t) \qquad i \in S$$

first is increasing, next decreasing towards 0 and if t_n is the maximum, then show: $t_1 < t_2 < t_3 < \dots$
(Hint: Use induction and Kolmogorov's forward differential system.)

(b) Show that if $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$ then $t_n \to \infty$.

Problem 12.3 *(Exploding birth processes).* In a population a new individual is born with probability $p \in]0,1]$, each time two individuals collide. The collision between two given individuals in a time interval of length t, happens with probability $\alpha t + o(t), \alpha > 0$. The number of possible collisions between k individuals is $\binom{k}{2}$, and it seems reasonable to describe the size of the population by a birth process with waiting time parameters

$$\lambda_i = \binom{i}{2} \alpha p \qquad i = 0, 1, 2, \dots$$

with $\lambda_0 = \lambda_1 = 0$. Show that the process explodes and find $E_i(S_\infty), i = 2, 3, \dots$. The model has been used to describe the growth of lemmings.

Problem 12.4 Consider two independent linear death processes both with same death intensity $\lambda_i = i\mu$. One of the populations consists of m men and k women. Determine the expected number of women left, when the men dies out.

Problem 12.5 Consider a birth and death process on the state space $M, M+1, \dots, N$ with

$$\lambda_n = \alpha n(N-n) \qquad \mu_n = \beta n(n-M)$$

where $M < N$ are interpreted as the upper and lower limits for the population. Show that the stationary distribution is proportional to

$$\frac{1}{j}\binom{N-M}{j-M}\left(\frac{\alpha}{\beta}\right)^{j-M} \qquad j = M, M+1, \ldots, N.$$

Problem 12.6 Consider a system consisting of N components, all working independent of each other, and with life spans of each component exponentially distributed with mean λ^{-1}. When a component breaks down, repair of the component starts immediately and independent of whether any other component has broken down. The repair time of each component is exponentially distributed with mean μ^{-1}. The system is in state n at time t, if there is exactly n components under repair at time t. This is a birth and death process.

(a) Determine the intensity matrix.

(b) Find the stationary initial distribution.

(c) Let $\lambda = \mu$ and assume that all N components are working. Find the distribution function $F(t)$ of the first time, when 2 components does not work.

Problem 12.7 Consider the linear birth and death process, i.e. the birth and death intensities are $\beta_i = i\beta$ and $\delta_i = i\delta$. Let

$$\begin{aligned} G_1 &= G_1(\theta, t) = \sum_{n=0}^{\infty} p_{1n}(t)\theta^n \\ G_2 &= G_2(\theta, t) = \sum_{n=0}^{\infty} p_{2n}(t)\theta^n \end{aligned}$$

(a) Show that $G_2(\theta, t) = (G_1(\theta, t))^2$.

(b) Write the backwards differential equations for $p_{1n}, n \geq 1$ and p_{10}.

(c) Show that

$$\frac{\partial G_1}{\partial t} = -(\beta + \delta)G_1 + \beta G_1^2 + \delta$$

(d) Show that for $\delta > \beta$

$$\frac{\partial}{\partial t}\log\left(\frac{\beta G_1 - \delta}{G_1 - 1}\right) = \delta - \beta$$

(e) Show that for $\delta > \beta$

$$G_1(\theta, t) = \frac{\delta(1-\theta) - (\delta - \beta\theta)e^{(\delta-\beta)t}}{\beta(1-\theta) - (\delta - \beta\theta)e^{(\delta-\beta)t)}}$$

This can also be shown for $\delta < \beta$.

(f) Show that for $\delta \neq \beta$ is

$$p_{10}(t) = \frac{\delta - \delta e^{(\delta-\beta)t)}}{\beta - \beta e^{(\delta-\beta)t)}}$$

and find $\lim_{t\to\infty} p_{10}(t)$.

(g) Show that

$$p_{10}(t) = \frac{\beta t}{1+\beta t} \qquad \text{for } \beta = \delta$$

(Hint: Taylor expand $p_{10}(t)$ from (f)).

