On truth, belief and knowledge

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***Abstract*— *Truth*, *belief* and *knowledge* are important aspects of the overall concept of *information*. Speculating over the possible connections between these entities, new concepts come up naturally and also, specific relationships among them are derived. In particular, a relatively narrow class of measures of *entropy* and *divergence* are isolated which have special interpretations related to the *world* of which we are a part. The classical *Shannon world* and the world of a *black hole* may be conceived as extreme examples.**

Emphasis is on the underlying philosophy and the resulting cognitive paradigm which allows for an interaction between truth, belief and knowledge. The specific concepts which emerge are basically known and, apart from the central concepts related to Shannon theory, they were introduced by Havrda and Charvát, by Daróczy, by Lindhard and Nielsen and by Tsallis.

I. The thoughtful observer

Let us have an *observer* in mind, say a physicist who is planning *experiments* from the *world* of which he is a part. The phenomena he wants to study are referred to as *situations*, following jargon borrowed from philosophy.

A particular situation is characterized by a set $\mathbb{A}$, the *alphabet* of *basic events*. These events are identified by an *index*, here typically denoted by i. The semiotic assignment of indices may well be quite important in individual cases. It should facilitate technical handling and catalyze semantic awareness. However, we shall not study this process further, but simply assume that the identification as well as the naming of basic events has been accomplished. Considering the inherent restrictions in mans ability, only situations with associated discrete finite alphabets are considered (later mathematical extensions should, however, be sought).

Following Shannon we will not express semantic differences in our general planning of experiments. On the contrary, we will seek those aspects which we find are common across semantic differences.

Our approach will be probabilistic. Though not the only possibility, this will enable a quantitative analysis.

The focus on what we seek, *truth*, makes us assign a *truth instance*, x, to each situation from the world, and our restriction to probabilistic modelling entices us to take probability distributions as truth instances. More specifically, a truth instance, x, pertaining to a situation with alphabet $\mathbb{A}$, is a probability distribution over $\mathbb{A}$, characterized by the associated point probabilities: $x = (x_i)_{i\in\mathbb{A}}$. Regarding belief, we assume that our beliefs are expressed by *belief instances*, taken to be of the same nature as the truth instances, probability distributions. A typical belief instance is $y = (y_i)_{i\in\mathbb{A}}$.

The last basic concept we shall worry about is related to *observations* of *experiments* from a situation of interest to the observer. The simplest we can think of is a small collection of *data*, say reflecting a moderate *sample*. However, such data may be a bit ad hoc (e.g. "3 heads, 1 tail") and hardly allow any firm insight to be formed. This points to problems of a statistical nature, problems we do not want to enter into. Instead, let us appeal to a standard frequential interpretation of probabilities and assume that we are in the ideal regime where the law of large numbers has taken over. Therefore, what we have in mind is *insight gained by extensive experience*, and this we refer to as *knowledge*. With the interpretations given, we also find it natural to consider a *knowledge instance*, z. Though this is again taken to be a set of numbers corresponding to the various basic events: $z = (z_i)_{i\in\mathbb{A}}$, we do not assume that these numbers represent probabilities, though often, this will be the case. The interpretation of z_i is the *weight* or *force* with which the basic event indexed by i will be presented to the observer.

The above reference to frequential thinking is not necessary and only serves as motivation. The modelling with belief instances of the same nature as truth instances points more in the direction of a Bayesian conception. We see no conflict regarding these considerations. For one thing, the observer really operates in two different "modes", one contemplative, "before observation, planning", the other "active, actually observing".

The beliefs held by the observer could depend on basic knowledge or on insight reflecting previous experience and thus vary with time. However, for the present study, we shall not introduce any dynamic elements. Accordingly, our modelling will be static in nature – except for the observation that an implicit succession in time is involved from truth to belief to knowledge.

Here follows the main – speculative – consideration: It is assumed that the world is characterized by an *interaction* between truth, belief and knowledge, expressed as an interaction between the three basic instances, x, y and z. The interaction is assumed to give z as a function of x and y and, moreover, it is assumed that this function acts *locally* in the sense that there exists a function $\pi : [0,1] \times [0,1] \to \mathbb{R}$, the *interactor*, such that $z_i = \pi(x_i, y_i)$ for each $i \in \mathbb{A}$.

