The game-theoretical approach to non-extensive entropy measures

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Goal: Operational definitions of entropy and related quantities covering the classical as well as non-extensive settings, thereby understanding which entropy measures are relevant for physics

classically

Given P and Q, $\sum p_i \kappa_Q(i)$, *average code length*, plays a central role. Here, κ_Q is the *code adapted to* Q: $\kappa_Q(i) = -\ln q(i)$. Denote this quantity $\Phi_{clas}(P\|Q)$. It has clear operational significance!
QUESTION: Other possibilities? Other sensible Φ's? We suggest a game theoretical approach.

the game theoretical setting:

alphabet: $\mathbb{A}$
strategy set for *Player I (Nature)*: $\mathcal{S}_I$ - a *preparation*
strategy set for *Player II (the physicist)*: $\mathcal{S}_{II}$;
complexity: $\Phi(P\|Q)$ with $P \in \mathcal{S}_I$, $Q \in \mathcal{S}_{II}$.

NOTE: “Complexity” just a word, in general no clear interpretation. We approach the problem of meaning by suggesting a natural two-person zero-sum game between the two players – a game which can be played for very general Φ's – and later by searching for special Φ's.

... continued

entropy: $S_\Phi(P) = \inf_Q \Phi(P\|Q)$, efficient viewing!
redundancy: $D_\Phi(P\|Q) = \Phi(P\|Q) - S_\Phi(P)$.

entropy = minimal complexity; redundancy = actual − minimal complexity

So: $\Phi(P\|Q) = S_\Phi(P) + D_\Phi(P\|Q)$, *linking identity*

AXIOMS: For $P \in \mathcal{S}_I$, minimum of $\Phi(P\|Q)$ w.r.t. Q is finite and assumed for $Q = P$ only.

Classically:

$$\Phi(P\|Q) = \sum_{i\in\mathbb{A}} p_i \ln\left(\frac{1}{q_i}\right)$$
$$\mathcal{S}(P) = -\sum_{i\in\mathbb{A}} p_i \ln(p_i)$$
$$D(P\|Q) = \sum_{i\in\mathbb{A}} p_i \ln\left(\frac{p_i}{q_i}\right)$$

the game

Pl.I (Nature) strives for high complexity,
Pl.II (the physicist) strives for low complexity.

If Pl.I plays $P \in \mathcal{S}_I$, best move by Pl.II is $Q = P$ leading to the complexity $\Phi(P\|P) = S_\Phi(P)$.

Therefore, $P^* \in \mathcal{S}_I$ is a Pl.I-*optimal strategy* if $\mathsf{S}_\Phi(P^*) = \mathsf{S}_\Phi^{\mathsf{max}}$, the *MaxEnt value* defined by $\mathsf{S}_\Phi^{\mathsf{max}} = \sup_{P\in\mathcal{S}_I} \mathsf{S}_\Phi(P)$

Similarly, $Q^* \in \mathcal{S}_{II}$ is a Pl.II-*optimal strategy* if $\mathsf{R}_\Phi(Q^*) = \mathsf{R}_\Phi^{\mathsf{min}}$, the *minimum risk value* defined by

$$\mathsf{R}_\Phi^{\mathsf{min}} = \inf_{Q\in\mathcal{S}_{II}} \mathsf{R}_\Phi(Q) \text{ with}$$

$$\mathsf{R}_\Phi(Q) = \sup_{P\in\mathcal{S}_I} \Phi(P\|Q)\,.$$

We have $\mathsf{S}_\Phi^{\mathsf{max}} \le \mathsf{R}_\Phi^{\mathsf{min}}$. *Equilibrium* if equality holds.

equilibrium achievable means maximum entropy = minimal risk

NOTE: *MaxEnt-principle* is *d e r i v e d* from the game!

ideal behaviour of the game

We consider it *ideal* if, for every convex *preparation* $\mathcal{S}_I$, the game is in equilibrium and has a unique *bi-optimal* strategy (optimal for both players).

ideal situation holds in the Tsallis case with $0 < q \leq 1$

... and not otherwise (presumably!).

Tsallis case given by

$$\Phi_q(P\|Q) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i^q\Big(1 - q_i^{1-q}\Big)\,,$$

$$\mathsf{S}_q(P) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i\Big(p_i^{q-1} - 1\Big)\,,$$

$$\mathsf{D}_q(P\|Q) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i\bigg(1 - \Big(\frac{p_i}{q_i}\Big)^{q-1}\bigg)\,.$$

NOTE: This Φ factorises! ...more later...

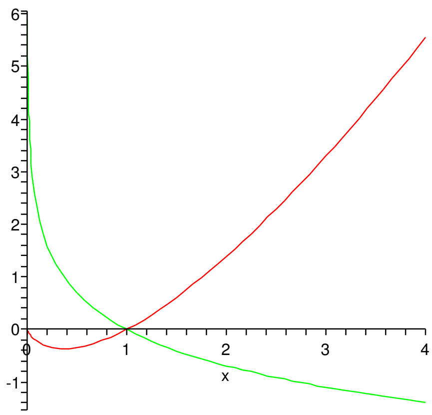
generation of Φ, S, D

Let f be a *(Csiszár)-generator*: analytic, convex function with $f(0) = f(1) = 0$ and $f'(1) = 1$. Define:

$$\Phi_f(P\|Q) = \sum_{i \in \mathbb{A}} \left(q_i f(\frac{p_i}{q_i}) - f(p_i) \right),$$

$$\mathsf{S}_f(P) = -\sum_{i \in \mathbb{A}} f(p_i)$$

$$\mathsf{D}_f(P\|Q) = \sum_{i \in \mathbb{A}} q_i f(\frac{p_i}{q_i}).$$



Graph shows typical f (red) and its *dual* $\widetilde{f}$ (green) given by $\widetilde{f}(x) = x f\left(\frac{1}{x}\right)$, $0 \le x \le \infty$.

