The Determinantal Formula in Schubert calculus

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ABSTRACT. We present a proof of the determinantal formula in Schubert calculus for general schemes. The result may be viewed as an appendix to the article [LT2].

6.0 Setup. Work with the notation introduced in [LT2]. In particular, $\mathcal{E}$ is a locally free module of rank $n \geq 1$ over the base scheme X, and $d \leq n$ is a fixed positive integer.

Let

$$F := \operatorname{Flag}^d(\mathcal{E}) \quad \text{and} \quad \pi_F \colon F \to X$$

be the d'th *partial flag scheme* and its structure map, parametrizing flags of corank d in $\mathcal{E}$. On F there is a universal flag of quotients of $\mathcal{E}_F$,

$$\mathcal{E}_F \twoheadrightarrow \mathcal{Q}_d \twoheadrightarrow \mathcal{Q}_{d-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_1, \tag{6.0.1}$$

with $\mathcal{Q}_i$ locally free of rank i. So, the successive kernels $\mathcal{L}_i := \operatorname{Ker}(\mathcal{Q}_i \to \mathcal{Q}_{i-1})$ for $i = 1, \dots, d$ (and $\mathcal{Q}_0 = 0$) are locally free of rank 1.

Let

$$G := \operatorname{Grass}^d(\mathcal{E}) \quad \text{and} \quad \pi_G \colon G \to X$$

be the d'th Grassmannian and its structure map, parametrizing rank-d quotients of $\mathcal{E}$. On G there is a universal rank-d quotient $\mathcal{Q}$ of $\mathcal{E}_G$. It defines a short exact sequence,

$$0 \to \mathcal{R} \to \mathcal{E}_G \to \mathcal{Q} \to 0 \tag{6.0.2}$$

with $\mathcal{R}$ locally free of corank d. On the flag scheme F, the quotient $\mathcal{E}_F \twoheadrightarrow \mathcal{Q}_d$ defines an X-morphism $g \colon F \to G$ under which the quotient $\mathcal{Q}_d$ of $\mathcal{E}_F$ is the pull-back of the quotient $\mathcal{Q}$ of $\mathcal{E}_G$. In fact, it follows from the functorial properties of flags that the map $g \colon F \to G$ identifies F and the complete flag scheme of $\mathcal{Q}$ over G:

$$\operatorname{Flag}^d_X(\mathcal{E}) = \operatorname{Flag}^d_G(\mathcal{Q}). \tag{6.0.3}$$

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Under this identification we have the equality $\mathcal{Q}_F = \mathcal{Q}_d$ and the universal (complete) flag of quotients of $\mathcal{Q}_F$ is tail of the flag (6.0.1).

In the case $d = 1$, the two schemes F and G reduce to projective space: Let

$$P := I\!P(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}\mathcal{E}) \quad \text{and} \quad \pi : P \to X$$

be the projective bundle and its structure map. On P there is an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_P \to \mathcal{O}_P(1) \to 0. \tag{6.0.4}$$

Clearly, the flag schemes may be defined inductively as a chain of schemes and maps,

$$F_d \to \cdots \to F_2 \to F_1 \to X,$$

with $F_1 := I\!P(\mathcal{E})$, and the following inductive description: Assume that $F = \operatorname{Flag}^d(\mathcal{E})$ is given with the universal flag (6.0.1). Let $\mathcal{R}_d \subseteq \mathcal{E}_F$ be the corank-d submodule corresponding to the quotient $\mathcal{Q}_d$. Set $F' := I\!P(\mathcal{R}_d)$, let $\mathcal{L}' := \mathcal{O}_{F'}(1)$ be the universal rank-1 quotient of $\mathcal{R}_{d,F'}$, let $\mathcal{R}_{d+1} \subseteq \mathcal{R}_{d,F'}$ be the kernel of the canonical epimorphism $\mathcal{R}_{d,F'} \to \mathcal{L}'$, and let $\mathcal{Q}_{d+1} := \mathcal{E}_{F'}/\mathcal{R}_{d+1}$. Then $\mathcal{Q}_{d+1}$ is a rank-$(d+1)$ quotient of $\mathcal{E}_{F'}$, and there is a canonical quotientmorphism $\mathcal{Q}_{d+1} \twoheadrightarrow \mathcal{Q}_{d,F'}$ with kernel $\mathcal{L}'$. Clearly, $F' = \operatorname{Flag}^{d+1}(\mathcal{E})$, with the corank-$(d+1)$ flag obtained from $\mathcal{Q}_{d+1}$ and the pull back to F' of the flag (6.0.1) as the universal corank-$(d+1)$ flag.