(h) From (f) it is known that for $\delta > \beta$ the process will reach state 0 (the population dies out) at some time. Let T_0 be the waiting time for this event. Find the distribution and mean of T_0.

(i) Show that for $\delta = \beta$ it holds that

$$G_1(\theta, t) = \frac{\beta t + \theta(1 - \beta t)}{1 + \beta t - \theta\beta t}$$

and consequently that

$$p_{1n}(t) = \frac{(\beta t)^{n-1}}{(1+\beta t)^{n+1}} \qquad n \geq 1$$

(Hint: Use e.g. Taylor expansion of $G_1(\theta, t)$ from (e)).

(j) Calculate $p_{n0}(t), n \geq 2$.

(k) Let F_m be the distribution function for $\frac{T_0^{(m)}}{m}$, where $T_0^{(m)}$ is the waiting time until the population has died out when it starts with m individuals. Show that

$$\lim_{n\to\infty} F_n(t) = \exp\left(-\frac{1}{t}\right)$$

13. Queuing processes

Problem 13.1 The modified Bessel function of order n is given by

$$I_n(y) = \sum_{j=0}^{\infty} \frac{(\frac{y}{2})^{n+2j}}{j!(n+j)!}$$

Put

$$\Phi_n(t) = \exp(-(\beta+\delta)t)\left(\frac{\beta}{\delta}\right)^{\frac{n}{2}} I_n(2t\sqrt{\delta\beta})$$

for $\delta, \beta > 0$.

The explicit specification of the transition probabilities for M/M/1-queues is a mathematically complicated matter. For $n \geq 1$ and with arrival and service times respectively β and δ, it holds that

$$p_{0n}(t) = \sum_{k=0}^{\infty}\left(\left(\frac{\beta}{\delta}\right)^{-k}\Phi_{n+k}(t) - \left(\frac{\beta}{\delta}\right)^{-k-1}\Phi_{n+k+2}(t)\right)$$

Show that $p_{0n}(t)$ satisfies the forward equations.
(Hint: Use, without proving it, that

$$\frac{d}{dy}(I_n(y)) = \frac{1}{2}(I_{n-1}(y) + I_{n+1}(y)) \quad \textit{)}.$$

Problem 13.2 Consider a M/M/1-queue with parameters (β, δ), where it is assumed that $\beta < \delta$ and that the stationary initial distribution is used. The total waiting time for a customer is the waiting time in the queue plus the service time. Show that the total waiting time for a customer is exponentially distributed with mean $(\delta - \beta)^{-1}$.

Problem 13.3 Consider a M/M/1-queue with parameters (β, δ) and with the change that customers are not going into the queue, unless they are being attended to immediately. Hence $p_{00}(t), p_{01}(t), p_{10}(t)$ and $p_{11}(t)$ are the only transition probabilities not equal to zero.

(a) Show that

$$\frac{d}{dt}(\exp((\beta+\delta)t)p_{01}(t)) = \beta\exp((\beta+\delta)t)$$

use the forward equations and that P(t) is a stochastic matrix.

(b) Find $p_{01}(t)$ and find $\lim_{t\to\infty} p_{01}(t)$.

Problem 13.4 We shall in this problem consider a M/M/s-queue, $s \in \mathbb{N}$, i.e. instead of one service station, there is now s service stations. Assume that the stations work independently of each other, and that the service times at all stations are independent and identical exponentially distributed with mean δ^{-1}. The customers arrives according to a Poisson process with intensity β. We assume that the service stations are optimally used, such that a customer is not queuing at a busy station if another station is available. Assume that $\rho = \frac{\beta}{s\delta} < 1$.

(a) Show that the stationary initial distribution $\{\pi_n\}_{n\in\mathbb{N}_0}$ is given by

$$\pi_0 = \left(\frac{(s\rho)^s}{s!(1-\rho)} + \sum_{i=0}^{s-1} \frac{(s\rho)^i}{i!} \right)^{-1}$$

$$\pi_n = \begin{cases} \dfrac{(s\rho)^n}{n!} a_0 & 1 \le n \le s \\[2ex] \dfrac{\rho^n s^s}{s!} a_0 & s < n < \infty \end{cases}$$

Let $Q = \max(X_t - s, 0), n = 0, 1, \ldots$ be the size of the queue, not counting those being served at the moment. Assume that the stationary initial distribution is used.