It is easy to imagine a situation where truth is a solid

Table I
SKETCH OF A TYPICAL SITUATION

$\mathbb{A}$	Truth (x)	Belief (y)	Knowledge (z)
.	.	.	.
i	x_i	y_i	z_i
.	.	.	.

constant entity in the sense that what you see (observe) is a faithful reflection of truth, i.e. the interactor, denoted π_1, is given by

$$\pi_1(x, y) = x\,.$$

A world where this is the case is referred to as the *classical world* or the *Shannon world*.

But one should be open to other possibilities where your belief also enters in the final determination of what you experience, of the knowledge you can obtain. An extreme instance of this is the *black hole* characterized by the interactor, denoted π_0, defined by

$$\pi_0(x, y) = y\,.$$

In such a world no matter what you do, you will only experience what you yourself put into the situations in the form of your beliefs. There is no basis for scientific inference in such a world. You will never see a reflection of anything that can be termed "truth". With a risk of over-interpretation you may argue that examples of black holes are provided in situations where extreme religious fanatism is present. A philosophical question triggered by such a scenario is that perhaps the world you are a part of is to a great extent (even entirely!) a construct of your own, rather than a reflection of a higher truth which lies outside you.

We can also imagine mixtures of the two extreme worlds identified. By π_q we denote the interactor given by

$$\pi_q(x, y) = qx + (1 - q)y\,. \tag{1}$$

For $0 \leq q \leq 1$ these interactors are consistent in the sense that probability distributions x and y lead to probability distributions z via the interaction. However, let us be open for other interactors corresponding to other values of q, though they may be hard to accept when we try to explain rationally what goes on. Interactors different from any of those given by (1) may also be of interest. We should, however, always assume that the intaractor is *sound*, i.e. that $\pi(x, x) = x$ holds for all $x \in [0, 1]$.

Let us consider what the observer can do. In our basic line of thought, the observer cannot change the world. But he can prepare for the observations he is up to. As our second main consideration, we want to give room for human ingenuity. Speculating about how the observer can influence the observation process, we focus on the conviction that each individual observation requires an *effort* or has a *cost*. Inspired by the experience we have with *coding* in Shannons world, we conceive the effort as related to *description*. The observer should attempt to diminish this effort. The *total description effort*, which we may also refer to as *total description cost*, is assumed to be the sum of the *local description efforts* associated with the various basic events.

Without really knowing how the observer can construct suitable devices, say measuring instruments, to enable observation, we imagine that, somehow, this can be accomplished and that the performance of the resulting device is characterized by a function on $[0, 1]$, the *descriptor* κ. The value $\kappa(y)$ for $y \in [0, 1]$, is interpreted as the effort associated with a single observation of a basic event with belief-assignment y. Note that the observer can only prepare for observations based on his belief-assignments. The total description effort, denoted Φ is assumed to be the sum of the local efforts, where, in calculating these efforts, one must take into account the weight with which the various basic events are presented to the observer. Having a concrete situation in mind, the contribution from the basic event indexed by i is $z_i\kappa(y_i)$. Thus, introducing the interactor into this expression and accumulating the contributions, we find that the total description effort is given by

$$\Phi(x, y) = \sum_{i \in \mathbb{A}} \pi(x_i, y_i)\kappa(y_i)\,. \tag{2}$$

Clearly, as $y = 1$ expresses certainty, $\kappa(1) = 0$ must hold. We also expect that κ is a strictly decreasing function on $[0, 1]$, possibly assuming the value ∞ at $y = 0$. If κ is a feasible descriptor, so is any positive multiple of κ. Which one to select essentially amounts to a choice of unit. We therefore agree only to accept descriptors for which the following condition of normalization holds:

$$\kappa'(1) = -1\,. \tag{3}$$

We suggest that the units resulting from this normalization are called *natural units*, "nats". As we shall see, this will not conflict with previous usage.

