Interaction between Truth, Belief and Knowledge as the key to entropy measures of statistical physics

Flemming Topsøe
University of Copenhagen
Department of Mathematical Sciences
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What is entropy?

Entropy operates in the interface
God $\leftrightarrow$ man or Nature $\leftrightarrow$ man or
system $\leftrightarrow$ physicist studying or actually observing.

Entropy assigns to the true state x of a system the complexity of the system in that state as seen by man. More precisely, the entropy of x is given by

$$\mathsf{S}(x) = \min_y \Phi(x, y) \quad \text{where}$$

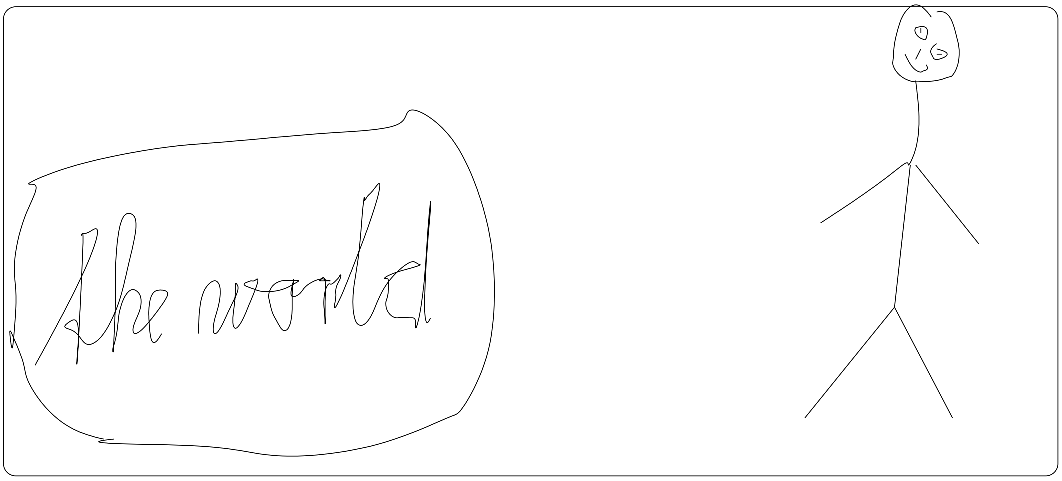
$\Phi(x, y) =$ description cost (or effort) needed to describe the system when the truth is x and your belief is y.

So: Entropy is minimal description effort

Note: Focus on Φ with S as a derived concept. Another important derived concept is redundancy (divergence) D defined by $\mathsf{D}(x, y) = \Phi(x, y) - \mathsf{S}(x)$.

Problem: Φ =?

Answer requires knowledge about nature, the world we operate in, and input from man who should design efficient descriprion strategies via descriptors (see later).



truth x belief y knowledge z

Regarding z: outcome from extended observations, synthesis of extended experience, knowledge .

x and y will be distributions over some alphabet, $\mathbb{A}$: $x = (x_i)_{i \in \mathbb{A}}$, $y = (y_i)_{i \in \mathbb{A}}$ and similarly for y.

Possible worlds

Key postulate: There is an interaction between x, y, z of the form $z = \Pi(x, y)$. Π characterizes the world. Some possibilities:

$\Pi(x, y) = x$: classical world,
$\Pi(x, y) = y$: black hole,
$\Pi(x, y) = qx + (1 - q)y$: mixtures.

Assume that interaction acts locally via interactor π: $z = \Pi(x, y)$ with $z_i = \pi(x_i, y_i)$. Always assume that $\pi(s, t) = s$ if $t = s$ (soundness).

Examples: $\pi_q(s, t) = qs + (1 - q)t$ corresponding to classical world ($q = 1$), black hole ($q = 0$) or mixtures.

There are many other possible interactors, but:

under consistency the π_q's are the only ones.

Consistency:
$\sum z_i = 1$, for x, y probability distributions, $z = \Pi(x, y)$.

What can the physicist do?

We assume that the world is known to the physicist through the interactor π.

