Paradigms of Cognition

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Abstract

Cognition concerns *truth*, *belief* and *knowledge*. It involves *perception*, *description*, *control*, *inference* and more. An abstract quantitative theory connecting these notions is presented, centred around a concept of *information triples*. Inference is facilitated via a notion of *core*, in classical settings related to *exponential families*. Justification of the theory is provided by pointing to possible applications within e.g. Hilbert space theory and Shannon theory. More concrete considerations include a study of *Tsallis entropy* and an extension of the familiar *Bregman construction*. This depends on a generalized notion of *proper scoring rules*. Philosophical considerations lead the way, game theory provides the technical tool!

Keywords—truth, knowledge, entropy, divergence, information triples, proper effort functions, fundamental inequality, core, Bregman construction, Tsallis entropy.

Contents

Introduction

A word of warning is in place in relation to the unusual emphasis on philosophical considerations, often of a speculative non-professional nature, often devoid of detailed reference to or discussion of the relevant philosophical or psychological literature. The aim has not been to produce a scholarly work. Rather, the many loose remarks of a non-mathematical nature are intended as a source of inspiration with indications of possibilities for further study.

Originally, the driving force behind the study was to extend the clear and convincing operational interpretations associated with classical information theory as developed by Shannon [1] and followers, to the theory promoted by Tsallis for statistical physics and thermodynamics, cf. [2], [3]. That there are difficulties is witnessed by the fact that, despite its apparent success, some well known physicists still find grounds for criticism. Evidence of this attitude may be found in Gross [4].

As it will turn out, if you accept a certain kind of interaction between truth, belief and knowledge, you are led in a natural way to the family of *Tsallis entropies*, cf. Section 22. Further study revealed that the philosophical elements of the indicated approach make sense in a much wider setting than originally intended. One does not achieve the same degree of clarity as in classical Shannon theory, where coding provides a solid reference. However, we shall demonstrate that the extension to a more abstract framework is meaningful and opens up for new areas of research. In addition, known results are consolidated and unified.

In the first many sections, Sections 1 - 15, an abstract theory of *information without probability* is presented. It is based on somewhat speculative considerations which, taken together, constitute possible paradigms of cognition. Inspiration from Shannon Theory and from the theory of inference within statistics and statistical physics is apparent. However, the ideas are here presented as an independent theory.

Previous endeavours in the direction taken includes research by Ingarden and Urbanik [5] who wrote *"... information seems intuitively a much simpler and more elementary notion than that of probability ... [it] represents a more primary step of knowledge than that of cognition of probability ..."*. We also point to Kolmogorov, cf. [6] and [7] who in the latter reference (but going back to 1970 it seems) stated *"Information theory must precede probability theory and not be based on it"*. The ideas by Ingarden and Urbanik were taken up by Kampé de Fériet, see the survey [8]. The work of Kampé de Fériet is rooted in logic. Logic is also a key ingredient in comprehensive studies over some 40 years by Jaynes, collected posthumously in [9]. Though many philosophically oriented discussions are contained in the work of Jaynes, the situations he deals with are limited to probabilistic models and intended mainly for a study of statistical physics.

The work by Amari and Nagaoka in information geometry, cf. [10] may also be viewed as a broad attempt to free oneself from a tie to probability. Among many followers we here only point to the recent thesis of Anthonis [11] with a base in physics.

In *complexity theory* as developed by Solomonoff, Kolmogorov and others, cf. the recent survey [12] by Rathmanner and Hutter, we have a highly theoretical discipline which aims at inference not necessarily tied to probabilistic modelling. The *Minimum Description Length Principle* may be considered an important spin-off of this theory. It is mainly directed at problems of statistical inference and was developed, primarily, by Rissanen and by Barron and Yu, cf. [13]. We also point to the treatise [14] by Grünwald. There you find discussions of many of the issues dealt with here, including a discussion of the work of Jaynes.

Still other areas of research have a bearing on "information without probability", e.g. semiotics, philosophy of information, pragmatism, symbolic linguistics, placebo research, social information and learning theory. Many areas within psychology are also of relevance. Some specific works of interest include Jumarie [15], Shafer and Vovk [16], Gernert [17], Bundesen and Habekost [18], Benedetti [19] and Brier [20]. The handbook [21] edited by Adriaans and Bentham and the encyclopedia article [22] by Adriaans collect views on the very concept of *"information"*. Over the years, an overwhelming amount of thoughts has been devoted to that concept in one form or another. Most of this bulk of material is entirely philosophical and not open to quantitative analysis. Part of it is impractical and presently mainly of theoretical interest. And some is far from Shannon's theory which we hold as a corner stone of quantitative information theory. In fact, we consider it a requirement of any quantitative theory of information to be downward compatible with basic parts of Shannon theory. This requirement is largely respected in the present work. But not entirely. For example, it is doubtful if one can meaningfully lift the concept of coding as known from Shannon theory to a more abstract level.

We thus attempt to go "beyond Shannon". So does e.g. Brier in his development of cybersemiotics, cf. [23], [20]. Brier goes deeper into some of the philosophical aspects than we do and also attempts a broad coverage by incorporating not only the exact natural sciences but also life science, the humanities and the social sciences. Though not foreign to such a wider scope, our study aims at more concrete results by basing the study more directly on quantitative elements. Both studies emphasize the role of the individual in the cognitive process.

A special feature of our development is the appeal to basic game theoretical considerations, cf. especially Sections 10 and 12. To illuminate the importance we attach to this aspect we quote from Jaynes preface to [9] where he comments on the maximum entropy principle, the central principle of inference promoted by Jaynes:

"*... it [maximum entropy] predicts only observable facts (functions of future or past observations) rather than values of parameters which may exist only in our imagination ... it protects us against drawing conclusions not warranted by the data. But when the information is extremely vague, it may be difficult to define any appropriate sample space, and one may wonder whether still more primitive principles than maximum entropy can be found. There is room for much new creative thought here.* "

This is one central place where game theory comes in. It represents a main addition, we claim, to Jaynes' work [1]. The merits of game theory in relation to information theoretical inference were presented in the probabilistic, Shannon-like setting, independently of each other, by Pfaffelhuber [25] and the author [26]. These works were often overlooked by subsequent authors. More recent references include Harremoës and Topsøe [27], Grünwald and Dawid [28], Friedman et al [29] (a utility-based work) and Dayi [30]. As sources of background material, [31], [32] and [33] may be helpful.

The quantitative elements we work with are brought into play via a focus on *effort*. From this concept general notions of *entropy* and *redundancy* – or the essentially equivalent notion of *divergence* – are derived. The *information triples* we have taken as the key object of study is an expression of the triple concepts *effort/entropy/redundancy* (or *effort/entropy/divergence*). By a "change of sign" , the triples may, just as well, concern *utility/max-utility/divergence*.

Apart from introducing game theory into the picture, a main feature of the present work lies in its abstract nature with a focus on interpretations rather than on axiomatics which was the emphasis of many previous authors, including Jaynes.

[1] At the conference "Maximum Entropy and Bayesian Methods", Paris 1993, the author had much hoped to discuss the impact of game theoretical reasoning with professor Jaynes. Unfortunately, Jaynes, who died in 1998, was too ill at the time to participate. In fact, he never incorporated arguments such as those in [24] which may be seen as supportive of his theory.

The later parts of the study may be viewed as a justification of the partly speculative deliberations of the first 15 sections. Sections 16 and 17 show instances where the main interpretations of the preceding abstract development may favorably be changed. Section 18 represents the natural building stones of the information triples. This is closely related to the well-known construction associated with Bregmans name. Section 19 which follows, depends on a weakened concept of "properness" developed in Section 11. It provides a principally far reaching generalization in that the Bregman generator, normally required to be smooth and concave (convex for utility-based considerations) may now be arbitrary as long as it is dominated by a linear function. The ideas behind this is a substitute for the process of differentiation in that the aim is focused on global (rather than local) optimization. Properly expanded, this idea may be of more wide applicability.

The applications presented – or indications of potential applications – come from combinatorial geometry, probabilistic information theory, statistics and statistical physics. For most of them, we focus on providing the key notions needed for the theory to work, thus largely leaving concrete applications aside. The aim is to provide enough details in order to demonstrate that our modelling can be applied in quite different contexts. For the case of discrete probabilistic models we do, however, embark on a more thorough analysis. The reason is, firstly, that this is what triggered the research reported on and, secondly, with a thorough discussion of modelling in this context, virtually all elements introduced in the first 15 sections have a clear and natural interpretation. In fact, full appreciation of the abstract theory may only be achieved after reading the material in Section 22.

Our treatment is formally developed independently of previous research. However, unconsciously or not, it depends on earlier studies as referred to above and on the tradition developed over time. More specifically, we mention that our focus on *description effort*, especially the notion of *properness*, cf. Section 6, is closely related to ideas first developed for areas touching on meteorology, statistics and information theory.

Finally, we mention that [34], [35], [36] and [37] are forerunners of the present work.

Part I
Information without probability

1 The world and you

By Ω we denote the *world*, more precisely the actual world, perhaps one among several conceivable worlds. Two fictive persons play a major role in our modelling, *"Nature"* and *"Observer"*. The interplay between the two takes place in relation to studies of *situations* from the world. Nature is seen as an expression of the world itself and reflects the rules of the world. Observer seeks knowledge about situations studied. It may be helpful to think of Observer as "you".

The knowledge sought by Observer aims at inference concerning particular situations under study. A higher form of inference may also be possible if Observer does not know the rules of the world, in other words, does not know which world he is placed in. Then, having a reservoir of conceivable worlds in mind, and based on experience from the study of several situations, Observer may attempt to infer which one is the actual world.

We think of Ω as limited in some sense, a partial world. This appears to be the most realistic. In principle, one could consider all kinds of phenomena at the same time, say of a statistical, physical, social, psychological or other nature. However, the rules of the world may vary from context to context and – if you do not take these rules as absolutes – even from one Observer to another. A finer modelling than here considered may bring the notion of *context* more prominently into the picture.

The notions introduced are left as loose indications. They will take more shape as the modelling progresses. The terminology chosen here and later on is intended to provoke associations to common day experiences of the cognitive process. In addition, the terminology is largely consistent with usage in philosophy.

2 Truth and Belief

Nature, as an expression of the fixed rules of the world, does not have a mind. Nature is the holder of *truth*. Observer seeks the truth but is relegated to *belief*. However, Observer possesses a conscious and creative mind which can be exploited with the goal to obtain knowledge as effortlessly as possible.

We introduce a set, X, the *state space*, and a set Y, the *belief reservoir*. Elements of X, generically denoted by x, are *truth instances* or *states of truth* or just *states*, whereas elements of Y, generically denoted by y, are *belief instances*. We assume that $Y \supseteq X$. Therefore, in any situation, it is conceivable that Observer actually believes what is true. Often, $Y = X$ will hold. Then, whatever Observer believes, could be true.

Typically, in any situation, we imagine that Nature chooses a state and that Observer chooses a belief instance. This leads to the introduction of certain games which will be studied systematically later on, starting with Section 10.

Though there may be no such thing as *absolute truth*, it is tempting to imagine that there is and to think of Natures choice as an expression of just that. This then helps to maintain a distinction between Nature and Observer. However, a closer analysis reveals that what goes on at Natures side is perhaps best thought of as another manifestation of Observer. Thus the two parts cannot be separated. Rather, a key to our modelling is the interplay between Nature and Observer.