In the observers attempts to diminish total description cost he may well argue that the smallest value, for x fixed, should be obtained when there is a *perfect match* between truth and belief, i.e. when $y = x$. This variational principle we shall call the *perfect match principle*. The quantity

$$\sum_{i \in \mathbb{A}} \pi(x_i, y_i)\kappa(y_i) - \sum_{i \in \mathbb{A}} x_i\kappa(x_i) \tag{4}$$

represents a kind of *frustration*, as it compares the actual description cost with the smallest possible cost, if only the observer had known the truth. The perfect match principle may, therefore, also be formulated by saying that frustration is the least, in fact disappears, when $y = x$.

Given x, *minimal description effort* is what the observer should aim at. We call this quantity *entropy* and denote it by the letter H:

$$\mathrm{H}(x) = \inf_{y=(y_i)_{i \in \mathbb{A}}} \Phi(x, y) = \sum_{i \in \mathbb{A}} x_i\kappa(x_i)\,.^{1} \tag{5}$$

[1] In order to allow the singular case corresponding to a black hole, the infimum should be restricted to run over probability distributions y with a support which contains the support of x.

The quantity (4) is, following tradition as known from the classical case, called *divergence*. It may also be called *redundancy*, a terminology which is actually closer to our line of thought. Indeed, it represents the redundant effort for the observer when he believes y, and prepares for observation based on this belief, though the truth is x. The quantity can also be written as

$$\mathrm{D}(x,y) = \Phi(x,y) - \mathrm{H}(x)\,, \tag{6}$$

which points to the fundamental relationship between description effort, entropy and divergence, actually best written in the form $\Phi = \mathrm{D} + \mathrm{H}$ to avoid problems with infinite entropy. See also (8) further on.

II. Quantitative reasoning

In the previous section philosophical contemplation led to certain basic principles. It is a key point that this kind of "soft" analysis is strong enough to imply precise quantitative results by standard mathematical arguments. For the result below it is an implicit assumption that natural technical assumptions as to continuity and differentiability of the interactor and the descriptor are understood to hold. Furthermore, the condition of *weak consistency* means that whenever x and y are probability distributions, then $\sum_{i\in\mathbb{A}} z_i = 1$ with $z_i = \pi(x_i, y_i)$. Also, to make precise the perfect match principle, it amounts to the assumption that for any probability distribution x with all point probabilities x_i positive, $\Phi(x,y)$ is minimal for $y = x$ when y runs over all probability distributions which have only positive point probabilities. Finally, for the convenient formulation of the main result, we need the *deformed logarithms* introduced by Tsallis, cf. [17]. They are given by the expression

$$\ln_q x = \begin{cases} \ln x \text{ if } q = 1, \\ \frac{x^{1-q}-1}{1-q} \text{ if } q \neq 1\,, \end{cases}$$

with q a real parameter.

Theorem 1: Assume that the interactor is weakly consistent and that the perfect match principle holds. Then $q = \pi(1,0)$ must be non-negative and, to each $q \in [0,\infty[$, there is only one interactor and one descriptor which fulfill the conditions imposed. These functions are the previously given interactor π_q from (1) and the descriptor κ_q determined by the formula

$$\kappa_q(y) = \ln_q \frac{1}{y}\,. \tag{7}$$

Regarding the proof, we give a brief outline: The formula (1) is readily derived from the assumption of weak consistency. Then, the only possible form for the descriptor is derived from pretty standard variational arguments involving Lagrange multipliers. Introducing these multipliers, one is led to the differential equation

$$(1-q)\kappa(x) + x\kappa'(x) = -1\,,$$

and (7) follows as $\kappa(1) = 0$. Simple examples show that if $q < 0$, the perfect match principle does not hold. That this principle does hold when $q \geq 0$, and still with interactor and descriptor given by (1) and (7), may be seen by observing that the entropy- and divergence measures you arrive at can also be derived from a well known approach due to Bregman, cf. [14] and [11].

