Transience and recurrence

1 Decomposition

Brémaud’s Theorem 2.7.2 has some useful consequences for non-irreducible Markov chains. They are more or less given at the end of section 3.1.3 but we choose to state them as a corollary for clarity:

Corollary 1 Suppose that $j$ is accessible from $i$ ($i \rightarrow j$).

1. If $i$ is recurrent, then
   
   (a) $f_{ji} = 1$
   
   (b) $j \rightarrow i$
   
   (c) $j$ is recurrent
   
   (d) $f_{ij} = 1$

2. If $i$ is not accessible from $j$ ($j \not\rightarrow i$), then $i$ is transient.

Proof As $i \rightarrow j$ there is a $n \in \mathbb{N}$ such that $P_i\{T_j = n\} > 0$. Now

$$P_i \left( \{T_j = n\} \cap \bigcup_{k=n+1}^{\infty} \{X_k = i\} \right) = P_i\{T_j = n\} \text{ and } (X_k)_k \text{ visits } i \text{ after time } n$$

$$= P_i\{T_j = n\}$$

as the recurrence of $i$ ensures that the chain will return infinitely often to $i$. The set $\{T_j = n\}$ may be written as $\{X_1 \neq i, \ldots, X_{n-1} \neq i, X_n = i\}$ so that

$$P_i\{T_j = n\} = P_i \left( \{T_j = n\} \cap \bigcup_{k=n+1}^{\infty} \{X_k = i\} \right)$$

$$= P_i \left( \bigcup_{k=n+1}^{\infty} \{X_k = i\} | T_j = n \right) P_i\{T_j = n\}$$

$$= P_i \left( \bigcup_{k=n+1}^{\infty} \{X_k = i\} | X_n = j \right) P_i\{T_j = n\}$$

$$= P_j\{T_i < \infty\} P_i\{T_j = n\}$$

$$= f_{ji} P_i\{T_j = n\}$$
using the Markov property. It follows that \( f_{ji} = 1 \), proving part (a). In particular we must have \( j \to i \), proving part (b).

Part (c) of 1 now follows from Brémaud Theorem 3.1.2 and then part (d) follows from part (a).

The second claim of the theorem follows from the first, part (c).

Note that we now have the following facts:

- If \( i \) is recurrent and \( i \to j \) then \( j \) is recurrent and \( i \leftrightarrow j \).
- If \( i \to j \) and \( j \not\to i \) then \( i \) is transient.

It follows immediately that any recurrent (communication) class of states is closed. Hence we may decompose the state space as

\[
E = T \bigcup \cup_k R_k
\]

where \( T \) is the set of transient states and each \( R_k \) is a closed, recurrent communication class. There may be finitely many or countably many of these recurrent classes, or none at all. Note that the transient set \( T \) may consist of more than one communication class, it may be empty and it could be a closed set of states.

### 2 Fox-Landi algorithm

Let \( B = [b_{i,j}]_{i,j \in E} \) be given by

\[
b_{i,j} = 1_{\{p_{i,j} > 0\}} \quad i, j \in E
\]

This matrix is a representation of the transition graph; \( b_{i,j} = 1 \) if and only if there is an edge from \( i \) to \( j \).

The Fox-Landi algorithm is a method for decomposing the state space of a Markov chain with finite state space. It consists of three steps that are iterated a finite number of times:

1. **[Classification]** Any \( i \) with \( b_{i,i} = 1 \) and \( b_{i,j} = 0 \) for \( j \neq i \) is a closed recurrent set of states (in the first run-through an absorbing state). For any such \( i \) classify any \( j \) with \( b_{j,i} > 0 \) as transient.

2. **[Chaining]** Take an unclassified \( i_0 \) (if any remain). Since \( i_0 \) is not closed, there is an \( i_1 \) such that \( b_{i_0,i_1} = 1 \) and an \( i_2 \) such that \( b_{i_1,i_2} = 1 \) and so on. Proceeding in this manner will give a chain of states. Proceed until either

   (a) A transient state is found; then all the states \( i_0, i_1, \ldots \) must be transient and are classified accordingly. Now repeat this step of the algorithm (take a new unclassified \( i_0 \ldots \)).
(b) $i_s = i_r$ for some $r < s$, then $i_r, i_{r+1}, \ldots, i_{s-1}$ are in the same communication class. Now proceed to the next step of the algorithm.