Some models will involve a set $X_0 \subseteq X$ of *realistic states*. States not in X_0 are considered unrealistic, out of reach for Observer. And some models involve a set $Y_{\text{det}} \subseteq Y$ of *certain beliefs*. Beliefs from Y_{det} are chosen by Observer if he is quite determined on what is going on – but of course, he could be wrong.

In a specific situation, Natures choice may not be free within all of X. Rather, it may be restricted to a non-empty subset $\mathcal{P}$ of X, the *preparation*. This set depends on the particular situation studied. The idea is that Observer, perhaps a physicist, can "prepare" a situation, thereby forcing Nature to restrict the choice of state accordingly. For instance, by placing a gas in a heat bath, Nature is restricted to states which have a mean energy consistent with the prescribed temperature.

A situation is normally characterized by specifying a preparation. However, further details, especially regarding Observers behaviour may also be included in the modelling of "a situation". A state x is *consistent* – viz. consistent with the preparation $\mathcal{P}$ of the situation – if $x \in \mathcal{P}$. Later on, we shall consider *preparation families* which are sets, generically denoted by $\mathbb{P}$, whose members are preparations.

Faced with a specific situation with preparation $\mathcal{P}$, Observer speculates about the state of truth chosen by Nature. Observer may express hes opinion by assigning a belief instance to the situation. If this is always chosen from the preparation $\mathcal{P}$, Observer will only believe what *could* be true. Sometimes, Observer may prefer to assign a belief instance in $Y \setminus \mathcal{P}$ (or even in $Y \setminus X$) to the situation. Then this instance cannot possibly be one chosen by Nature. Nevertheless, it may be an adequate choice if an instance inside $\mathcal{P}$ would contradict Observers subjective beliefs. Therefore, the chosen instance may be the "closest" to the actual truth instance in some subjective sense. Anyhow, Observers choice of belief instance is considered a subjective choice which takes available information into account such as general insight and any *prior knowledge*. Qualitatively, these thoughts agree with Bayesian thinking, and as such enjoy the merits, but are also subject to the standard criticism, which applies to this line of thought, cf. [12] and [38].

3 A tendency to act, a wish to control

Two ways will lead us to new and important structural elements.

First, we point to the mantra that *belief is a tendency to act.* This is a rewording taken from Good [39] who suggested this point of view as a possible interpretation of the notion of belief. In daily life, action appears more often than not to be a spontaneous reaction in situations man is faced with, rather than a result of rational considerations. Or reaction depends on psychological factors or brain activity largely outside conscious control. In contrast, we shall rely on rational thinking based on quantitative considerations. This may at times bring order to apparent irrational behaviour. Precise details of the modelling we shall promote will have to wait until Section 6. As a preparation, we introduce in this section a set $\hat{Y}$, the *action space*, and a map from Y into $\hat{Y}$, referred to as the *response*. Elements of $\hat{Y}$ are *actions* and are, generically, indicated by the letter w. We use the notation $y \mapsto \hat{y}$ to indicate the action which is Observers response in situations where Observers belief is represented by the belief instance y.

Response need not be injective, thus it is in general not possible to infer Observers beliefs from Observers actions. Response need not either be surjective, though for most applications it will be so. Actions not in the range are idle for the actual model under discussion but may become relevant if the setting is later expanded. Belief instances, say y_1 and y_2, with the same response are *response-equivalent*, notationally written $y_1 \hat{\sim} y_2$.

If the model contains certain beliefs, i.e. if $Y_{\text{det}} \neq \emptyset$, we assume that $\hat{Y}$ contains a special element, $w_\emptyset$, the *empty action*, and that this action is chosen by Observer in response to any certain belief instance. Thus $\hat{y} = w_\emptyset$ for every $y \in Y_{\text{det}}$. In such cases, Observer sees no reason to take any action. If Observer finds several actions equally attractive, one could allow response to be a set-valued map. However, for the present study we insist that response is an ordinary map defined on all of Y.

Let us turn to another tendency of man, the wish to control. This makes us introduce a set W, the *control space*. The elements of W are referred to as *controls*. For the present modelling, we take W and $\hat{Y}$ to be identical: $W = \hat{Y}$. This simplification may be defended by taking the point of view that in order to exercise control, you have to act, typically by setting up appropriate experiments, and in a rough mathematical model as here suggested we simply identify the two aspects. Later elaborations of the modelling may lead to a clear distinction between action and the more passive concept of control.

The simplest models are obtained when response is an injection or even a bijection. And simplest among these models are the cases when $Y = \hat{Y} = W$ and response is the identity map. This corresponds to a further identification of belief, action and control. Even then it makes a difference if you think about an element as an expression of belief, an expressions of action or as an expressions of control.

Though many models do not need the introduction of $\hat{Y}$ (or W), the further development will to a large extent refer first and foremost to $\hat{Y}$-related concepts. Technically, this results in greater generality, as response need not be injective. Belief-type concepts, often indicated by referring to the "Y-domain" will then be derived from action- or control-based concepts, often indicated by pointing to the "$\hat{Y}$-domain". The qualifying indication may be omitted if it is clear from the context whether we work in the one domain or the other.

4 Atomic situations, Controllability and Visibility

Two relations will be introduced. *Controllability* is the primary one from which the other one, *visibility* will be derived. These relations constitute refinements which may be disregarded at a first reading. This can be done by taking the relations to be the *diffuse relations* (in notation below, $X \otimes \hat{Y} = X \times \hat{Y}$ and $X \otimes Y = X \times Y$).

Pairs of states and belief instances or pairs of states and controls are key ingredients in situations from the world. However, not all such pairs will be allowed. Instead, we imagine that offhand, Observer has some limited insight into Natures behaviour and therefore, Observer takes care not to associate "completely stupid" belief instances or controls, as the case may be, with situations of interest.

We express these ideas in the $\hat{Y}$-domain by introducing a relation from X to $\hat{Y}$, called *controllability* and denoted $X \otimes \hat{Y}$. Thus $X \otimes \hat{Y}$

is a subset of the product set $X \times \hat{Y}$. Elements of $X \otimes \hat{Y}$ are *atomic situations* (in the $\hat{Y}$-domain). Given a preparation $\mathcal{P}$, we may consider the restriction $\mathcal{P} \otimes \hat{Y}$ which consists of all atomic situations (x, w) with $x \in \mathcal{P}$.

For an atomic situation (x, w), we write $w \succ x$ and say that w *controls* x or that x can be *controlled* by w. An atomic situation (x, w) is an *adapted pair* if w is *adapted* to x in the sense that $w = \hat{x}$.

For a preparation $\mathcal{P}$ we write $w \succ \mathcal{P}$, and call w a *control* of $\mathcal{P}$, if $w \succ x$ for every $x \in \mathcal{P}$. By $\hat{}[\mathcal{P}]$ we denote the set of all control points of $\mathcal{P}$[2]. We write $\hat{}[x]$ if $\mathcal{P}$ is the singleton set $\{x\}$. Note that if $X \otimes \hat{Y}$ is the diffuse relation, $\hat{}[\mathcal{P}] = \hat{Y}$ for any preparation $\mathcal{P}$.

For $\mathcal{Q} \subseteq \hat{Y}$, $]\mathcal{Q}[$ denotes the *control region* of $\mathcal{Q}$, the set of $x \in X$ for which $w \succ x$ for some $w \in \mathcal{Q}$. We write $]w[$ if $\mathcal{Q}$ is the singleton set $\{w\}$. Clearly, $w \in \hat{}[\mathcal{P}]$, $w \succ \mathcal{P}$ and $\mathcal{P} \subseteq]w[$ are equivalent statements.

We assume that the following conditions hold:

$$\forall x \in X : \hat{x} \succ x \,, \tag{1}$$

$$\forall w \in \hat{Y} :]w[\neq \emptyset \,, \tag{2}$$

and normally also that

$$\exists y \in Y : \hat{y} \succ X. \tag{3}$$

The first condition is essential and the second one is rather innocent. The third condition is introduced when we want to ensure that X is not "too large". Models where (3) does not hold are considered unrealistic, beyond what man (Observer) can grasp. If response is surjective, it amounts to the condition $\hat{}[X] \neq \emptyset$. It is illuminating to have models of classical Shannon theory in mind, cf. Section 23.

From controllability we derive the relation of *visibility* for the Y-domain, denoted $X \otimes Y$, and given by

$$X \otimes Y = \{(x, y) \in X \times Y | \hat{y} \succ x\} \,. \tag{4}$$

Restrictions $\mathcal{P} \otimes Y = \{(x, y) \in X \otimes Y | x \in \mathcal{P}\}$ are at times of relevance. We use the same sign, $\succ$, for visibility and for controllability. The context will have to show if we work in the Y- or in the $\hat{Y}$-domain. We see that $y \succ x$ if and only if $\hat{y} \succ x$. If this condition holds, we say that y *covers* x or that x is *visible* from y. Pairs in $X \otimes Y$ are *atomic situations* (in the Y-domain). An atomic situation (x, y) is an *adapted pair* if $(x, \hat{y})$ is so in the $\hat{Y}$-domain, i.e. if $y \hat{\sim} x$. And (x, y) is a *perfect*

[2] here we use a "hat" merely to indicate that we work in the $\hat{Y}$-domain. Thus, $\hat{}[\mathcal{P}]$ should not be confused with the set of $\hat{x}$ for a state x in $\mathcal{P}$.

match if $y = x$. The two notions coincide if response is injective. An atomic situation (x, y) is a *situation of certainty* if $y \in Y_{\mathrm{det}}$.

By (1) and the defining relation (4), $x \succ x$ for all $x \in X$, thus $X \otimes Y$ contains the diagonal $X \times X$. The *outlook* (or *view*) from $y \in Y$ is the set $]y[= \{x | y \succ x\}$. Clearly, $]y[=]\hat{y}[$. By (2) and (4), this set is non-empty and, when (3) holds, for at least one belief instance, the outlook is all of X.

For a preparation $\mathcal{P}$ we write $y \succ \mathcal{P}$, and call y a *viewpoint* of $\mathcal{P}$, if $y \succ x$ for every $x \in \mathcal{P}$. The set of all viewpoints of $\mathcal{P}$ is denoted $[\mathcal{P}]$. We write $[x]$ if $\mathcal{P}$ is the singleton $\mathcal{P} = \{x\}$. By $\mathrm{ctr}(\mathcal{P})$, the *centre of* $\mathcal{P}$, we denote the set of viewpoints in the preparation, i.e. $\mathrm{ctr}(\mathcal{P}) = \mathcal{P} \cap [\mathcal{P}]$. This set may be empty. Note that if response is surjective, $\hat{}[\mathcal{P}]$ is the image of $[\mathcal{P}]$ under response.

In any situation, Observer should ensure that from his chosen belief instance, every state which could conceivably be chosen by Nature is visible. Therefore, in a situation where the preparation $\mathcal{P}$ is known to Observer, Observer should only consider belief instances in $[\mathcal{P}]$. Indeed, if Observer chooses a belief instance $y \in Y \setminus [\mathcal{P}]$, there is a risk that Natures choice will be a truth instance which is not visible from y.