6.1 Setup continued. Consider an increasing sequence of d linear subschemes of $I\!P(\mathcal{E})$:

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathrm{A}_d \subseteq I\!P(\mathcal{E}),$$

say $A_i = I\!P(\mathcal{A}_i)$ given by a decreasing sequence of d locally free quotients of $\mathcal{E}$:

$$\mathcal{E} \twoheadrightarrow \mathcal{A}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{A}_2 \twoheadrightarrow \mathcal{A}_1, \tag{6.1.1}$$

say $A_i := I\!P(\mathcal{E}/\mathcal{V}_i)$ with a decreasing sequence $\mathcal{E} \supseteq \mathcal{V}_1 \supseteq \cdots \supseteq \mathcal{V}_d$ of d locally split submodules of $\mathcal{E}$. The associated *Schubert subscheme* is the closed subscheme of the Grassmannian,

$$\Omega(A_1, \ldots, A_d) = \Omega(\mathcal{A}_1, \ldots, \mathcal{A}_d) \subseteq \operatorname{Grass}_X^d(\mathcal{E}),$$

defined informally, see [LT2], by the following conditions on a $(d-1)$-plane Q in $I\!P(\mathcal{E})$:

$$\dim Q \cap A_i \geq i - 1 \quad \text{for } i = 1, \ldots, d.$$

Over the flag scheme $F = \operatorname{Flag}^d(\mathcal{E})$ we have two decreasing sequences of quotients of $\mathcal{E}_F$:

$$\begin{array}{ccccccccc}
\mathcal{E}_F & \longrightarrow & \mathcal{A}_{d,F} & \longrightarrow & \mathcal{A}_{d-1,F} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}_{1,F} \\
\| & & & & & & & & \\
\mathcal{E}_F & \longrightarrow & \mathcal{Q}_d & \longrightarrow & \mathcal{Q}_{d-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}_1.
\end{array}$$

Denote by $Z = \operatorname{Flag}(\mathcal{A}_1, \ldots, \mathcal{A}_d) = \operatorname{Flag}(A_1, \ldots, A_d)$ the closed subscheme $Z \subseteq F$ where the top sequence dominates the bottom sequence, that is, the quotient $\mathcal{E}_F \twoheadrightarrow \mathcal{A}_{i,F}$ dominates the quotient $\mathcal{E}_F \twoheadrightarrow \mathcal{Q}_i$ for $i = 1, \ldots, d$:

$$\begin{array}{ccccccccc}
\mathcal{E}_F & \longrightarrow & \mathcal{A}_{d,F} & \longrightarrow & \mathcal{A}_{d-1,F} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}_{1,F} \\
\| & & \downarrow & & \downarrow & & & & \downarrow \\
\mathcal{E}_F & \longrightarrow & \mathcal{Q}_d & \longrightarrow & \mathcal{Q}_{d-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}_1.
\end{array}$$

So, if $\mathcal{A}_i = \mathcal{E}/\mathcal{V}_i$, then Z is the intersection of the zero schemes of the d compositions $\mathcal{V}_{i,F} \to \mathcal{E}_F \to \mathcal{Q}_i$.