Every observation entails a cost (or effort). The cost depends on the event being observed. An event with high probability has little cost. Thus define a descriptor κ to be a decreasing function on $[0, 1]$ with $\kappa(1) = 0$. Also insist that κ satisfies the normalization condition $\kappa'(1) = -1$, corresponding to a choice of unit.

Interpretation: $\kappa(t)$ is the effort associated with observations from an event which you believe occurs with probability t (equal to some y_i, say).

Total description cost, $\Phi(x, y)$, when truth is x and belief is y is given by

$$\Phi(x, y) = \sum_{i \in \mathbb{A}} z_i \kappa(y_i) = \sum_{i \in \mathbb{A}} \pi(x_i, y_i) \kappa(y_i) \, .$$

The perfect match principle (PMP)

To design κ, hence Φ, apply perfect match principle:

PMP: Φ is the smallest when *belief matches truth*: $\Phi(x, y) \geq \Phi(x, x)$, i.e. $\Phi(x, y) \geq \mathsf{S}(x)$.

Theorem Given π, the only *possible* descriptor satisfying PMP is the solution to

$$\frac{\partial \pi}{\partial t}(t, t)\kappa(t) + t\kappa'(t) = -1 \; ; \; \kappa(1) = 0 \, .$$

If $\pi = \pi_q$, this becomes

$$(1 - q)\kappa(t) + t\kappa'(t) = -1 \; ; \; \kappa(1) = 0$$

with solution $\kappa = \kappa_q$ given by

$$\kappa_q(t) = \ln_q \frac{1}{t} \text{ with } q\text{-logarithm given by}$$

$$\ln_q x = \begin{cases} \ln x \text{ if } q = 1 \, , \\ \frac{1}{1-q}\left(x^{1-q} - 1\right) \text{ if } q \neq 1 \, . \end{cases}$$

However, this only satisfies PMP if $q \geq 0$.

Comments on the proof

The differential equation comes up via standard variational principles (introduce Lagrange multipliers!).

Re π_q, κ_q: Failure of PMP for $q < 0$: simple direct counter examples with a 3-element alphabet.

Validity of PMP for $q \geq 0$: via PFI, the pointwise fundamental inequality, which states that $d(s,t) \geq 0$ where

$$d(s,t) = \Big(\pi(s,t)\kappa(t) + t\Big) - \Big(s\kappa(s) + s\Big) .$$

Indeed,

$$\mathrm{D}(x,y) = \sum d(x_i, y_i)$$

and PMP really says that $\mathrm{D}(x,y) \geq 0$ (with "$=$" iff $y = x$). Writing up the formulas with π_q and κ_q in place of π and κ one finds that a simple application of the geometric/arithmetic mean inequality leads to the desired inequality $d(s,t) \geq 0$.

Comments on result:

• Strong argument in favour of the view that only possible entropy measures of statistical physics are those in the Tsallis family: $S_q(x) = \sum x_i \kappa_q(x_i)$ $= \left(\sum x_i^q - 1\right)/(1-q)$ for $q \geq 0$. Classical value for $q = 1$, and for black hole, $\mathrm{S}(x) = n - 1$, number of degrees of freedom ($n =$ size of alphabet).

• Φ more important than S, gives more, e.g. shows which are the feasable preparations, viz. those of the form $\mathcal{P} = \{x | \Phi(x, y_0) = c\}$, and Φ also assists greatly in finding equilibrium distributions (via Max-Ent, and even without introducing Lagrange multipliers – instead a natural, therefore better, intrinsic approach is used). (more on next slides)

• The roles of God and man (nature and man, system and man) clearly separated!

• Formulas are mathematically attractive, e.g. lines up with popular Bregman divergencies... there "must be some truth in them"!

• main outstanding issues are: interaction, how? description, how – via coding as in classical case (à la Shannon) ?

Equilibrium calculations

Background theorem $\mathcal{P}$ any preparation (set of x's). Given x^*, y^* such that: $x^* \in \mathcal{P}$, y^* robust, i.e. $\exists h \forall x \in \mathcal{P} : \Phi(x, y^*) = h$, and $y^* = x^*$. Then x^* is the MaxEnt distribution of $\mathcal{P}$ (and y^* the MinRisk strategy for the physicist, i.e. $\text{argmin}\left(\max_{x \in \mathcal{P}} \Phi(x, y)\right) = y^*$).