In the sequel we shall often consider bivariate functions, generically denoted by either $\hat{f}$ ($\hat{Y}$-domain) or by f (Y-domain). The $\hat{f}$-type functions are defined either on $X \otimes \hat{Y}$ or on some subset of $X \otimes \hat{Y}$ of the form $\mathcal{P} \times \hat{}[\mathcal{P}]$ for some preparation $\mathcal{P}$. The range of $\hat{f}$ may be any abstract set but will often be a subset of the extended real line. Given $\hat{f}$, it is understood that f without the hat denotes the *derived function* defined by $f(x, y) = \hat{f}(x, \hat{y})$ for pairs (x, y) for which $(x, \hat{y})$ is in the domain of definition of $\hat{f}$. The domain of definition of the derived function is either $X \otimes Y$ or the set $\mathcal{P} \times [\mathcal{P}]$ if $\hat{f}$ is defined on $\mathcal{P} \times \hat{}[\mathcal{P}]$.

Every derived function *depends only on response* in the sense that $f(x, y_1) = f(x, y_2)$ if only $y_1 \hat{\sim} y_2$. If response is a surjection, there is a natural one-to-one relation between $\hat{Y}$-type functions and Y-type functions which depend only on response.

Consider an f-type function defined on all of $X \otimes Y$. For $y \in Y$, f^y denotes the *marginal function given* y, defined on $]y[$ by $f^y(x) = f(x, y)$. And the *marginal function given* $x \in X$ is the function f_x defined by $f_x(y) = f(x, y)$ for $y \in [x]$. We write $f^y < \infty$ on $\mathcal{P}$ to express, firstly, that $y \succ \mathcal{P}$ so that f^y is well defined on all of $\mathcal{P}$ and, secondly, that this marginal function is finite on $\mathcal{P}$. We write $f^y < \infty$ if $f^y < \infty$ on X.

5 Knowledge, Perception and Interaction

Observer strives for *knowledge*, conceived as the *synthesis of extensive experience*. Referring to probabilistic thinking, we could point to situations where accidental experimental data are smoothed out over time as you enter the regime of the law of large numbers. However, Observers endeavours may result in less definitive insight, a more immediate reaction which we refer to as *perception*. It reflects how Observer perceives situations from the world or, with a different focus, how situations from the world are presented to Observer.

In the same way as we have introduced truth- and belief instances, we consider *knowledge instances*, also referred to as *perceptions*. Typically, they are denoted by z and taken from a set denoted Z, the *knowledge base* or *perception base*.

A simplifying assumption for our modelling is that the rules of the world Ω contain a special function, $\hat{\Pi}$, which maps $X \otimes \hat{Y}$ into Z, generically, $z = \hat{\Pi}(x, w)$. The derived function, Π, then maps $X \otimes Y$ into Z. Both functions are referred to as the *interactor*. The context will show which one we have in mind, $\hat{\Pi}$ or Π.

Thus knowledge can be derived deterministically from truth and belief alone, and as far as belief is concerned, we only have to know the associated response. In terms of perception, Observers perception z of an atomic situation (x, y) is given by $z = \Pi(x, y) = \hat{\Pi}(x, \hat{y})$.

We consider the world as characterized by the interactor. We may thus talk about the world with interactor Π, $\Omega = \Omega_\Pi$. The rules of the world may contain other structural elements, but such elements are not specified in the present study. Possibilities which could be considered in future developments include *context*, *noise from the environment*, and *dynamics*. To some extent, such features can be expressed in the present modelling by defining X, Y and Z appropriately and by introducing suitable interpretations.

In case response is a bijection and Z contains X as well as Y we may consider the interactors Π_1 and Π_0 defined by $\Pi_1(x, y) = x$, respectively $\Pi_0(x, y) = y$. The associated worlds are $\Omega_1 = \Omega_{\Pi_1}$ and $\Omega_0 = \Omega_{\Pi_0}$. In Ω_1, *"what you see is what is true"*, whereas in Ω_0, *"you only see what you believe"*. The world Ω_1 is the *classical world* where, optimistically, *truth can be learned*, whereas, in Ω_0, you cannot learn anything about truth. We refer to Ω_0 as a *black hole*. It is a narcissistic world, a world of extreme scepticism, only reflecting Observers beliefs and bearing no trace of Nature. If Z is provided with a linear structure, we can consider further interactors Π_q depending on a parameter q by putting $\Pi_q(x, y) = qx + (1 - q)y$. Worlds associated with interactors of this type are denoted Ω_q. These are the worlds we find of relevance regarding Tsallis entropy, cf. Section 22.

The simplest world to grasp is the classical world, but also the worlds Ω_q and even a black hole contain elements which are familiar to us from daily experiences, especially in relation to certain psychological phenomena. In this connection we point to *placebo effects*, cf. Benedetti [19], and to *visual attention*, cf. Bundesen and Habekost [18]. Presently, the relevance of our modelling in relation to these phenomena is purely qualitative.

Considering examples as indicated above, it is natural to expect that knowledge is of a nature closely related to the nature of truth and of belief. A key case to look into is that $Z = X = Y$. However, we shall not make any general assumption in this direction. What we shall do is to follow the advice of Shannon, as far as possible avoiding assumptions which depend on concrete semantic interpretations. As a consequence we shall only in Section 22 introduce more specific assumptions about the representation of knowledge. Until then, and starting with the section to follow, necessary quantitative structures will be introduced without explicit reference to knowledge.

6 Effort and Description

We turn to the introduction of the key quantitative tool we shall work with. In so doing, we will be guided by the view that *perception requires effort*. Expressed differently, *knowledge is obtained at a cost*. Since, according to the previous section, knowledge can be derived from truth and belief, in fact from truth and action, no explicit reference to knowledge is necessary. Instead, we model effort by a certain bivariate function defined on $X \otimes \hat{Y}$, the *effort function*. The rules of the world Ω may not point directly to an effort function which Observer can favorably work with. Or there may be several sensible functions to choose from. The actual selection is considered a task for Observer.

Effort, description, experiment and measurement are related concepts. Description is intended to aid Observer in his encounters with situations from the world. Logically, description comes before effort. Effort arises when specific ideas about description are developed into a method of description, an experiment. The implementation of such a method or the performance of the associated experiment involves a cost and this is what we conceive as specified quantitatively by the effort function.

Description depends on semantic interpretations and is often thought of in loose qualitative terms. However, in order to develop precise concepts which can be communicated among humans, quantitative elements will invariably appear, typically through a finite set of certain real-valued functions (descriptors). The descriptors of Section 22 is an indication of what could be involved.

Imagine now that somehow Observer has chosen all elements needed – response, actions, experiments – and settled for an effort function, $\hat{\Phi} = \hat{\Phi}(x, w)$ defined on $X \otimes \hat{Y}$. Let us agree on what a "good" effort function should mean. Generally speaking, Observer should aim at experiments with low associated effort. Consider a fixed truth instance x and the various possible actions, in principle free to be any action which controls x. It appears desirable that the action adapted to x will be the one preferred by Observer. Thus effort should be minimal in this case, i.e. $\hat{\Phi}(x, w) \geq \hat{\Phi}(x, \hat{x})$ should hold. Further, if the inequality is sharp except for the adapted action, this will have a training effect which, over time, will encourage Observer to choose the optimal action, $\hat{x}$.

Formally, we define a $\hat{Y}$*-effort function* as a function $\hat{\Phi}$ on $X \otimes \hat{Y}$ with values in $]-\infty, +\infty]$ such that

$$\hat{\Phi}(x, w) \geq \hat{\Phi}(x, \hat{x}) \text{ for all } (x, w) \in X \otimes \hat{Y} . \tag{5}$$

Thus, for all $x \in X$, $\hat{x} \in \arg\min \hat{\Phi}_x$. The actual minimum value of $\hat{\Phi}_x$ is the *entropy* ($\hat{Y}$-domain) of x for which we use the notation $\hat{\mathrm{H}}(x)$. This quantity will be discussed more thoroughly in the sequel. If $w_\emptyset \in \hat{Y}$, it is to be expected that $\hat{\Phi}(x, w_\emptyset) = 0$ when $w_\emptyset \succ x$.

The effort function is *proper*, if, for any $x \in X$ with $\hat{\mathrm{H}}(x) < \infty$, the minimum value of $\hat{\Phi}_x$ is only achieved for the control $\hat{x}$ adapted to x.[3] As opposed to this notion we have the notion of a *degenerate* effort function which is one which only depends on the first argument x, i.e., for all $x \in X$, $\hat{\Phi}_x$ is a constant function.

Note that effort may be negative (but not $-\infty$). This flexibility will later be convenient as it will allow us to pass freely from notions of effort to notions of utility by a simple change of sign. But normally, effort functions will be non-negative.

The set of effort functions and the set of proper effort functions over $X \otimes \hat{Y}$ are ordered positive cones in a natural way. You may note that if, in a sum of effort functions, one of the summands is proper, so is the sum. Two effort functions $\hat{\Phi}_1$ and $\hat{\Phi}_2$, which only differ from each other by a positive finite factor are *scalarly equivalent*. If an effort function is proper, so is every scalarly equivalent one. There may be many non-scalarly equivalent effort functions. The choice among scalarly equivalent ones amounts to a choice of unit.

[3] A more general notion of properness is suggested in Section 11.

Proper effort functions could have been taken as the key primitive concept on which other concepts, especially response, can be based. To illustrate this, assume that $Y = X$ and consider a function $\hat{\Phi} : X \otimes \hat{Y} \mapsto]-\infty, \infty]$ such that, for every state x for which $\hat{\Phi}_x$ is not identically $+\infty$, $\arg\min \hat{\Phi}_x$ is a singleton. The minimal value of $\hat{\Phi}_x$ is again the entropy $\hat{\mathrm{H}}(x)$ and we may define the set of realistic states by $X_0 = \{\hat{\mathrm{H}} < \infty\}$ and, more importantly, response $x \mapsto \hat{x}$ by the requirement that $\hat{\Phi}(x, \hat{x}) = \min \hat{\Phi}_x$. This defines response uniquely on X_0 and for $x \notin X_0$, the definition of $\hat{x}$ is really immaterial and any element in $\hat{Y}$ which controls x will do.

Turning to the Y-domain, we define a *Y-effort function*, as a function $\Phi : X \otimes Y \mapsto]-\infty, \infty]$ such that

$$\Phi(x, y) \geq \Phi(x, x) \text{ for all } (x, y) \in X \otimes Y\,. \qquad (6)$$

If the model contains atomic situation of certainty, it is natural to expect that effort vanishes for such situations.

The effort function is *proper* if equality holds in (6) only if either $\Phi(x, x) = \infty$ or else $y = x$. We also express this by saying that Φ satisfies the *perfect match principle*. An effort function is *degenerate* if, for every $(x, y) \in X \otimes Y$, $\Phi(x, y) = \Phi(x, x)$. *Entropy* (in the Y-domain) is given by $\mathrm{H}(x) = \Phi(x, x)$.