(b) Show that

$$\gamma = P(Q = 0) = \frac{\displaystyle\sum_{i=0}^{s} \frac{(s\rho)^i}{i!}}{\displaystyle\sum_{i=0}^{s} \frac{(s\rho)^i}{i!} + \frac{(s\rho)^s \rho}{s!(1-\rho)}}$$

(c) Show that $E(Q) = \dfrac{1-\gamma}{1-\rho}$.

Problem 13.5 Consider a M/M/∞-queue, i.e. a birth and death process with birth intensities given by $\beta_i = \alpha$ and death intensities by $\delta_i = i\delta$. Let

$$G(\theta, t) = \sum_{n=1}^{\infty} p_{mn}(t)\theta^n$$

for fixed $m > 0$.

(a) Write the forward differential equations for $p_{mn}(t), n \geq 0, m$ fixed, where $p_{m,-1}(t) \equiv 0$.

(b) Show that

$$\frac{\partial G}{\partial t} = -\left(\alpha G + \delta\theta\frac{\partial G}{\partial \theta}\right) + \delta\frac{\partial G}{\partial \theta} + \alpha\theta G \qquad (*)$$

(c) Show that

$$G(\theta, t) = \exp\left\{\frac{\alpha}{\delta}(\theta - 1)[1 - \exp(-\delta t)]\right\}[1 + (\theta - 1)\exp(-\delta t)]^m$$

is the solution of $(*)$. *(Hint: Use the boundary conditions* $p_{mn}(0) = 0 \quad (n \neq m)$ *and* $p_{mm}(0) = 1$.*)*

(d) Determine $p_{m0}(t)$.

(e) Show that

$$p_{0n}(t) = \frac{\left(\frac{\alpha}{\delta}[1 - \exp(\delta t)]\right)^n}{n!}\exp\left(-\left(\frac{\alpha}{\delta}[1 - \exp(\delta t)]\right)\right)$$

(f) Consider a linear death process with death intensity $\delta_i = i\delta$, with transition probabilities $p^{\star}_{ij}(t)$. Show that

$$p_{mn}(t) = \sum_{i=0}^{\infty} p_{0i}(t)p^{\star}_{m,n-i}(t)$$

in the M/M/∞-queue.

14. Markov chains in continuous time with finite state space.

Problem 14.1 Consider 2 cables, A and B, transmitting signals across the Atlantic. The waiting times until cable A or cable B breaks are independent, exponential distributed with mean λ^{-1}. We assume that as soon as a cable breaks, the repair starts immediately. The repair times for cable A and B are independent exponential distributed random variables with mean μ^{-1}. This is a homogeneous Markov chaining continuous time with 3 states: 1 = {Both cables function}, 2 = {Exactly one cable functions}, 3 = {Neither cable functions}.

(a) Show that $\Pi = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} \\ 0 & 1 & 0 \end{pmatrix}$ and $(\lambda_i)_{i\in S} = (2\lambda, \lambda+\mu, 2\mu)$

(b) Given that both cables work at time 0, show that the probability of both cables working at time $t > 0$ is

$$\frac{\mu^2}{(\mu+\lambda)^2} + \frac{\lambda^2 \exp\{-2(\mu+\lambda)t\}}{(\mu+\lambda)^2} + \frac{2\mu\lambda}{(\mu+\lambda)^2} \exp\{-(\mu+\lambda)t\}$$

(c) Find the stationary initial distribution for the number of cables out of order and show that it has mean $\frac{2\lambda}{\mu+\lambda}$