Classically: $f(x) = x\ln(x)$, $\widetilde{f}(x) = \ln(\frac{1}{x})$.

deformed logarithms

$$\ln_{\alpha,\beta} x = \frac{x^{\beta} - x^{\alpha}}{\beta - \alpha}$$

$$f_{\alpha,\beta}(x) = x \ln_{\alpha,\beta}(x) \leftrightarrow \widetilde{f}_{\alpha,\beta}(x) = \ln_{\alpha,\beta} \frac{1}{x}$$

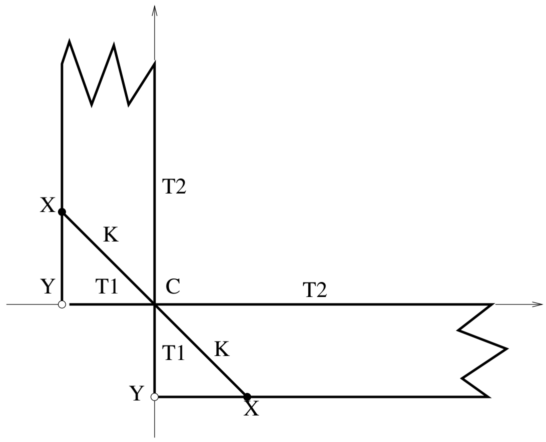


Figure shows possible choices of (α, β)

C: classical,

T: Tsallis (T1: $(0, -1 + q)$ with $0 < q \leq 1$,

T2: same, but with $q > 1$),

K: Kaniadakis ($(-\kappa, \kappa)$ with $0 \leq \kappa \leq 1$).

general expressions

$$\Phi = \sum_{i\in\mathbb{A}} \Big(q_i f(\frac{p_i}{q_i}) - f(p_i)\Big) = \sum_{i\in\mathbb{A}} p_i\Big(\widetilde{f}\Big(\frac{q_i}{p_i}\Big) - \widetilde{f}\Big(\frac{1}{p_i}\Big)\Big)$$

$$S = \sum_{i\in\mathbb{A}} \Big(- f(p_i)\Big) = \sum_{i\in\mathbb{A}} \Big(- p_i\widetilde{f}\Big(\frac{1}{p_i}\Big)\Big)$$

$$\mathsf{D} = \sum_{i\in\mathbb{A}} q_i f(\frac{p_i}{q_i}) = \sum_{i\in\mathbb{A}} p_i\widetilde{f}\Big(\frac{q_i}{p_i}\Big)$$

Tsallis expressions

$$\Phi_q(P\|Q) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i^q\Big(1 - q_i^{1-q}\Big)$$

$$\mathsf{S}_q(P) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i\Big(p_i^{q-1} - 1\Big)$$

$$\mathsf{D}_q(P\|Q) = \frac{1}{1-q}\sum_{i\in\mathbb{A}} p_i\Big(1 - \Big(\frac{p_i}{q_i}\Big)^{q-1}\Big)$$

factorisation, escorting

PROBLEM: For which f does $\Phi = \Phi_f$ factorise, hence open up for "escorting"?

In detail: To find ξ and ζ s.t. $\Phi_f(P\|Q) = \sum \xi(p_i)\zeta(q_i)$ holds generally.

ξ: *escort function* or *mean-value generator*

ζ: *surprise factor* or *self-information*

Assume: $\xi(1) = 1$, $\zeta(1) = 0$, both functions analytic in $[0, \infty[$.

Looking at D you find $\xi(x)\Big(\zeta(y) - \zeta(x)\Big) = x\widetilde{f}(\frac{y}{x})$. Putting first $x = 1$, then $y = 1$ you find expressions for ξ and ζ, especially, $\zeta = \widetilde{f}$. Expression for D gives

$$\widetilde{f}(\frac{1}{x})\Big(\widetilde{f}(x) - \widetilde{f}(y)\Big) = \widetilde{f}(x)\widetilde{f}(\frac{y}{x}) \,.$$

This then leads to

$$\frac{\widetilde{f}(x) + \widetilde{f}(y) - \widetilde{f}(xy)}{\widetilde{f}(x)\widetilde{f}(y)} = \frac{1}{\widetilde{f}(x)} + \frac{1}{\widetilde{f}(\frac{1}{x})} \,.$$

rounding up, conclusions, outlook

By symmetry (+analyticity!), expression is constant, say $= \alpha$. $\alpha = 0$ gives the classical case, the other cases give the (other) Tsallis quantities.

(i) Escorting (or factorisation) is only possible for Tsallis quantities.
(ii) The Tsallis family can be derived mathematically in a natural way, searching for sensible complexity measures.
(iii) Factorisation appears important as it separates the two sides, that of Nature (the "system" or that part of the world you are looking at) and the physicist, the person looking at the world.

... but, but, but, several open problems: Operational interpretation not yet in place, and if you accept the game theoretical approach, the notion of equilibrium appears too strict and should be loosened allowing for asymmetry of the players, Nature and the Physicist (for suggestion see FT: Physica A, 340, 11-31 (2004)).