The notions just introduced were defined directly with reference to the Y-domain. However, it lies nearby also to consider functions which can be derived from $\hat{Y}$-effort functions $\hat{\Phi}$. They are *derived effort functions* and, in case $\hat{\Phi}$ is proper, *proper derived effort functions*. The two strategies for definitions, intrinsic and via derivation, give slightly different concepts. In case response is injective, the resulting notions are equivalent. In general, derived effort functions are *response-dependent* by which we mean that if $y_1 \succ x$ and $y_2 \succ x$ and if $y_1 \hat{\sim} y_2$ then $\Phi(x, y_1) = \Phi(x, y_2)$. In the other direction, for a proper derived effort function, you can only conclude response-equivalence, $y \hat{\sim} x$, if $\Phi(x, y) = \Phi(x, x)$ and $\Phi(x, x) < \infty$.

Formally, the definitions related to Y-effort functions may be conceived as a special case of the definitions pertaining to the $\hat{Y}$-domain (put $\hat{Y} = Y$ and take the identity map for response).

We shall talk about effort functions without a qualifying prefix, $\hat{Y}$ or Y, if it is clear from the context what we have in mind. We shall always point out if we have derived functions in mind. And from now on, we shall only use H, not $\hat{\mathrm{H}}$, as notation for entropy.

The effort functions introduced determine *net effort*. However, the implementation of the method of description – which we imagine lies behind – may, in addition to a specific cost, entail a certain *overhead* and, occasionally, it is appropriate to include this overhead in the effort. We refer to Section 22 for instances of this.

We imagine that the choice of effort function involves considerations related to knowledge and to the rules of the world. However, once $\hat{\Phi}$, hence also Φ are fixed, these other elements are only present indirectly. The ideas of Section 5 have thus mainly served as motivation for the further abstract development. The ideas will be taken up again when in Section 22 we turn to a study of probabilistic models.

The author was led to consider proper effort functions in order to illuminate certain aspects of statistical physics, cf. [34], [37]. However, the ideas have been around for quite some time, especially among statisticians. For them it has been more natural to work with functions taken with the reverse sign by looking at "score" rather than effort. Our notion of proper effort functions, when specialized to a probabilistic setting, matches the notion of *proper scoring rules* as you find it in the statistical literature. As to the literature, Csiszár [40] comments on the early sources, including Brier [41], a forerunner of research which followed, and Good [39], Savage [42] (see e.g. Section 9.4) and Fischer [43]. See also the reference work [44] by Gneiting and Raftery. For research of Dawid and collaborators – partly in line with what you find here – see [45], [28], [46] and [47].

7 Basic Concepts of Information, Information Triples

As advocated in the last section, effort is a notion of central importance. However, this notion should not stand alone but be discussed together with other fundamental concepts of information. This point of view will be emphasized by the introduction of a notion of *information triples*, the main notion of the present study. We start by philosophizing over the very concept of information.

Information in any particular situation concerns truth. If $\mathcal{P}$ is a preparation, "$x \in \mathcal{P}$" signifies that the true state is to be found among the states in $\mathcal{P}$. If $\mathcal{P}$ is a singleton, we talk about *full information* and use the notation "x" rather than "$x \in \{x\}$"; otherwise, we talk about *partial information*.

We shall not be concerned with how information can be obtained – if at all. Perhaps, Observer only speculates about the potential possibility of acquiring information, either through his own activity or otherwise, e.g. via the involvement of an aid or a third party, an informer

Information will be related to quantitatively defined concepts. As our basis we take a proper effort function $\hat{\Phi}$. Following Shannon we disregard semantic content. Instead, we focus on the possibility for Observer to benefit from information by a saving of effort. Accordingly, we view $\hat{\Phi}(x, w)$ as the information content of "x" in an atomic situation with x as truth instance and w as action or control – indeed, if you are told that x is the true state, you need not allocate the effort $\hat{\Phi}(x, w)$ to the situation which you were otherwise prepared to do. The somewhat intangible and elusive concept of "information" is thus measured by the more concrete and physical notion of effort. The unit of information is, therefore, the same as the unit used for effort.

There is a huge literature elucidating what information really "*is*". Suffice it here to refer to [21] and, as an example of a discussion more closely targeted on our main themes, we refer to Caticha [48] who maintains that "*Just as a force is defined as that which induces a change in motion, so information is that which induces a change in beliefs*". One may just as well – or even better – focus on action. Then we can claim that *"information" is that which induces a change of action.*

The undisputed central concept of the theory developed by Shannon is that of *entropy.* This concept was already introduced in the preceding section. Here, we elaborate on possible interpretations. One view is that entropy is *guaranteed saving of effort.* With effort given by $\hat{\Phi}$ we are led to define the entropy associated with the information "x" as the minimum over w of $\hat{\Phi}(x, w)$. By (5), this equals $\hat{\Phi}(x, \hat{x})$. Denoting, as agreed, entropy by H, we therefore have

$$\mathrm{H}(x) = \hat{\Phi}(x, \hat{x})\,. \tag{7}$$

The considerations above make most sense if, one way or another, Observer eventually obtains full information about the true state. However, if instead you view entropy as *necessary allocation of effort*, understood as the effort you have to allocate in order to have a chance to obtain full information, it does not appear important actually to obtain that information.[4] As yet a third route to entropy we suggest to view it as a quantitative expression of the *complexity* of the various states, maintaining that to evaluate complexity, Observer may use *minimal accepted effort*, the effort he is willing to allocate to the various states in order to obtain the information in question.

Entropy may also be obtained with reference only to the Y-domain.

[4]Instead of "entropy" one could have suggested a more neutral terminology such as "necessity". This may be considered less awkward when you turn to other applications of the abstract theory than classical Shannon theory.

Indeed, with Φ the derived effort function, for each state x,

$$\mathrm{H}(x) = \Phi(x, x)\,.$$

Whichever route to entropy you take – including the game theoretical route of Section 10 – subjective elements will be involved, typically through Observers choice of description and associated experiments. If, modulo scalar equivalence, the actual world only allows one proper effort function, then entropy and notions related to entropy, are of a more objective nature. We shall later see examples of such worlds but even then, subjective elements may enter through inference by Observer regarding which world is the actual one.

Apart from effort itself, and the derived notion of entropy, we turn to the introduction of two other basic concepts which make sense in our abstract setting, viz. *redundancy* for the $\hat{Y}$-domain and its counterpart, *divergence*, for the Y-domain.

To define redundancy, consider an atomic situation $(x, w) \in X \otimes \hat{Y}$. Then *redundancy* $\hat{\mathrm{D}}$ between x and w is measured by the difference between actual and minimal effort, i.e., ideally, as

$$\hat{\mathrm{D}}(x, w) = \hat{\Phi}(x, w) - \mathrm{H}(x)\,. \tag{8}$$

Assume, for a moment, that entropy is finite-valued. Then redundancy in (8) is well defined. Furthermore, redundancy is non-negative and only vanishes under a perfect match: $\hat{\mathrm{D}}(x, w) \geq 0$ and $\hat{\mathrm{D}}(x, w) = 0 \Leftrightarrow w = \hat{x}$.

However, we find it important to be able deal with models for which entropy may be infinite. We do that by simply assuming that appropriate versions of redundancy and divergence exist with desirable properties. The simple device we shall apply in order to reach a sensible definition is to rewrite the defining relation (8), isolating effort on the left hand side.

With the above preparations, we are ready to introduce the key concepts of our study. We start with concepts for the $\hat{Y}$-domain and follow up after that by parallel concepts for the Y-domain.

We shall study certain triples $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ of functions taking values in $]-\infty, \infty]$ with $\hat{\Phi}$ and $\hat{\mathrm{D}}$ defined on $X \otimes \hat{Y}$ and H defined on X. If need be we may talk about triples over $X \otimes \hat{Y}$ or we may point to the $\hat{Y}$-domain. Such triples must satisfy special conditions in order to be of interest. The four most important properties are *linking* (L), *the fundamental inequality* (F), *soundness* (S) and *properness* (P). *Linking* is the property:

$$\forall (x, w) \in X \otimes \hat{Y} :\ \hat{\Phi}(x, w) = \mathrm{H}(x) + \hat{\mathrm{D}}(x, w), \tag{9}$$

also referred to as the *linking identity*. It may be written shortly as $\hat{\Phi} = \mathrm{H} + \hat{\mathrm{D}}$ or, formally correct with $\hat{\mathrm{pr}}$ the projection of $X \otimes \hat{Y}$ onto X, as $\hat{\Phi} = \mathrm{H} \circ \hat{\mathrm{pr}} + \hat{\mathrm{D}}$.

The *fundamental inequality* is the property:

$$\forall (x, w) \in X \otimes \hat{Y} : \hat{\mathrm{D}}(x, w) \geq 0; \tag{10}$$

– *soundness* the property:

$$\forall x \in X : \hat{\mathrm{D}}(x, \hat{x}) = 0; \tag{11}$$

– and *properness* the property:

$$\forall (x, w) \in X \otimes \hat{Y},\ w \neq \hat{x} : \hat{\mathrm{D}}(x, w) > 0. \tag{12}$$

An *information triple* is a triple $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ which satisfies the three first conditions. For such triples the function $\hat{\Phi}$ is the associated *effort function*, H the associated *entropy* and $\hat{\mathrm{D}}$ the associated *redundancy*. This does not conflict with previous terminology. In particular, the associated effort function is indeed an effort function in the sense of Section 6.

Information triples with the same redundancy are said to be *equivalent*. Equivalent triples may have quite different properties and one may search for representatives with good properties.

A process of addition or, more generally, integration, of information triples is defined in the natural way and will be taken up in Section 18.

A *proper information triple* is an information triple for which redundancy is proper, i.e. (12) holds. Clearly, the effort function of a proper information triple is proper in the sense of Section 6. And if a triple is proper, so is any equivalent one.

An information triple is *degenerate* if redundancy vanishes: $\hat{\mathrm{D}}(x, w) = 0$ for all $(x, w) \in X \otimes \hat{Y}$. The effort function of a degenerate information triple is degenerate.

Among the four defining properties, the last three (FSP) only involve redundancy. A function $\hat{D}$ defined on $X \otimes \hat{Y}$ is a *general redundancy function* if it satisfies the fundamental inequality as well as the requirements of soundness and properness. Note that for such a redundancy function, $(\hat{\mathrm{D}}, 0, \hat{\mathrm{D}})$ is a proper information triple and that any equivalent information triple may be obtained from $(\hat{\mathrm{D}}, 0, \hat{\mathrm{D}})$ by adding any function on X with values in $]-\infty, \infty]$, taking this function as the entropy function. Structurally, what is involved is that you add information triples, one of which is proper, viz. you add $(\hat{\mathrm{D}}, 0, \hat{\mathrm{D}})$ and $(\mathrm{H}, \mathrm{H}, 0)$, cf. also Section 18.

Normally, given a proper effort function $\hat{\Phi}$, there is a natural way to extend the redundancy function as defined by (8) when $\mathrm{H}(x) < \infty$, so that a proper information triple emerges. For this reason, we may talk about the information triple *generated by* $\hat{\Phi}$. And then, the problem of indeterminacy of redundancy disappears. The slightly strengthened assumption that redundancy can be defined "appropriately" on all of $X \otimes \hat{Y}$ will, as it turns out, present no limitation in most concrete cases of interest.