We shall dwell on another approach to the last part of the proof which illuminates the well known *fundamental inequality* of information theory ($\mathrm{D}(x,y) \geq 0$ with equality if and only if $y = x$). Let us return to the consideration of any candidates for an interactor and a descriptor, π and κ, and let Φ, H and D be the associated functions as introduced above. Clearly, the perfect match principle is equivalent with the validity of the fundamental inequality for D (an exception corresponds to singular cases related to black holes). Writing D in the form

$$\mathrm{D}(x,y) = \sum_{i\in\mathbb{A}} \Big(\big(\pi(x_i,y_i)\kappa(y_i) + y_i\big) - \big(x_i\kappa(x_i) + x_i\big) \Big)\,, \tag{8}$$

we are tempted to introduce the following definition: We say that the *pointwise version of the fundamental inequality* (PFI) holds if, for each $x \in [0,1]$, the function $y \mapsto \pi(x,y)\kappa(y) + y$ assumes its minimal value for $y = x$ and not for any other value. Then, if the pointwise version of the fundamental inequality holds, so does the classical integrated version, hence also the perfect match principle.

As regards the situation treated in Theorem 1, we may assume that $q > 0$. The validity of PFI for $q = 1$ is a standard fact. Regarding the remaining cases, we note that

$$\pi_q(x,y)\kappa_q(y) + y - x\kappa_q(x) - x = \frac{q}{1-q} x y^{q-1} + y^q - \frac{1}{1-q} x^q\,.$$

Then, by an application of the geometric-arithmetic mean inequality, PFI also follows in these cases (consider the cases $q < 1$ and $q > 1$ separately and collect the two positive terms).

In order to be explicit, we write down the formulas for the key quantities associated with the result of Theorem 1. The formulas (1) and (7) give us the interactor and the descriptor. Regarding the other quantities, we obtain the Shannon-type quantities: *Kerridge inaccuracy*, *Shannon entropy* and *Kullback-Leibler divergence* (not displayed here), for $q = 1$, and for $q \geq 0$, $q \neq 1$ we find the following central quantities:

$$\Phi_q(x,y) = \textstyle\sum_{i\in\mathbb{A}} \left(\frac{q}{1-q} x_i y_i^{q-1} + y_i^q - \frac{1}{1-q} x_i \right), \tag{9}$$

$$\mathrm{H}_q(x) = \tfrac{1}{1-q} \textstyle\sum_{i\in\mathbb{A}} (x_i^q - x_i) = \tfrac{1}{1-q} \Big(\sum_{i\in\mathbb{A}} x_i^q - 1 \Big), \tag{10}$$

$$\mathrm{D}_q(x,y) = \textstyle\sum_{i\in\mathbb{A}} \left(\frac{q}{1-q} x_i y_i^{q-1} + y_i^q - \frac{1}{1-q} x_i^q \right). \tag{11}$$

Note that the formula for divergence, written in a way related to PFI, allows an extension to the continuous case.

In (9) the linearity in x is evident. This is important as it leads to a relatively easy approach to key optimization problems. For an indication of this, see [14] and [11]. In (10) we recognize the family of *Tsallis entropies*, cf. Tsallis [16]. Note the special case $q = 0$ corresponding to a black hole. There, the entropy only depends on the number n of elements in the support of x, indeed, $\mathrm{H}_0(x) = n - 1$. In (11) the main convenience of the formula is due to the fact, made possible by PFI, that the summands are non-negative.

The general formulas (2), (5) and (6) indicate that for the determination of the quantities involved one needs to know the interactor π as well as the descriptor κ. Two facts should be emphasized: Firstly, through the perfect match principle, the descriptor is uniquely determined from the interactor. Therefore, in principle, only the interactor needs to be known. Secondly, different interactors may well determine the same descriptor. Thus, knowing only the descriptor, you cannot know which world you operate in, in particular, you cannot determine divergence or description effort. But you *can* determine the entropy function. This points to a general thesis, that *entropy should never be considered alone*. Experience says that even when entropy can be considered by itself in interesting connections – a key example being the *maximum entropy principle* – full understanding and easy technical handling is always accomplished by introducing also other basic quantities in the discussion, typically description effort.