In precise geometric terms, the flag scheme F represents the functor whose value at an X-scheme T is the set of all flags,

$$Q_1 \subset Q_2 \subset \cdots \subset Q_d \subseteq \mathbb{P}(\mathcal{E}_T), \tag{6.1.2}$$

where Q_i is a linear subspace of relative dimension $i-1$ in $\mathbb{P}(\mathcal{E}_T) = \mathbb{P}(\mathcal{E}) \times_X T$, and Z respresents the subfunctor whose value is the subset of flags (6.1.2) where

$$Q_i \subseteq A_i \times_X T \text{ for } i = 1, \dots, d.$$

Clearly, the canonical morphism $g\colon \operatorname{Flag}^d(\mathcal{E}) \to \operatorname{Grass}^d(\mathcal{E})$ induces a morphism of closed subschemes,

$$f\colon \operatorname{Flag}(A_1, \dots, A_d) \to \Omega(A_1, \dots, A_d) \tag{6.1.3}$$

6.2 Lemma. *The morphism* (6.1.3) *induces an isomorphism between the open dense subset of* $\operatorname{Flag}(A_1, \dots, A_d)$ *where* $A_i \cap Q_d = Q_i$ *for* $i = 1, \dots, d$*, and the open dense subset of* $\Omega(A_1, \dots, A_d)$ *where* $A_i \cap Q$ *is a linear subscheme of dimension* $i-1$ *for* $i = 1, \dots, d$*.*

Proof. The assertion is obvious.

6.3 Iterative construction. Generalizing the inductive definition of the flag scheme, the subscheme $Z = \operatorname{Flag}(A_1, \dots, A_d)$ of the d'th flag scheme $F = \operatorname{Flag}^d(\mathcal{E})$ may be constructed inductively as follows:

First, if $d = 0$, then $Z = F = X$. To describe the inductive step, passing from d to $d+1$, assume that Z and F are constructed for d linear subspaces $A_1, \dots, A_d$ of $\mathbb{P}(\mathcal{E})$, and consider an additional linear subspace $A_d \subseteq A_{d+1} \subseteq \mathbb{P}(\mathcal{E})$, say $A_{d+1} = \mathbb{P}(\mathcal{E}/\mathcal{V}_{d+1})$ with $\mathcal{V}_{d+1} \subseteq \mathcal{V}_d$. Consider the universal flag on F:

$$\mathcal{E}_F \twoheadrightarrow \mathcal{Q}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_1, \tag{6.3.1}$$

where $\mathcal{Q}_i$ is locally free of rank i. Assume that the quotient $\mathcal{Q}_d$ corresponds to the submodule $\mathcal{R}_d$, say $\mathcal{Q}_d = \mathcal{E}_F/\mathcal{R}_d$. Consider the projective scheme $h\colon F' := \mathbb{P}(\mathcal{R}_d) \to F$, and on F' the tautological exact sequence,

$$0 \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R}_{d,F'} \longrightarrow \mathcal{O}_{F'}(1) \longrightarrow 0. \tag{6.3.2}$$

Then the quotient $\mathcal{Q}' := \mathcal{E}_{F'}/\mathcal{R}'$ dominates the quotient $\mathcal{Q}_{d,F'} = \mathcal{E}_{F'}/\mathcal{R}_{d,F'}$, and $\mathcal{L}' := \mathcal{O}_{F'}(1)$ is the kernel of the surjection $\mathcal{Q}' \twoheadrightarrow \mathcal{Q}_{d,F'}$. Clearly, $F' = \operatorname{Flag}^{d+1}(\mathcal{E})$ with

$$\mathcal{E}_{F'} \twoheadrightarrow \mathcal{Q}' \twoheadrightarrow \mathcal{Q}_{d,F'} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{1,F'}$$

as the universal flag.

Over F' we have two $(d+1)$-flags of quotients,

$$\begin{array}{ccccccccc}
\mathcal{E}_{F'} & \longrightarrow & \mathcal{A}_{d+1,F'} & \longrightarrow & \mathcal{A}_{d,F'} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}_{1,F'} \\
\| & & & & & & & & \\
\mathcal{E}_{F'} & \longrightarrow & \mathcal{Q}' & \longrightarrow & \mathcal{Q}_{d,F'} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}_{1,F'}.
\end{array}$$

Over Z, and hence over the preimage $h^{-1}Z$, we have that $\mathcal{E} \twoheadrightarrow \mathcal{A}_i$ dominates $\mathcal{E} \twoheadrightarrow \mathcal{Q}_i$ for $i = 1, \dots, d$:

$$\begin{array}{ccccccccc}
\mathcal{E}_{h^{-1}Z} & \longrightarrow & \mathcal{A}_{d+1,h^{-1}Z} & \longrightarrow & \mathcal{A}_{d,h^{-1}Z} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}_{1,h^{-1}Z} \\
\| & & \downarrow & & \downarrow & & & & \downarrow \\
\mathcal{E}_{h^{-1}Z} & \longrightarrow & \mathcal{Q}' & \longrightarrow & \mathcal{Q}_{d,h^{-1}Z} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}_{1,h^{-1}Z}.
\end{array}$$