We turn briefly to Y-type triples. They are triples $(\Phi, \mathrm{H}, \mathrm{D})$ with Φ and D defined on $X \otimes Y$ and H defined on X. Key properties to consider are quite parallel to what we have discussed for the $\hat{Y}$-domain:

$$\forall (x,y) \in X \otimes Y : \ \Phi(x,y) = \mathrm{H}(x) + \mathrm{D}(x,y) \text{ (linking, L);} \tag{13}$$

$$\forall (x,y) \in X \otimes Y : \ \mathrm{D}(x,y) \geq 0 \text{ (fundamental inequality, F);} \tag{14}$$

$$\forall x \in X : \ \mathrm{D}(x,x) = 0 \text{ (soundness, S);} \tag{15}$$

$$\forall (x,y) \in X \otimes Y,\ y \neq x : \ \mathrm{D}(x,y) > 0 \text{ (properness, P).} \tag{16}$$

An *information triple* in the Y-domain is a triple which satisfies the conditions L, F and S. For such triples, Φ is the associated *effort*, H the associated *entropy* and D the associated *divergence*.

A *proper information triple* is an information triple $(\Phi, \mathrm{H}, \mathrm{D})$ for which divergence is proper. If divergence vanishes, the triple is *degenerate*. The effort function of a proper information triple is proper in the sense of Section 6 and the effort function of a degenerate triple is degenerate.

A triple $(\Phi, \mathrm{H}, \mathrm{D})$ is a *derived information triple*, respectively a *derived proper information triple*, if there exists a triple $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ satisfying the corresponding properties for the $\hat{Y}$-domain such that Φ is derived from $\hat{\Phi}$ and D from $\hat{\mathrm{D}}$.

If response is injective, the two types of information triples for the Y-domain – directly defined or defined via derivation – are equivalent notions.

A *general divergence function* D on $X \otimes Y$ is a function on $X \otimes Y$ which satisfies the F, S and P-requirements. A *general derived divergence function* is one which can be derived from a general redundancy function.

For the Y-domain, notions of equivalence (same divergence!) and of addition of information triples are defined in the obvious manner.

Instead of taking triples as introduced above as the basis, it is sometimes more natural to focus on triples of the "opposite nature". This refers to situations where it is appropriate to focus on a positively oriented quantity such as *utility* or *pay-off* rather than on effort. This is often the case for studies of economy, meteorology and statistics where one meets the notion of "score" as previously indicated. In order to distinguish the two types of triples from each other, we may refer to them as being *effort-based*, respectively *utility-based*.

For a utility-based triple $(\hat{\mathrm{U}}, \mathrm{M}, \hat{\mathrm{D}})$ in the $\hat{Y}$-domain or for a utility-based triple $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ in the Y-domain, $\hat{\mathrm{U}}$ and U are *utility*, M is *max-utility*, and $\hat{\mathrm{D}}$ is *redundancy* and D *divergence*. The linking identity takes the form $\hat{\mathrm{U}} = \mathrm{M} - \hat{\mathrm{D}}$ $(\mathrm{U} = \mathrm{M} - \mathrm{D})$ which can never result in the indeterminate form $\infty - \infty$ since, by definition, $\hat{\mathrm{U}}$ and U, hence also M, can never assume the value $+\infty$.

For the $\hat{Y}$-domain, $(\hat{\mathrm{U}}, \mathrm{M}, \hat{\mathrm{D}})$ is an *utility-based information triple* if $(-\hat{\mathrm{U}}, -\mathrm{M}, \hat{\mathrm{D}})$ is an effort-based information triple. And a *proper utility-based information triple* or a *proper derived utility-based information triple* or a *degenerate* utility-based triple is defined in the obvious way, again by reference to the associated effort-based triple $(-\hat{\mathrm{U}}, -\mathrm{M}, \hat{\mathrm{D}})$.

In view of the main examples we have in mind, we have found it most illuminating to take effort rather than utility as the basic concept to work with, and hence to develop the main results for effort-based quantities. Anyhow, even if you are primarily interested in considerations based on effort, you are easily led to consider also utility-based quantities as we shall see in Section 8.

The concept of proper information triples is, except for minor technical details, equivalent to the concept of proper effort functions. Apart from a slight technical advantage, the triples constitute a preferable base for information theoretical investigations as the three truly basic notions of information are all emphasized together with their basic interrelationship – the linking identity. Historically, the notions arose for classical probabilistic information theoretical models, cf. Section 23. Effort functions go back to Kerridge [49] who coined the term *inaccuracy*, entropy goes back to Shannon [1] and divergence to Kullback [50]. The term "redundancy" which we have used for another side of divergence, corresponds to one usage in information theory, though there the term is used in several other ways which are not expressed in our abstract setting.[5]

Our way to information triples was through effort and one may ask why we did not go directly to the triples. For one thing, triples lead

[5] It is tempting for the author to point to the pioneering work of Edgar Rubin going back to the twenties. Unfortunately, this was only published posthumously in 1956, cf. [51] and [52]. Rubin made experiments over human speech and focused on what he called *the reserve of understanding*. This is a quantitative measure of the amount you can cut out of a persons speech without seriously disrupting a listeners ability to understand what has been said. It can be conceived as a forerunner of the notion of redundancy.

to a smooth axiomatic theory, for the beginnings of which see Topsøe [53]. However, though axiomatization can be technically attractive, we find that a focus on interpretation as in our more philosophical and speculative approach, is of primary importance and contributes best to an understanding of central concepts of information. Axiomatics only comes in after basic interpretations are in place.

8 Relativization, Updating

In this section we shall work entirely in the Y-domain. We start by considering a proper effort-based information triple $(\Phi, \mathrm{H}, \mathrm{D})$ over $X \otimes Y$. Often, it is natural to measure effort relative to some standard performance rather than by Φ itself. An especially important instance of this kind of *relativization* concerns situations where Observer originally fixed a *prior*, say $y_0 \in Y$, but now wants to update his belief by replacing y_0 with a *posterior* y. Perhaps, Observer – through his own actions or via an informer – has obtained the information "$x \in \mathcal{P}$" for some preparation $\mathcal{P}$. If $y_0 \notin \mathcal{P}$, Observer may want to replace y_0 by a posterior $y \in \mathcal{P}$. In a first attempt of a reasonable definition, the associated *updating gain* is given by the quantity $\mathrm{U}_{|y_0}$ obtained by comparing performance under the posterior with performance under the prior:

$$\mathrm{U}_{|y_0}(x, y) = \Phi(x, y_0) - \Phi(x, y)\,. \tag{17}$$

A difficulty with (17) concerns the possible indeterminate form $\infty - \infty$. If we ignore the difficulty and apply the linking identity (13) to both terms in (17), entropy $\mathrm{H}(x)$ cancels out and we find the expression

$$\mathrm{U}_{|y_0}(x, y) = \mathrm{D}(x, y_0) - \mathrm{D}(x, y)\,. \tag{18}$$

This is less likely to be indeterminate. When not of the indeterminate form $\infty - \infty$, we therefore agree to use (18) as the formal definition of updating gain, more precisely of *relative updating gain with* y_0 *as prior*. For the present study, we shall only work with updating gain when D^{y_0} is finite on some preparation $\mathcal{P}$ under consideration. Assuming that this is the case, we realize that

$$(\mathrm{U}_{|y_0}, \mathrm{D}^{y_0}, \mathrm{D}) \tag{19}$$

is a utility-based information triple over $\mathcal{P} \otimes Y$. For such triples we put $Y_{\mathrm{det}} = \{y_0\}$, i.e. we take y_0 as the only certain belief instance. Max-utility is identified as the marginal function D^{y_0} on $\mathcal{P}$ and divergence is the original divergence function restricted to $\mathcal{P} \otimes Y$.

It is important to note that the triples which occur in this way by varying y_0 and $\mathcal{P}$ do not require the full effort function Φ in order to make sense. It suffices to start out with a general divergence function on $X \otimes Y$ in order for the construction to make sense. When the construction is based on a general divergence function D, we refer to (19) as the *updating triple* generated by D and with y_0 as prior.

Though rather trivial, the observations regarding updating gain are important as they show that results in that setting may be obtained from results based on effort. To emphasize this, we introduce – based only on a general divergence function D – the effort-based information triple *associated with* (19) as the triple

$$(\Phi_{|y_0}, -\mathrm{D}^{y_0}, \mathrm{D}) \tag{20}$$

with $\Phi_{|y_0}$, given by

$$\Phi_{|y_0}(x,y) = \mathrm{D}(x,y) - \mathrm{D}(x,y_0)\,. \tag{21}$$

This is a perfectly feasible effort-based triple over $\mathcal{P} \otimes Y$ whenever D^{y_0} is finite on $\mathcal{P}$. Clearly, it is proper.

In Sections 12 and 15 we shall derive results about minimum divergence (information projections) from results about maximum entropy by exploiting the simple facts here uncovered.

As we have seen, natural information triples may be derived from a general divergence function by a simple process of *relativization*. While we are at it, we note that in case $Y = X$, also *reverse divergence* $(x,y) \mapsto \mathrm{D}(y,x)$ defines a genuine divergence function on $X \otimes Y$[6]. Therefore, if $\mathrm{D}_{y_0} < \infty$ and we put $\Phi^r_{|y_0}(x,y) = \mathrm{D}(y,x) - \mathrm{D}(y_0,x)$,

$$\big(\Phi^r_{|y_0}(x,y), -\mathrm{D}(y_0,x), \mathrm{D}(y,x)\big) \tag{22}$$

defines a genuine proper information triple (when restricting the variables x and y appropriately). However, these triples are not found to be that significant.

9 Feasible Preparations

We claim that description is the key to what can be known, a key to the "knowable". Not every possible information "$x \in \mathcal{P}$" for any odd preparation $\mathcal{P}$ can be expected to reflect a realistic situation. The question we ask is "what can Observer know?" or "what kind of information can Observer hope to obtain?". We thus want to investigate

[6] in contrast, reverse description effort need not define a genuine effort function.

"limits to knowledge" and "limits to information". In order to provide an answer, we shall identify classes of preparations which represent *feasible information*. These classes will be defined with reference to an effort function $\hat{\hat{\Phi}}$. For this section, $\hat{\hat{\Phi}}$ need not be proper.

Given $w \in \hat{Y}$ and a *level* $h < \infty$, we define the *level set* $\mathcal{P}^w(h)$ and the *sublevel set* $\mathcal{P}^w(h^\downarrow)$ by

$$\mathcal{P}^w(h) = \{\hat{\Phi}^w = h\}\,;\ \mathcal{P}^w(h^\downarrow) = \{\hat{\Phi}^w \leq h\}\,, \tag{23}$$

i.e. as the set of states which are controlled by w, either at the *level* h or at the *maximum level* h. These sets are genuine preparations whenever they are non-empty. When w is the response of a state $x \in X$, $\mathcal{P}^w(h^\downarrow)$ is non-empty whenever $h \geq \mathrm{H}(x)$. As level- and sublevel sets for other functions will appear later on, cf. Section 13, we may for clarity refer to $\mathcal{P}^w(h)$ and to $\mathcal{P}^w(h^\downarrow)$ as, respectively, $\hat{\Phi}^w$*-level sets* and $\hat{\Phi}^w$*-sublevel sets*.

The preparations in (23) are *primitive strict*, respectively *primitive slack preparations*. A *general strict*, respectively a *general slack preparation* is a finite non-empty intersection of primitive strict, respectively primitive slack preparations. The *genus* of these preparations is the smallest number of primitive preparations (either strict or slack as the case may be) which can enter into the definition just given. Thus primitive preparations are of genus 1.