Proof: Assume $\Phi(x, y^*) = h$ for all $x \in \mathcal{P}$. Then $\mathsf{S}(x^*) = \Phi(x^*, x^*) = \Phi(x^*, y^*) = h$, and, for $x \in \mathcal{P}$ and $x \neq x^*$ we have $\mathsf{S}(x) < \mathsf{S}(x) + \mathsf{D}(x, y^*) = h$. The min-risk part is proved just as easily. □

Define: $y^* \in \mathcal{E}(y)$, the exponential family of y, if $\Phi(x, y^*)$ only depends on $\Phi(x, y)$. In short:

$$\mathcal{E}(y) = \{y^* | \exists \xi : \Phi(x, y^*) = \xi\big(\Phi(x, y)\big)\}.$$

Corollary Put $L^y(h) = \{x | \Phi(x, y) = h\}$. If $\mathcal{P} = L^y(h)$ and x^*, y^* are given such that: $x^* \in \mathcal{P}$, $y^* \in \mathcal{E}(y)$ and $y^* = x^*$, then x^* is the MaxEnt distribution of $\mathcal{P}$.

... continued

Let $\mathcal{L}^y$ be the class of non-empty models of the form $L^y(h)$ with $h \in \mathbb{R}$. The associated exponential family or equilibrium generating family, is the family

$$\mathcal{E}(y) = \{x | \forall L \in \mathcal{L}^y \exists c \in \mathbb{R} : \ L \subseteq L^x(c)\} \,.$$

$x \in \mathcal{E}(y)$, $\mathsf{S}(x) = h \ \Rightarrow x$ is MaxEnt dist. of $L^x(h)$

Thus one should try and determine the elements in $\mathcal{E}(y)$. Looking at it, you find that for the worlds determined by one of the interactions π_q, every x for which there exist constants α and β such that

$$\forall i \in \mathbb{A} : \ \kappa(x_i) = \alpha + \beta\kappa(y_i)$$

are in $\mathcal{E}(y)$. For $q = 1$ this leads to classical analysis (with partition function etc.) and even without the use of Lagrange multipliers. For the general case, you have to adjust constants so that

$$\sum_{i \in \mathbb{A}} \kappa^{-1}\Big(\alpha + \beta\kappa(y_i)\Big) = 1 \,.$$