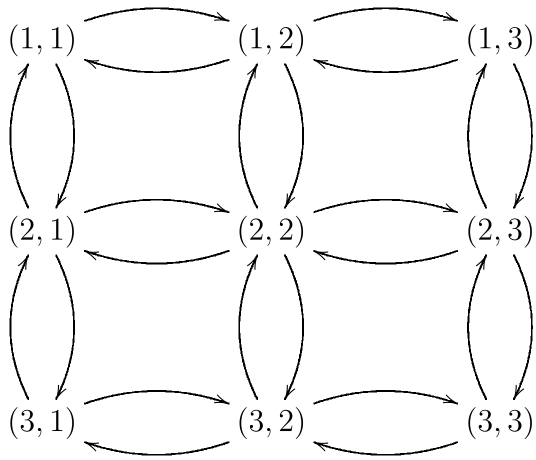
Problem 14.2 Let $(X_t)_{t\geq 0}$ be a (regular jump) homogeneous continuous-time Markov chain on

$$E = \{1,2,3\}^2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

The corresponding infinitesimal generator has

$$q_{(i,j),(k,l)} = \begin{cases} 1 & \text{if } |i-k|+|j-l| = 1 \\ 0 & \text{if } |i-k|+|j-l| > 1 \end{cases}$$

Thus the local characteristics are exactly 1 when there is an arrow in the graph below:



Note that $q_{(i,j),(i,j)}$ is neither 0 or 1 but must be found from the rest.

(a) Find the transition matrix for the embedded discrete-time Markov chain $(Y_n)_{n\in\mathbb{N}}$.

(b) Find the invariant distribution for the embedded discrete time Markov chain.

(c) Find the invariant distribution for $(X_t)_{t\geq 0}$.

Let f and g be functions defined on E by

$$f((i,j)) = i \qquad g((i,j)) = j \qquad (i,j) \in E$$

and let $U_t = f(X_t)$ and $V_t = g(X_t)$. Hence $X_t = (U_t, V_t)$. One may show that $(U_t)_{t\geq 0}$ is a Markov chain with infinitesimal generator

$$A_U = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Let $p_{i,j}(t) = P\{U_t = j | U_0 = i\}$ and note that by symmetry

$$p_{1,1}(t) = p_{3,3}(t) \quad p_{1,2}(t) = p_{3,2}(t) \quad p_{1,3}(t) = p_{3,1}(t) \quad p_{2,1}(t) = p_{2,3}(t)$$

(d) Find $p_{2,2}(t)$.

(e) Find the remaining transition probabilities.

Assume that the initial distribution of $(X_t)_{t\geq 0}$ is such that

$$P\{X_0 = (i,j)\} = P\{(U_0, V_0) = (i,j)\} = P\{U_0 = i\}P\{V_0 = j\} \quad (i,j) \in E$$

Then one may show $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are independent, i.e. that

$$\begin{aligned} &P\{U_{t_1} = i_1, \dots, U_{t_k} = i_k, V_{t_1} = j_1, \dots, V_{t_k} = j_k\} \\ &\qquad = P\{U_{t_1} = i_1, \dots, U_{t_k} = i_k\}P\{V_{t_1} = j_1, \dots, V_{t_k} = j_k\} \end{aligned}$$

for any $k \in \mathbb{N}$, $0 \leq t_1 \leq t_k$, $i_1, \dots, i_k, j_1, \dots, j_k \in \{1,2,3\}$. Use this to find the transition probabilities for $(X_t)_{t\geq 0}$.

(f) Are the discrete time Markov chains $(f(Y_n))_{n\in\mathbb{N}_0}$ and $(g(Y_n))_{n\in\mathbb{N}_0}$ independent? I.e. is

$$\begin{aligned} &P\{f(Y_0) = i_0, \dots, f(Y_k) = i_k, g(Y_0) = i_0, \dots, g(Y_k) = i_k\} \\ &\qquad = P\{f(Y_0) = i_0, \dots, f(Y_k) = i_k\}P\{g(Y_0) = i_0, \dots, g(Y_k) = i_k\} \end{aligned}$$

for all $k \in \mathbb{N}_0$,$i_0, \dots, i_k, j_0, \dots, j_k \in \{1,2,3\}$.