If $\mathbf{w} = (w_1, \cdots, w_n)$ are elements of $\hat{Y}$ and $\mathbf{h} = (h_1, \cdots, h_n)$ are real numbers, the sets

$$\mathcal{P}^{\mathbf{w}}(\mathbf{h}) = \bigcap_{i \leq n} \mathcal{P}^{w_i}(h_i) \text{ and } \mathcal{P}^{\mathbf{w}}(\mathbf{h}^\downarrow) = \bigcap_{i \leq n} \mathcal{P}^{w_i}(h_i^\downarrow) \tag{24}$$

define strict, respectively slack preparations whenever they are non-empty. When using these expressions, it is natural to assume that n is the genus of the preparations considered. The set $\mathcal{P}^{\mathbf{w}}(\mathbf{h})$ is the *corona* of $\mathcal{P}^{\mathbf{w}}(\mathbf{h}^\downarrow)$ whenever it is non-empty.

The preparations introduced are those we consider to be feasible and we formally refer to them as the *feasible preparations*. They provide the answer to the question about what can be known. They are the key ingredients in situations which Observer can be faced with. In any such situation a main problem concerns *inference*, an issue we shall take up in the next section.

Of special interest are families of feasible preparations. Given $\mathbf{w} = (w_1, \cdots, w_n)$, we denote by $\mathbb{P}^{\mathbf{w}}$, respectively $\mathbb{P}^{\mathbf{w}\downarrow}$, the families which consist of all preparations $\mathcal{P}^{\mathbf{w}}(\mathbf{h})$, respectively $\mathcal{P}^{\mathbf{w}}(\mathbf{h}^\downarrow)$, which can be obtained by varying $\mathbf{h}$.

Clearly, the feasible preparations can also be expressed by reference to the derived effort function Φ rather than $\hat{\hat{\Phi}}$. We use the notation

$\mathcal{P}^y(h)$ and $\mathcal{P}^y(h^{\downarrow})$ for, respectively, the Φ^y-*level set* $\{\Phi^y = h\}$ and the Φ^y-*sublevel set* $\{\Phi^y \leq h\}$. If $\hat{y} = w$, $\mathcal{P}^y(h) = \mathcal{P}^w(h)$ and $\mathcal{P}^y(h^{\downarrow}) = \mathcal{P}^w(h^{\downarrow})$[7]. For finite sequences $\mathbf{y} = (y_1, \cdots, y_n)$ of elements of Y and $\mathbf{h} = (h_1, \cdots, h_n)$ of real numbers, the sets $\mathcal{P}^{\mathbf{y}}(\mathbf{h})$ and $\mathcal{P}^{\mathbf{y}}(\mathbf{h}^{\downarrow})$ are defined in the obvious manner as are the families of preparations $\mathbb{P}^{\mathbf{y}}$, respectively $\mathbb{P}^{\mathbf{y}\downarrow}$.

From a formal point of view, it does not matter if we use $\mathcal{P}^w$-type sets or $\mathcal{P}^y$-type sets as the basis for the definition of feasible preparations. However, entering into more speculative interpretations, the $\mathcal{P}^w$-type sets which emphasize control seem preferable. Individual controls $w \in \hat{Y}$ or a collection of such controls point to experiments which Observer may perform. An experimental setup identifies a certain preparation, the preparation of states consistent with the setup, and thus determines what is known to Observer. Determining all preparations which can arise in this way, we are led to the class of feasible preparations as defined above.

As to the nature of the various controls, we imagine that they are derived from description. To control a situation, you must be able to describe it, and with a description you have the key to control. We might imagine that, corresponding to a control w, Observer can realize a certain experimental setup consisting of various parts – measuring instruments and the like. In particular, there is a special handle which is used to fix the level of effort. If the level, perhaps best thought of as a kind of temperature, is fixed to be h, the states available to Nature are those in the appropriate feasible preparation. Several experiments can be carried out with the same equipment by adjusting the setting of the handle. If Observer wants to constrain the states by other means, he can add equipment corresponding to another control w' and choose a level h' for the experimental setup constructed based on w'. The result is a restriction of the available states to the intersection of the two preparations involved. If the preparation is $\mathcal{P}^w(h^{\downarrow})$ and the actual state is not inside this preparation, you may imagine that the result is overheating and breakdown of the experimental setup! Thus you must keep the state inside the preparation and this may well be what requires an effort as specified by $\hat{\Phi}$.

10 Inference via Games

For this section, $(\hat{\Phi}, \mathrm{H}, \hat{D})$ is a proper effort-based information triple over $X \otimes \hat{Y}$ and $(\Phi, \mathrm{H}, \mathrm{D})$ the derived triple over $X \otimes Y$.

[7] note that for an expression such as $\mathcal{P}^q(h)$, the nature of q determines if this is a $\hat{\Phi}$- or a Φ-level set.

Consider partial information "$x \in \mathcal{P}$". In practice, $\mathcal{P}$ will be a feasible preparation, but we need not assume so for this section.

The process of *inference* concerns the identification of "sensible" states in $\mathcal{P}$ – ideally only one such state, the *inferred state*. In many cases, this can be achieved by game theoretical methods involving a two-person zero-sum game. As it turns out, this will result in *double inference* where also either control instances or belief instances will be identified – ideally, only one such instance, the *inferred control* or the *inferred belief instance* as the case may be.

An inferred state, say x^*, brings Observer as close as possible to the truth in a way specified by the method applied. On the other hand, focusing on control, an inferred control instance w^* is more of an instruction to Observer on how to act regarding the setup of experiments and performance of subsequent observations. You may say that actions by Observer as dictated by the control w^* is what is needed for Observer in order to justify the inference x^* about truth. In short, double inference gives Observer information both about *what* can be inferred about truth and *how*.

Given $\mathcal{P}$, we shall study two closely related two-person zero-sum games, the *control game* $\hat{\gamma}(\mathcal{P})$, and the *derived game* $\gamma(\mathcal{P})$. If need be, we may write $\hat{\gamma}(\mathcal{P}\,|\hat{\Phi})$ and $\gamma(\mathcal{P}\,|\Phi)$. The games have Nature and Observer as players and $\hat{\Phi}$, respectively Φ as *objective function*. Nature is understood to be a *maximizer*, Observer a *minimizer*. For both games, *strategies* involve the choice by Nature of a state in $\mathcal{P}$. Observer strategies for $\hat{\gamma}(\mathcal{P})$ are controls from which every state in $\mathcal{P}$ can be controlled. And for $\gamma(\mathcal{P})$, Observer strategies are belief instances from which every state in $\mathcal{P}$ is visible. Then pairs of *permissible strategies* for the two games are either pairs $(x, w) \in X \otimes \hat{Y}$ with $x \in \mathcal{P}$ and $w \succ \mathcal{P}$ or pairs $(x, y) \in X \otimes Y$ with $x \in \mathcal{P}$ and $y \succ \mathcal{P}$. In consistency with the discussion in Section 2, an observer strategy may be thought of as a strategy which is not "completely stupid" whatever the strategy of Nature as long as that strategy respects the requirement $x \in \mathcal{P}$. The choice of strategy for Observer may be a real choice, whereas, for Nature, it is often more appropriate to have a fictive choice in mind which reflects Observers thoughts about what the truth could be.

A trivial remark is in order regarding models where it is unnatural to work with controls and only belief, through Y and $X\otimes Y$, is involved. Then the basis will be a proper effort-based information triple $(\Phi, \mathrm{H}, \mathrm{D})$ over $X \otimes Y$ and only one type of game, $\gamma(\mathcal{P})$ will be studied. Formally, this may be considered a derived game by artificially introducing $\hat{Y} = Y$, $X \otimes \hat{Y} = X \otimes Y$, by taking response to be the identity map and by taking $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ to be identical with $(\Phi, \mathrm{H}, \mathrm{D})$. Thus the approach taken, based on objects for the $\hat{Y}$ domain, is the more general one.

Following standard philosophy of game theory, Observer should always be prepared for a choice by Nature which is least favourable. One can argue that in our setting anything else would mean that Observer would not have used all available information. The line of thought goes well with Jaynes thinking as collected in [9], though there you find no reference to game theory.

In order for our exposition to be self-contained and also because our games are slightly at variance with what is normally considered, we shall here give full details regarding definitions and proofs. As a general reference to game theory we point to [32].

Let us introduce basic notions for the control game and then comment more briefly on the derived game.

The two *values* of $\hat{\gamma}(\mathcal{P})$ are, for Nature,

$$\sup_{x\in\mathcal{P}} \inf_{w\succ x} \hat{\Phi}(x,w) \tag{25}$$

and, for Observer,

$$\inf_{w\succ\mathcal{P}} \sup_{x\in\mathcal{P}} \hat{\Phi}(x,w)\,. \tag{26}$$

Note the slight deviation from usual practice in that w in the infimum in (25) varies over $\hat{}[x]$ and not just over $\hat{}[\mathcal{P}]$ or some other set independent of x. Philosophically, one may argue that Nature does not know of the restriction to $\mathcal{P}$ – this is something Observer has arranged – and hence cannot know of any restriction besides the natural one $w\succ x$. As the infimum in (25) is nothing but the entropy $\mathrm{H}(x)$, the value for Nature, denoted $\mathrm{H}_{\max}(\mathcal{P})$, is the *maximum entropy value*

$$\mathrm{H}_{\max}(\mathcal{P}) = \sup_{x\in\mathcal{P}} \mathrm{H}(x)\,, \tag{27}$$

also referred to as the MaxEnt-*value*.

Problems on the determination of this value and associated strategies achieving the value (if any) are maximum entropy problems, MaxEnt-problems. If $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$, x^* is an *optimal strategy for Nature*, also referred to as a MaxEnt-*state* or MaxEnt-*strategy*. The archetypal concrete problems of this nature are discussed in Section 23.

As to the value for Observer, we identify the supremum in (26) with the *risk* associated with the strategy w and denote it by $\hat{\mathrm{Ri}}(w|\,\mathcal{P})$:

$$\hat{\mathrm{Ri}}(w|\,\mathcal{P}) = \sup_{x\in\mathcal{P}} \hat{\Phi}(x,w)\,. \tag{28}$$

The value for Observer then is the *minimal risk* of the game, also referred to as the MinRisk-*value*:

$$\hat{\mathrm{Ri}}_{\min}(\mathcal{P}) = \inf_{w\succ\mathcal{P}} \hat{\mathrm{Ri}}(w|\,\mathcal{P})\,. \tag{29}$$

An *optimal strategy for Observer* is a strategy $w^* \succ \mathcal{P}$ with $\hat{\text{Ri}}(w^*|\,\mathcal{P}) = \hat{\text{Ri}}_{\min}(\mathcal{P})$, also referred to as a MinRisk-*control* or a MinRisk-*strategy*.

The reader will easily verify the general validity of the *minimax inequality*:

$$\text{H}_{\max}(\mathcal{P}) \leq \hat{\text{Ri}}_{\min}(\mathcal{P})\,. \tag{30}$$

If this inequality holds with equality and defines a finite quantity, the game is said to be in *game theoretical equilibrium*, or just in *equilibrium*, and the common value of $\text{H}_{\max}(\mathcal{P})$ and $\hat{\text{Ri}}_{\min}(\mathcal{P})$ is the *value* of the game.