Moreover, for any scheme T over $h^{-1}Z$ we have that $\mathcal{A}_{d+1,T}$ dominates $\mathcal{Q}'_T$ if and only if the composition $\mathcal{V}_{d+1,T} \to \mathcal{E}_T \to \mathcal{Q}'_T$ vanishes. Since $\mathcal{V}_{d+1,T} \subseteq \mathcal{V}_{d,T}$, the composition maps into the kernel $\mathcal{L}'_T$ of $\mathcal{Q}'_T \twoheadrightarrow \mathcal{Q}_{d,T}$. So the composition defines over the preimage $h^{-1}Z$ of Z in F a natural morphism,

$$\mathcal{V}_{d+1,h^{-1}Z} \to \mathcal{L}'_{h^{-1}Z},$$

or a section,

$$\mathcal{O}_{h^{-1}Z} \to \mathcal{V}^*_{d+1,h^{-1}Z} \otimes L_{d+1,h^{-1}}; \tag{6.3.3}$$

clearly, $Z' = \mathrm{Flag}(A_1, \dots, A_{d+1})$ is the scheme of zeros of the latter section.

The section (6.3.3) is regular, since the structure map $h^{-1}Z \to Z$ identifies $h^{-1}Z$ with $\mathbb{P}(\mathcal{R}_{d,Z})$ and $\mathcal{R}_{d,h^{-1}Z} \twoheadrightarrow \mathcal{L}'_{h^{-1}Z}$ with the universal rank-1 quotient(?). As a consequence, the class of Z' in $\mathrm{A}(h^{-1}Z)$ is obtained by applying the top Chern class $c_{\mathrm{top}}(\mathcal{V}^*_{d+1} \otimes \mathcal{L}')$ to $[h^{-1}Z]$. The top Chern class is, as a consequence of the Splitting Principle, equal to the value of the Chern polynomial $C_{\mathcal{V}_{d+1}}$ at the first Chern class $x = c_1\mathcal{L}_{d+1}$. Hence, under the inclusion $j \colon Z' \to h^{-1}Z$,

$$j_*[Z'] = C_{\mathcal{V}_{d+1}}(x)[h^{-1}Z]. \tag{6.3.4}$$

6.4 Proposition. *Consider the d'th flag scheme $F = \mathrm{Flag}^d(\mathcal{E})$ with its universal flag* (6.0.1), *let $\mathcal{L}_i := \mathrm{Ker}(\mathcal{Q}_i \to \mathcal{Q}_{i-1})$ and $\xi_i := c_1\mathcal{L}_i$ for $i = 1, \dots, d$. Let $Z := \mathrm{Flag}(A_1, \dots, A_d)$ and, more generally, let Z_i be the preimage in F of $\mathrm{Flag}(A_1, \dots, A_i)$ under the natural map $F = \mathrm{Flag}^d(\mathcal{E}) \to \mathrm{Flag}^i(\mathcal{E})$. Then, for $i = 1, \dots, d$, we have the equation in $\mathrm{A}(F)$:*

$$[Z_i] = \prod_{j=1}^{i} C_{\mathcal{V}_j}(\xi_j) \cap [F].$$

Note that, by the Whitney Sum formula, $C_{\mathcal{V}_j}(T) = C_{\mathcal{E}}(T)/C_{\mathcal{A}_j}(T)$.