3. [Merging] Replace the rows and columns in $B$ corresponding to the states $i_r, i_{r+1}, \ldots, i_{s-1}$ with a single row $(b_{i^*, j})$ and column $(b_{j, i^*})$ where

$$b_{i^*, j} = \max_{k = r, r+1, \ldots, s-1} b_{i, j}, \quad b_{j, i^*} = \max_{k = r, r+1, \ldots, s-1} b_{j, i_k}$$

Now go back to the first step with the new $B$-matrix.

Note that the third step of the algorithm changes the $B$-matrix. Thus the entries no longer correspond to single states but to (communicating) sets of states. In the first application of the merging-step we put

$$b_{i^*, j} = \begin{cases} 1 & \text{if } p_{i_k, j} > 0 \text{ for some } k = r, \ldots, s - 1 \\ 0 & \text{otherwise} \end{cases}$$

In each run-through (at least) one new state is classified or (at least) two states are merged. Consequently, the algorithm terminates after a finite number of iterations and then all states are classified and we have the desired decomposition.

For an infinite state space the algorithm does not terminate in finite time. But even so, the steps of the algorithm may be a useful help in classifying the states of the Markov chain.

### 3 Positive and null recurrent communication classes

Let $R$ be a closed recurrent communication class. Then $[p_{i, j}]_{i, j \in R}$ as an irreducible, recurrent transition matrix on $R$. We say that $R$ is a positive recurrent communication class if each state in $R$ is positive recurrent; if not, the it is null recurrent.

By the results in Brémaud’s section 3.2, it has a unique invariant measure (Theorems 3.2.1 and 3.2.2) and $[p_{i, j}]_{i, j \in R}$ is positive recurrent if and only if the invariant measure is finite (Theorem 3.2.3). Hence the expected return time for any state in $R$ is finite if and only if the invariant measure of $[p_{i, j}]_{i, j \in R}$ is finite. We state this as a corollary:

**Corollary 2** A closed recurrent communication class $R$ is positive recurrent if and only if the invariant measure of $[p_{i, j}]_{i, j \in R}$ is finite.

Hence to classify the recurrent communication classes $R_k$ of the decomposition (1), one way to proceed is to find the invariant measure of $[p_{i, j}]_{i, j \in R_k}$. Note in particular, that if $R_k$ is finite, then it must be positive recurrent.

We have now seen that if there is a positive recurrent communication class, then there is a finite invariant measure: If $\pi$ is a finite invariant measure on $R$ then $\pi^*$ given by

$$\pi^*(i) = \begin{cases} \pi(i) & \text{if } i \in R \\ 0 & \text{otherwise} \end{cases}$$

is a finite invariant measure on $E$. The contrary is also true:
**Theorem 3** If $P$ is a transition matrix on $E$ and $\pi$ is a finite invariant measure, then

$$\text{supp } \pi = \{ i \in E : \pi(i) > 0 \}$$

is a union of positive recurrent communication classes.

**Proof** Suppose $\pi(i) > 0$ and $i \to j$ so that $p_{i,j}(n) > 0$ for some $n \in \mathbb{N}$. Then

$$\pi(j) = \sum_{k \in E} \pi(k)p_{k,j}(n) \geq \pi(i)p_{i,j}(n) > 0$$

showing that $j \in \text{supp } \pi$. Hence $\text{supp } \pi$ is a union of closed communication classes. Now let $C$ be a communication class contained in $\text{supp } \pi$. Then for any $i \in C$ we have

$$\pi(i) = \sum_{j \in E} \pi(j)p_{j,i} = \sum_{j \in C} \pi(j)p_{j,i}$$

We see that $(\pi(i))_{i \in C}$ is a finite invariant measure on $[p_{i,j}]_{i,j \in C}$ and conclude that $C$ is a positive recurrent communication class. The result follows. $\square$

We conclude that if $P$ has a finite invariant measure, then there must be at least one positive recurrent communication class. It is not, however, very clear how to find a finite invariant measure without knowledge of the communication classes.

Finally we make the following observation. If $P$ has (at least) two positive recurrent communication classes, then $P$ has an infinite number of invariant distributions: The two positive recurrent classes gives rise to one each, $\pi_1^*$ and $\pi_2^*$ say, and then

$$(\lambda \pi_1^* + (1 - \lambda)\pi_2^*)^T P = \lambda \pi_1^T P + (1 - \lambda)\pi_2^T P = (\lambda \pi_1^* + (1 - \lambda)\pi_2^*)^T$$

showing that $\lambda \pi_1^* + (1 - \lambda)\pi_2^*$ is an invariant distribution of $P$ for any $\lambda \in [0; 1]$. 