We need one more notion of equilibrium which we associate with the name of Nash[8]. A pair of permissible strategies (x^*, w^*) is a *Nash equilibrium pair for* $\hat{\gamma}(\mathcal{P})$ if, with these strategies, none of the players have an incentive to change strategy – provided the opponent does not do so either. This means, for Nature, that

$$\forall x \in \mathcal{P} : \hat{\Phi}(x, w^*) \leq \hat{\Phi}(x^*, w^*)\,, \tag{31}$$

and, for Observer, that

$$\forall w \succ \mathcal{P} : \hat{\Phi}(x^*, w) \geq \hat{\Phi}(x^*, w^*)\,. \tag{32}$$

The inequalities (31) and (32) constitute (a special case of) the celebrated *saddle-value inequalities* of game theory. Note that, in our case, one of these inequalities, (32), is automatic if (x^*, w^*) is an adapted pair. This implies that $x^* \in \text{ctr}(\mathcal{P})$ as follows from the following trivial observation:

Lemma 10.1. *If x^* and w^* are permissible strategies for the two players in $\hat{\gamma}(\mathcal{P})$ and if w^* is adapted to x^*, then $x^* \in \text{ctr}(\mathcal{P})$.*

Proof. By hypothesis, $x^* \in \mathcal{P}$, $w^* \succ \mathcal{P}$ and $w^* = \hat{x^*}$. Then $\hat{x^*} \succ \mathcal{P}$, hence, switching to the Y-domain, $x^* \succ \mathcal{P}$. As also $x^* \in \mathcal{P}$, $x^* \in \text{ctr}(\mathcal{P})$. □

We find that the game $\hat{\gamma}(\mathcal{P})$ is the most natural one to consider in view of the associated interpretations. However, we shall also formulate results for $\gamma(\mathcal{P})$. Then the focus is on belief instances and no mention of controls is needed. The values of $\gamma(\mathcal{P})$ are $\sup_{x\in\mathcal{P}} \inf_{y\succ x} \Phi(x, y)$ and $\inf_{y\succ\mathcal{P}} \sup_{x\in\mathcal{P}} \Phi(x, y)$ and notions of strategies and optimal strategies are defined in an obvious manner. We use the notation Ri for *risk* in this game, defined, for $y \succ \mathcal{P}$, as $\text{Ri}(y|\mathcal{P}) = \sup_{x\in\mathcal{P}} \Phi(x, y)$. Clearly, $\text{Ri}(y|\mathcal{P}) = \hat{\text{Ri}}(\hat{y}|\mathcal{P})$. Therefore, if $y_1 \hat{\sim} y_2$ with $y_1 \succ \mathcal{P}$ and $y_2 \succ \mathcal{P}$, the

[8]It should, however, be said that for the relatively simple case here considered (two players, zero sum), the ideas we need originated with von Neumann, see [54] and [55] and, for a historical study, see Kjeldsen [56].

associated risks are the same. The value for Nature in $\gamma(\mathcal{P})$ is $\mathrm{H}_{\max}(\mathcal{P})$, the same as for $\hat{\gamma}(\mathcal{P})$ and optimal strategies for Nature agree in the two games. The value for Observer in $\gamma(\mathcal{P})$ is $\mathrm{Ri}_{\min}(\mathcal{P}) = \sup_{y \succ \mathcal{P}} \mathrm{Ri}(y|\mathcal{P})$ which agrees with the value for Observer in $\hat{\gamma}(\mathcal{P})$ in case response is surjective – as well as in key cases considered in the theorems below. In general only $\mathrm{Ri}_{\min}(\mathcal{P}) \geq \hat{\mathrm{Ri}}_{\min}(\mathcal{P})$ can be asserted. We leave it to the reader to transform the notions of equilibrium and the form of the saddle-value inequalities to the game $\gamma(\mathcal{P})$.

Important results of game theory are non-constructive by nature and aim at securing equilibrium and existence of optimal strategies for wide classes of games. For our setting, this will be taken up in Section 15. However, for the present section we shall focus on the possibility to identify optimal strategies. This leads to problems which are easy to handle technically and yet, it may be argued that from an applied point of view such results are the more important ones.

In our first result, we point out basic properties of optimal strategies for the key case we shall deal with, that of a game in equilibrium for which both players have optimal strategies. The base is still a proper effort-based information triple together with a preparation[9].

Theorem 10.1. *[Basics] The game $\hat{\gamma}(\mathcal{P})$ is in equilibrium and both players have optimal strategies, if and only if these properties also hold for the game $\gamma(\mathcal{P})$. Assume now that this is the case.*

Then there is only one optimal strategy, say w^*, for Observer in $\hat{\gamma}(\mathcal{P})$. All optimal strategies for Nature in $\hat{\gamma}(\mathcal{P})$ lie in $\mathrm{ctr}(\mathcal{P})$ and are response-equivalent with w^* as response.

A belief instance y is an optimal strategy for Observer in $\gamma(\mathcal{P})$ if and only if it is permissible ($y \succ \mathcal{P}$) and has w^* as response. The optimal strategies for Nature in $\gamma(\mathcal{P})$ are the same as the optimal strategies for Nature in $\hat{\gamma}(\mathcal{P})$.

If response is injective, all optimal strategies considered are unique and the two optimal strategies for $\gamma(\mathcal{P})$ coincide.

Proof. Clearly, if $\gamma(\mathcal{P})$ is in equilibrium and both players have optimal strategies, these properties also hold for $\hat{\gamma}(\mathcal{P})$. Now assume that $\hat{\gamma}(\mathcal{P})$ is in equilibrium and that both players in this game have optimal strategies. Let x^*, w^* be any set of optimal strategies for the players in $\hat{\gamma}(\mathcal{P})$. By the defining relations (5) and (7), by the assumptions of equilibrium and by optimality of x^* and of w^*, and by the definition (28) of risk, we find that

$$\hat{\Phi}(x^*, w^*) \geq \hat{\Phi}(x^*, \hat{x^*}) = \mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P}) = \hat{\mathrm{Ri}}_{\min}(\mathcal{P}) = \hat{\mathrm{Ri}}(w^*|\,\mathcal{P}) \geq \hat{\Phi}(x^*, w^*)\,, \tag{33}$$

[9]the reader may wish to note that the results to follow hold in a slightly more general setting, cf. Section 11

hence $\hat{\Phi}(x^*, w^*) = \mathrm{H}(x^*)$ and, since $\hat{\Phi}$ is proper and since $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P}) < \infty$, it follows that $w^* = \hat{x^*}$. Then, by Lemma 10.1, $x^* \in \mathrm{ctr}(\mathcal{P})$.

Since x^* above was an arbitrary optimal strategy for Nature and w^* an arbitrary optimal strategy for Observer, we conclude from $w^* = \hat{x^*}$ that the optimal Observer strategy is unique and that all optimal strategies for Nature are response-equivalent, lie in $\mathrm{ctr}(\mathcal{P})$ and has the optimal Observer strategy as response.

We leave it to the reader to establish the results for $\gamma(\mathcal{P})$, say by noting that $y \succ \mathcal{P}$ is equivalent with $\hat{y} \succ \mathcal{P}$ and that $\mathrm{Ri}(y|\mathcal{P}) = \hat{\mathrm{Ri}}(\hat{y}|\mathcal{P})$ and by using the first facts established.

Assume now that response is injective. Then uniqueness of optimal strategies, say of (x^*, w^*) for $\hat{\gamma}(\mathcal{P})$ and of (x^*, y^*) for $\gamma(\mathcal{P})$ follows readily and the identity of x^* and y^* follows as these belief instances are response-equivalent. □

Note that we did not have to assume that response is surjective in order to prove that the optimal strategy w^* above is in the range of this map. Regarding the necessity of injectivity of response in the last part of the theorem, it should be noted that if this condition does not hold, there may be strategies for Nature with the optimal strategy w^* as response which are not optimal for Nature. Simple examples, say with "collapse of response", i.e. with $\hat{Y}$ a singleton, will demonstrate that.

When the games are in equilibrium and optimal strategies exist, we refer to any optimal strategy for Nature as a *a bi-optimal state*. The bi-optimality refers to the fact that also Observer-optimality is secured. Indeed, $\hat{x^*}$ is optimal for Observer in $\hat{\gamma}(\mathcal{P})$ and any $y^* \succ \mathcal{P}$ which is response-equivalent to x^* (including x^* itself) is optimal for Observer in $\gamma(\mathcal{P})$. If response is injective, there is only one bi-optimal state, *the* bi-optimal state. If we prefer, we may also talk of a *bi-optimal control* – with the advantage that this object is unique – or a *bi-optimal belief instance*. Whereas it may be difficult to find optimal strategies, it is often easy to check if given candidates are in fact optimal.

Theorem 10.2. *[Identification] Let (x^*, w^*) be permissible strategies for $\hat{\gamma}(\mathcal{P})$ with $x^* \in \mathrm{ctr}(\mathcal{P})$ and $\mathrm{H}(x^*) < \infty$.*

Then a necessary and sufficient condition that $\hat{\gamma}(\mathcal{P})$ and $\gamma(\mathcal{P})$ are in equilibrium with x^* as bi-optimal state is that (x^*, w^*) is a Nash equilibrium pair. If this is so, w^* is adapted to x^*.

Proof. The sufficiency follows since (31) is equivalent with the condition $\hat{\mathrm{Ri}}(w^*|\mathcal{P}) \le \hat{\Phi}(x^*, w^*)$ and, under the assumption $x^* \succ \mathcal{P}$ (hence also $\hat{x^*} \succ \mathcal{P}$), (32) is equivalent with the condition $\Phi(x^*, w^*) \le \mathrm{H}(x^*)$. Thus, when (x^*, w^*) is a Nash equilibrium pair, $\hat{\mathrm{Ri}}(w^*|\mathcal{P}) \le \mathrm{H}(x^*)$, hence, by the minimax inequality, x^* and w^* are optimal strategies and $\mathrm{H}_{\max}(\mathcal{P}) = \hat{\mathrm{Ri}}_{\min}(\mathcal{P})$. As we assumed that $\mathrm{H}(x^*) < \infty$, $\hat{\gamma}(\mathcal{P})$ is in equilibrium.