It is instructive to consider the family $(\kappa_q)_{0\leq q<\infty}$ of descriptors. This is a descending family of decreasing functions on $[0, 1]$. The largest descriptor, $\kappa_0(x) = \frac{1}{x} - 1$, is associated with a black hole. For $0 \leq q \leq 1$, the descriptors are convex and assume the value ∞ for $x = 0$. For $q = 1$, we find the descriptor $\kappa_1(x) = \ln \frac{1}{x}$ associated with the classical world. Then, for $1 < q < 2$ the descriptors are convex and finite valued, also for $x = 0$. The special descriptor $\kappa_2(x) = 1 - x$ is affine. For $2 < q < \infty$ we find descriptors which are concave with $\kappa_q'(0) = 0$. The zero function is not a descriptor covered by Theorem 1. It may be conceived as a limiting case corresponding to $q = \infty$. A world corresponding to this value of q would lead to situations with no outstanding issues, a world of wisdom (*paradise* or *hell* according to personal taste).

III. Notes and outlook

The essence of our findings is that the family of Tsallis entropies can be derived based on two principles, the essential principle which allows for an interaction between truth, belief and knowledge and then a more innocent and natural variational principle, that optimal performance is obtained when there is a perfect match between truth and belief. It should be emphasized that though these principles may be viewed as axioms, they are intended as key elements of an interpretation behind the quantities they lead to, typically entropy, divergence and description cost.

Further research on the fundamental nature of the quantities characterized is much desired. In particular, we need to understand the mechanisms behind interaction and also, there is a need for a more complete interpretation of descriptors, ideally as clear and convincing as the coding interpretation of the classical quantities due to Shannon, cf. [12]. In this connection, [18], [2] and references there as well as [13] may be relevant.

We continue by pointing to more specific issues, intended to indicate further possibilities. Partly due to restrictions of space, partly due to the fact that work is still in progress, this is done in an incomplete, somewhat sketchy manner.

First, some comments on description effort. We used $\Phi(x, y) = \sum \phi(x_i, y_i)$ with the *local description effort* given by $\phi(s, t) = \pi(s, t)\kappa(t)$. For all examples investigated for which the PMP (perfect match principle) was satisfied, the stronger property PFI was also satisfied. We believe that there is something fundamental behind this and, therefore, have defined *adjusted* notions of local as well as total description effort:

$$\tilde{\phi}(s, t) = \phi(s, t) + t$$
$$\tilde{\Phi}(x, y) = \sum_{i\in\mathbb{A}} \tilde{\phi}(x_i, y_i)\,.$$

The added term, t, in $\tilde{\phi}$ is interpreted as the contribution to the *total overhead* due to a basic event with believed probability t. Actually, the total overhead in any situation is $\sum y_i = 1$. In other words, the normalization (3) corresponds to choosing the overhead cost as the unit to work with. This makes good sense in the Shannon world since, apart from the necessary adjustment from nats to bits, the overhead in that case corresponds to taking the cost of having access to a binary memory cell as the basic unit.

Adjusting also the entropy function, one finds that adjusted entropy is always bounded below by the overhead cost, 1 nat.

From [14] the importance of the descriptors κ_q for the adequate handling of MaxEnt, the maximum entropy principle, can be seen (by the way, using intrinsic methods, devoid of any reference to Lagrange multipliers). Or rather, one finds that it is the inverses to the descriptors (*deformed exponentials*) that are important. These inverses, extended appropriately when $q > 1$, have an important interpretation as *probability checkers*: Indeed, if, in a Tsallis world with parameter q, you have access to a nats and ask how complex an event this will allow you to describe, the appropriate answer is “you can describe any event with a probability as low as $\kappa^{-1}(a)$”. Thus, when $q \leq 1$, however large your resources to nats are, there are events so complex that you cannot describe them, whereas, if $q > 1$ you can describe any event if you have access to K nats if only K is sufficiently large ($K \geq \frac{1}{q-1}$).