6.5 Conclusion. Recall that for the Chow group of the d'th flag scheme $F = \mathrm{Flag}^d(\mathcal{E})$ there is a canonical identification,

$$\mathrm{A}(F) = A^*(F) \otimes_{A^*(X)} A(X) = \mathrm{Split}^d_{A^*(X)}(C_{\mathcal{F}}) \otimes_{A^*(X)} A(X), \tag{6.5.1}$$

where $\mathrm{Split}^d_A(p)$ denotes the d'the *splitting algebra* for the polynomial p over A in d linear factors $p(T) = (T - \xi_1) \cdots (T - \xi_d)\tilde{p}(T)$, where $\xi_1, \dots, \xi_d$ are the *universal roots* in $\mathrm{Split}^d_A(p)$. By the identification (6.5.1), the Chern classes $x_i \in \mathrm{A}^*(X)$ correspond to the universal roots ξ_i.

Recall in addition that the A-subalgebra of $\mathrm{Split}^d_A(p)$ generated by the elementary symmetric polynomials in the $\xi_1, \dots, \xi_d$ is the *factorization algebra* $\mathrm{Fact}^d_A(p)$, universal with respect to factorizations $p = q\tilde{q}$ into factors of degree d and $n-d$. The Chern ring $A^*(G)$ of the d'th Grassmannian $G = \mathrm{Grass}^d(\mathcal{E})$ is in fact the d'th factorization algebra of $C_{\mathcal{E}}$, and the equation $C_{\mathcal{E}} = C_{\mathcal{Q}} C_{\mathcal{R}}$ in $A^*(G)[T]$ resulting from the exact sequence (6.0.2) is the universal factorization. Moreover the pull back of the morphism $g\colon F \to G$ and the inclusion of the factorization algebra into the splitting algebra induce a commutative diagram,

$$\begin{array}{ccc} A(F) & \longrightarrow & \mathrm{Split}^n_{A^*(X)}(C_{\mathcal{E}}) \otimes_{A^*(X)} A(X) \\ {\scriptstyle g^*}\uparrow & & \uparrow \\ A(G) & \longrightarrow & \mathrm{Fact}^n_{A^*(X)}(C_{\mathcal{E}}) \otimes_{A^*(X)} A(X) \end{array} \tag{6.5.2}$$

6.6 Theorem. *Consider over the base scheme X a locally free module $\mathcal{E}$ and a flag of d locally free quotients,*

$$\mathcal{E} \twoheadrightarrow \mathcal{A}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{A}_2 \twoheadrightarrow \mathcal{A}_1, \tag{6.6.1}$$

with $\mathrm{rk}\,\mathcal{A}_i \geq i$, *say* $\mathcal{A}_i := \mathbb{P}(\mathcal{E}/\mathcal{V}_i)$ *with a decreasing sequence,*

$$\mathcal{V}_d \subseteq \cdots \subseteq \mathcal{V}_1 \subseteq \mathcal{E}, \tag{6.6.2}$$

of d locally split submodules $\mathcal{V}_i$. Form the Grassmanian $G := \mathrm{Grass}^d(\mathcal{E})$ parametrizing rank-d locally free quotients of $\mathcal{E}$ with the universal quotient $\mathcal{E}_G \twoheadrightarrow \mathcal{Q}$, fitting into an exact sequence,

$$0 \to \mathcal{R} \to \mathcal{E}_G \to \mathcal{Q} \to 0. \tag{6.6.3}$$

Let $\Omega = \Omega(\mathcal{A}_1, \ldots, \mathcal{A}_d)$ be the corresponding Schubert subscheme of G. Then the class of Ω in the Chow group $\mathrm{A}(G)$ is given by the determinantal formula,

$$[\Omega] = \mathrm{Res}\Big(\frac{C_{\mathcal{V}_1}}{C_{\mathcal{Q}}}, \ldots, \frac{C_{\mathcal{V}_d}}{C_{\mathcal{Q}}}\Big)[G] = \mathrm{Res}\Big(\frac{C_{\mathcal{R}}}{C_{\mathcal{A}_1}}, \ldots, \frac{C_{\mathcal{R}}}{C_{\mathcal{A}_d}}\Big)[G]. \tag{6.6.4}$$

Proof. The Laurent series in the two resultants are the same since, by the Whitney Formula, $C_{\mathcal{E}}(T) = C_{\mathcal{V}_i}(T) C_{\mathcal{A}_i}(T) = C_{\mathcal{R}}(T) C_{\mathcal{Q}}(T)$ in $\mathrm{A}^*(G)[T]$. Note also that the Segre series is the inverse of the Chern polynomial so that $C_{\mathcal{V}_i}/C_{\mathcal{Q}} = C_{\mathcal{V}_i} S_{\mathcal{Q}} = C_{\mathcal{R}} S_{\mathcal{A}_i}$.