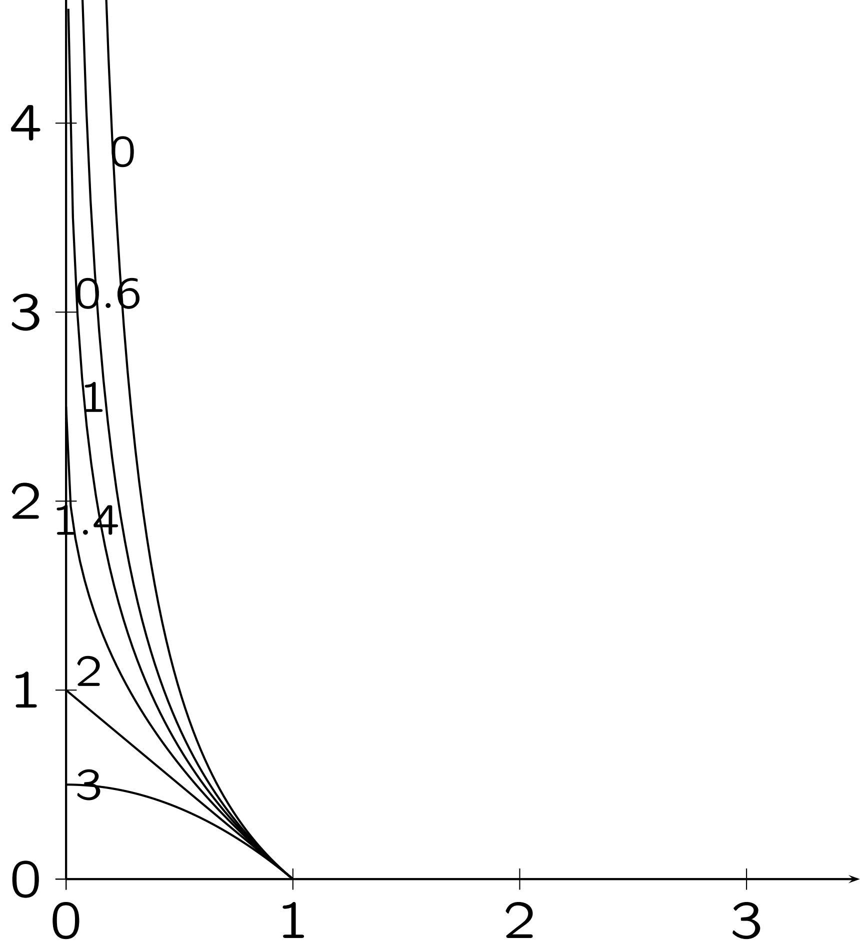


Figure shows family $(\kappa_q)_{q\geq 0}$ of descriptors

So, given a probability t, you can see what effort is needed, measured in nats (natural units) , in order to describe events with probability t when you use the chosen descriptor.

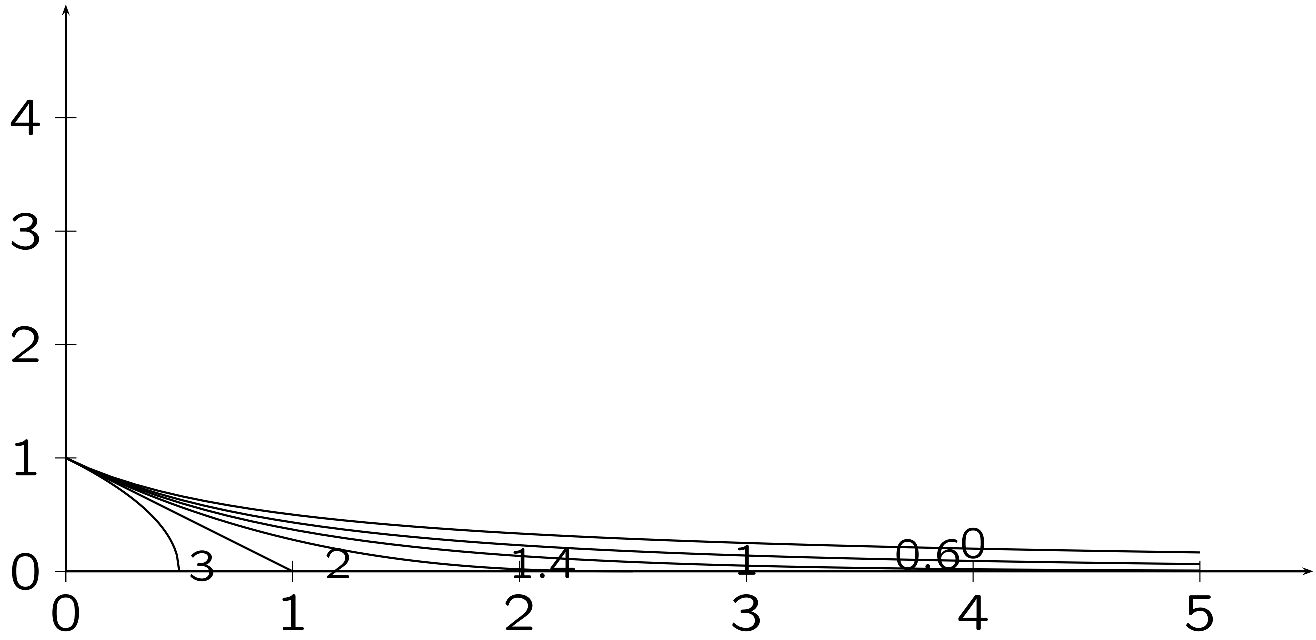


Figure shows inverses of descriptors.

"probability checkers"

You can use these functions (q-deformed exponentials) to check, for a chosen descriptor, how complicated events you can describe with a given number of nat's available, i.e. how low a probability an event can have and still be describable with the available number of nat's. This kind of consideration is important in order to carry out MaxEnt calculations indicated previously.