Other ideas may come from MaxEnt, in particular, you can ask which are the *feasible preparations* (sets of x’s). We claim that these are the sets for which finitely many function values $\Phi(x, y_\nu)$ have been fixed (or upper bounded). Roughly speaking, the view is that you can control description effort in as far as the choice of belief, typically transformed into an *observation strategy* is concerned. And you cannot control anything else. Considering just one believed distribution (observation strategy) y_0, the *level sets* for Φ, sets of the form $L^{y_0}(h) = \{x|\Phi(x, y_0) = h\}$, are the basic feasible preparations. Associated with these preparations is the *exponential family*, most conveniently defined as a family of beliefs (again, best understood as the associated observation strategies), viz. the family of all y for which, to any h, there exists c such that $L^{y_0}(h) \subseteq L^y(c)$. If y^* is an element in the exponential family and if, with $x^* = y^*$, you find that $x^* \in L^{y_0}(h)$, then x^* is the MaxEnt distribution of the preparation $L^{y_0}(h)$.

The connection (duality) between feasible preparations and the exponential family should fit into geometric ideas as developed by Amari and his school, cf. [3].

It is apparent that two sides, Nature and Observer, are involved in the general modelling and also in the indicated applications to MaxEnt. This invites for a game theoretical modelling. Actually, it turns out that much of the paradigm here put forward can be extended and provide a basis for a general theory not necessarily tied to probabilistic modelling, which attempts to capture essential aspects of the concept of "information". Research in this direction has recently been initiated, cf. [15].

Regarding the present research, developed over the last year, it may be appropriate to note that the basics have been presented at various meetings, including a poster presentation at ISIT2008. A manuscript will be published later as part of a "Festschrift" dedicated to Klaus Krickeberg. Some passages from Sections I and II are similar or identical with passages from that manuscript. A more technical paper will be worked out and submitted for publication later this year.

IV. Comments on the literature

The formula (10) for a measure of entropy first appeared in the mathematical literature in Havrda and Charvát [6] and, independently, in Daróczy [5]. The latter author emphasized the characterization via functional equations, cf. also [1] and the more recent reference work [4].

The first appearance in the physical literature is due to Lindhard and Nielsen [10], where the property of *composability* – the ability to determine the entropy of a combined system from the entropies of its component subsystems – was the motivating principle. Subsequently, Lindhard gave a careful treatment of aspects of the measuring process, cf. [9].

The trend-setting publication [16] from 1988 by Tsallis marks the efficient promotion within the physical community of the new entropy measures. The paper triggered much research as also witnessed by the more than 2000 entries in the database maintained by Tsallis. At the time of publication, Tsallis was unaware of the earlier research. Regarding [10] and [9], these papers were largely unnoticed, probably due to their mathematical and somewhat lengthy style. However, there is a casual reference to Lindhard's work in one of Jaynes' papers, [8].

The success of Tsallis in launching the entropy measures which now bear his name is due to the direct approach and the fact that when combined with Jaynes *Maximum Entropy Principle*, cf. [7], main problems of statistical physics lead to *power laws*, a class of distributions which was and still is very popular as the basis for modelling when heavy-tailed distributions are involved.

The present approach is in line with earlier game theoretical considerations, cf. [14]. Because of a relation to Bregman divergences, we also point the reader to [11] and works referred to there.

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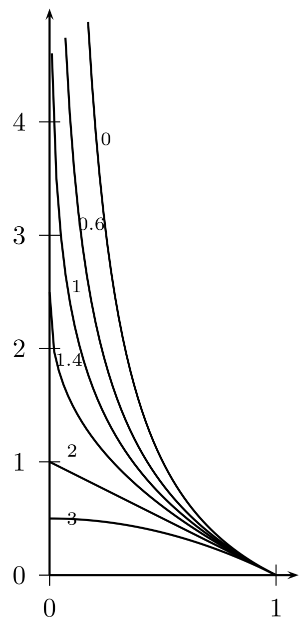

Figure 1. Descriptors

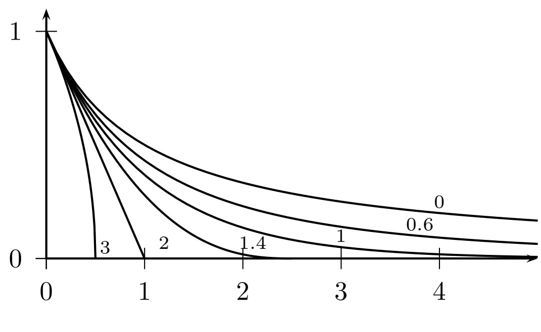

Figure 2. Probability-checkers