Let F be the d'th flag scheme, $F = \mathrm{Flag}^d(\mathcal{E})$; under the induced morphism $g\colon F \to G$, the flag scheme F is identified with the complete flag scheme $\mathrm{Flag}^d(\mathcal{Q})$ of $\mathcal{Q}$. Let $Z \subseteq F$ be the closed subscheme $Z = \mathrm{Flag}(\mathcal{A}_1, \ldots, \mathcal{A}_d)$. Consider the following diagram,

$$\begin{array}{ccccc} \mathrm{A}(Z) & \xrightarrow{i_*} & \mathrm{A}(F) & = & \mathrm{Split}^d(C_{\mathcal{Q}}) \otimes_{A^*(G)} A(G) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle \partial\otimes 1} \\ \mathrm{A}(\Omega) & \xrightarrow{j_*} & \mathrm{A}(G) & = & \mathrm{A}^*(G) \otimes_{A^*(G)} A(G). \end{array}$$

The left square of Chow groups corresponds to the natural commutative diagram of schemes. Hence the first square is commutative. The right vertical map ∂ in the second square is the map $\partial = \partial^{1,\ldots,d}$ considered in [LT1, Section 6]. It is a map from the splitting algebra $\mathrm{Split}^d(C_{\mathcal{Q}})$ of the Chern polynomial $C_{\mathcal{Q}} \in \mathrm{A}^*(G)$ to the base ring $A^*(G)$; the splitting algebra $\mathrm{Split}^d(C_{\mathcal{Q}})$ is the full splitting algebra since $C_{\mathcal{Q}}$ is of degree d.

The inductive constrution of full flag scheme $F = \mathrm{Flag}^d(\mathcal{Q})$ with the map $g\colon F \to G$ is recalled in Section 6.1. The inductive construction of the splitting algebra $\mathrm{Split}^d(C_{\mathcal{Q}})$ over the ring $\mathrm{A}^*(G)$ with its map $\mathrm{Split}^d(C_{\mathcal{Q}}) \to \mathrm{A}^*(G)$ is given in [LT1, Section 6]. It follows from the two constructions that the second diagram is commutative under the top horizontal identification given in [LT2, Formula (4.9.1)] with $X := G$ and $\mathcal{E} := \mathcal{Q}$.

To prove the formula in (6.6.4), consider the fundamental class $[Z] \in \mathrm{A}(Z)$ and its image in $A(G)$. First, $f_*[Z] = [\Omega]$ since the map $f\colon Z \to \Omega$ is birational by Lemma 6.2. Hence the image of $[Z]$ in $\mathrm{A}(G)$ is equal to the class $[\Omega]$ in $A(G)$.

Next, the inclusion $i\colon Z \to F$ is, by the iterative construction 6.3, a composition of d regular embeddings defined by zero schemes of regular sections correspoding to the one considered in (6.3.3). Hence,

$$j_*[Z] = C_{\mathcal{V}_d}(\xi_d) \cdots C_{\mathcal{V}_1}(\xi_1)[F]. \tag{6.6.5}$$

Under the identifications in the diagram, the product of Chern classes in (6.6.5) may be viewed as a product in the splitting algebra $\mathrm{Split}^d(C_{\mathcal{Q}})$. By the fundamental result in [LT1, Proposition 6.3], the image of the product under the map ∂ is the resultant in (6.6.4),

$$\partial\Big(C_{\mathcal{V}_d}(\xi_d) \cdots C_{\mathcal{V}_1}(\xi_1)\Big) = \mathrm{Res}\Big(\frac{C_{\mathcal{V}_1}}{C_{\mathcal{Q}}}, \dots, \frac{C_{\mathcal{V}_d}}{C_{\mathcal{Q}}}\Big).$$

Therefore $g_*i_*[Z]$ is equal to the resultant applied to $[G]$. Thus formula (6.6.4) has been proved.

References

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