The necessity and the last part of the theorem follow from Theorem 10.1 and the above noticed equivalent forms of the saddle-value inequalities. □

Elaborating slightly, we obtain the following corollary:

Corollary 10.1. *Let (x^*, w^*) be permissible strategies for $\hat{\gamma}(\mathcal{P})$ with $x^* \in \mathrm{ctr}(\mathcal{P})$, $\mathrm{H}(x^*) < \infty$ and with w^* adapted to x^*.*

Then a necessary and sufficient condition that $\hat{\gamma}(\mathcal{P})$ and $\gamma(\mathcal{P})$ are in equilibrium with x^* as bi-optimal state is that $\hat{\mathrm{Ri}}(w^*|\mathcal{P}) \leq \mathrm{H}(x^*)$, i.e. that

$$\forall x \in \mathcal{P} : \hat{\Phi}(x, w^*) \leq \mathrm{H}(x^*)\,. \tag{34}$$

Proof. Under the conditions stated, (32) is automatic and (34) is a reformulation of (31). Thus (34) implies that (x^*, w^*) is a Nash equilibrium pair and the result then follows from Theorem 10.2. □

An important consequence of the existence of a bi-optimal state is the validity of the *Pythagorean inequalities*. Let x^* be a bi-optimal state and w^* its response. The *direct Pythagorean inequality*, or just the *Pythagorean inequality*, is the inequality $\mathrm{H}(x) + \hat{\mathrm{D}}(x, w^*) \leq \mathrm{H}(x^*)$, typically considered for $x \in \mathcal{P}$. This is nothing but a trivial rewriting of (34). When it holds, $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$ and the inequality for an individual state $x \in \mathcal{P}$ is, therefore, a sharper form of the trivial inequality $\mathrm{H}(x) \leq \mathrm{H}_{\max}(\mathcal{P})$. The *dual Pythagorean inequality* is the inequality $\hat{\mathrm{Ri}}(w^*|\mathcal{P}) + \hat{\mathrm{D}}(x^*, w) \leq \hat{\mathrm{Ri}}(w|\,\mathcal{P})$, typically considered for $w \succ \mathcal{P}$. When it holds, $\hat{\mathrm{Ri}}(w^*|\,\mathcal{P}) = \hat{\mathrm{Ri}}_{\min}(\mathcal{P})$, and the inequality for an individual strategy $w \succ \mathcal{P}$ is, therefore, a sharper form of the trivial inequality $\hat{\mathrm{Ri}}_{\min}(\mathcal{P}) \leq \hat{\mathrm{Ri}}(w|\,\mathcal{P})$.

Theorem 10.3. *[Pythagorean inequalities] If $\hat{\gamma}(\mathcal{P})$ is in equilibrium with x^* as bi-optimal state then, with $w^* = \hat{x^*}$, the direct as well as the dual Pythagorean inequality holds:*

$$\forall x \in \mathcal{P} : \mathrm{H}(x) + \hat{\mathrm{D}}(x, w^*) \leq \mathrm{H}(x^*)\,, \tag{35}$$

$$\forall w \succ \mathcal{P} : \hat{\mathrm{Ri}}(w^*|\,\mathcal{P}) + \hat{\mathrm{D}}(x^*, w) \leq \hat{\mathrm{Ri}}(w|\,\mathcal{P})\,. \tag{36}$$

Proof. As to (35), this follows from Corollary 10.1. Also (36) must hold since, for $w \succ \mathcal{P}$,

$$\hat{\mathrm{Ri}}(w^*|\,\mathcal{P}) + \hat{\mathrm{D}}(x^*, w) = \mathrm{H}(x^*) + \hat{\mathrm{D}}(x^*, w) = \hat{\Phi}(x^*, w) \leq \hat{\mathrm{Ri}}(w|\,\mathcal{P})\,.$$

□

Let us elaborate on the direct Pythagorean inequality. First, let us agree that a control $w \succ \mathcal{P}$ is a *Pythagorean control for* $\hat{\gamma}(\mathcal{P})$ if, for every $x \in \mathcal{P}$,

$$\mathrm{H}(x) + \hat{\mathrm{D}}(x, w) \leq \mathrm{H}_{\max}(\mathcal{P})\,. \tag{37}$$

This notion will be used whether or not $\hat{\gamma}(\mathcal{P})$ is in equilibrium and whether or not this game has optimal strategies. In particular, it applies in cases when no MaxEnt-state exists. Of course, the notion is only of interest if $\mathrm{H}_{\max}(\mathcal{P}) < \infty$.

Translating to the Y-domain, we say that y is a *Pythagorean belief instance for* $\gamma(\mathcal{P})$ if $y \succ \mathcal{P}$ and if, for every $x \in \mathcal{P}$,

$$\mathrm{H}(x) + \mathrm{D}(x, y) \leq \mathrm{H}_{\max}(\mathcal{P})\,. \tag{38}$$

With these definitions, the part of Theorem 10.3 dealing with the direct Pythagorean inequality states that if $\hat{\gamma}(\mathcal{P})$ is in equilibrium and x^* is bi-optimal, then $\hat{x^*}$ is a Pythagorean control. It follows that x^* is a Pythagorean belief instance.

Elaborating, we find the following result:

Theorem 10.4. *Assume that a MaxEnt-state exists for the preparation* $\mathcal{P}$ *and that* $\mathrm{H}_{\max}(\mathcal{P}) < \infty$*. Then the following three conditions are equivalent:*

- *a Pythagorean control for* $\hat{\gamma}(\mathcal{P})$ *exists;*
- *a Pythagorean belief instance for* $\gamma(\mathcal{P})$ *exists;*
- *The games* $\hat{\gamma}(\mathcal{P})$ *and* $\gamma(\mathcal{P})$ *are in equilibrium and a bi-optimal state for these games exists.*

If these conditions are fulfilled, the Pythagorean control, w^**, is unique and identical to the optimal strategy for Observer in* $\hat{\gamma}(\mathcal{P})$*. Further, a belief instance,* y^* *with* $y^* \succ \mathcal{P}$ *is a Pythagorean belief instance if and only if it has* w^* *as response.*

Proof. Assume that w is a Pythagorean control. Choose a MaxEnt-state x^*. Apply (37) with $x = x^*$ and use the facts that $\mathrm{H}_{\max}(\mathcal{P}) = \mathrm{H}(x^*) < \infty$. You find that $\hat{\mathrm{D}}(x^*, w) = 0$. Thus, by properness, $w = \hat{x^*}$. Then, by Corollary 10.1, $\hat{\gamma}(\mathcal{P})$ and $\gamma(\mathcal{P})$ are in equilibrium with x^* as bi-optimal state. Appealing also to previous results, all statements of the theorem follow. □

Up to now, we have mainly worked in the $\hat{Y}$-domain based on the triple $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$. For the remaining results of this section we find it more natural to work in the Y-domain based on the derived triple $(\Phi, \mathrm{H}, \mathrm{D})$.

First we point to an extra property of bi-optimal states which follows from (35). In order to formulate this in a convenient way we need some definitions. A sequence (x_n) of states *converges in divergence* to the state x, written $x_n \overset{\mathrm{D}}{\to} x$, if $\lim_{n\to\infty} \mathrm{D}(x_n, x) = 0$. This requires that $(x_n, x) \in X \otimes Y$ for all n (or for all n sufficiently large). If $x_n \in \mathcal{P}$ for all n, we say that (x_n) is *asymptotically optimal*, more precisely *asymptotically optimal for Nature in the game* $\gamma(\mathcal{P})$, if $\mathrm{H}(x_n) \to \mathrm{H}_{\max}(\mathcal{P})$ as $n \to \infty$. Finally, a state x (not necessarily in $\mathcal{P}$) is a *maximum entropy attractor for* $\mathcal{P}$, or an $\mathrm{H}_{\max}$*-attractor*, if $x_n \overset{\mathrm{D}}{\to} x$ for every asymptotically optimal sequence. We can now state a trivial corollary to Theorem 10.3 (transformed to the Y-domain):

Corollary 10.2. *Any bi-optimal state x^* of a game $\gamma(\mathcal{P})$ in equilibrium, is a maximum entropy attractor for $\mathcal{P}$.*

One can establish existence of the attractor in many cases, even if the bi-optimal state does not exist. We shall return to this issue in Section 15.

Dual versions of the notions and results indicated above could be introduced, depending on (36) rather than on (35). However, it seems that the notions related to the direct Pythagorean inequality are the more useful ones.

The Pythagorean flavour of (35) is more pronounced when one turns to models of updating, cf. Sections 12 and 20.

For the corollary to follow we need an abstract version of *Jeffrey's divergence* given, for two states x_1 and x_2 by

$$\mathrm{J}(x_1, x_2) = \mathrm{D}(x_1, x_2) + \mathrm{D}(x_2, x_1)\,. \tag{39}$$

Corollary 10.3. *[transitivity inequality] If $\gamma(\mathcal{P})$ is in equilibrium with x^* as a bi-optimal state, then, for every state $x \in \mathcal{P}$ and every belief instance $y \succ \mathcal{P}$, the inequality*

$$\mathrm{H}(x) + \mathrm{D}(x, x^*) + \mathrm{D}(x^*, y) \leq \mathrm{Ri}(y|\,\mathcal{P}) \tag{40}$$

holds. In particular, for every $x \in \mathrm{ctr}(\mathcal{P})$,

$$\mathrm{H}(x) + \mathrm{J}(x, x^*) \leq \mathrm{Ri}(x|\,\mathcal{P}). \tag{41}$$

Proof. First note that also $\hat{\gamma}(\mathcal{P})$ is in equilibrium with x^* as bi-optimal state. Then, putting $w^* = \hat{x^*}$, (35) and(36) hold. Therefore, and as $\mathrm{H}(x^*) = \hat{\mathrm{Ri}}(w^*|\mathcal{P})$, for $x \in \mathcal{P}$ and $w \succ \mathcal{P}$,

$$\mathrm{H}(x) + \hat{\mathrm{D}}(x, w^*) + \hat{\mathrm{D}}(x^*, w) \leq \hat{\mathrm{Ri}}(w|\mathcal{P})\,. \tag{42}$$

To a given belief instance y with $y \succ \mathcal{P}$ we then apply (42) with $w = \hat{y}$. As $\hat{\mathrm{D}}(x, w^*) = \mathrm{D}(x, x^*)$, $\hat{\mathrm{D}}(x^*, w) = \mathrm{D}(x^*, y)$ and $\hat{\mathrm{Ri}}(w|\mathcal{P}) = \mathrm{Ri}(y|\mathcal{P})$, (40) follows. □

We refer to (40) as the *transitivity inequality*. It is a sharper version of the trivial inequality $\mathrm{H}(x) \leq \mathrm{Ri}(y|\mathcal{P})$. It combines both Pythagorean inequalities and these are easily derived from it. If $\mathrm{Ri}(y|\mathcal{P}) < \infty$, the inequality holds with equality if and only if both Pythagorean inequalities (35) and (36) hold with equality .

As to the last part of Corollary 10.3, we note that if you put $r = \mathrm{Ri}(x|\mathcal{P}) - \mathrm{H}(x)$, then the bi-optimal state has Jeffrey divergence at most r from x.

11 Possible refinements

The definition of proper information triples attempts to strike a balance between generality and simplicity. However, there are situations – as will be demonstrated in Section 19 – where more general concepts are appropriate, even if one has to sacrifice a bit on simplicity. The concepts we have in mind are here presented in an abstract context.

Let $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ be an information triple and $\mathcal{P}$ a preparation. We say that the triple is a *weakly proper information triple over* $\mathcal{P}$ if the following holds: For $x \in \mathcal{P}$, $w \succ \mathcal{P}$, $w \neq \hat{x}$ and $\hat{\mathrm{D}}(x, w) = 0$, $\hat{\mathrm{Ri}}(w|\mathcal{P}) > \mathrm{H}(x)$. A stronger form of weak properness works with the requirement $\hat{\mathrm{Ri}}(w|\mathcal{P}) > \hat{\mathrm{Ri}}(\hat{x}|\mathcal{P})$ in place of the requirement $\hat{\mathrm{Ri}}(w|\mathcal{P}) > \mathrm{H}(x)$. In some sense that condition has a simpler interpretation since it amounts to the requirement that in case there are several controls for which redundancy vanishes, response is selected as the one with the smallest risk. However, the chosen definition appears to be all that is needed. Clearly, a proper information triple is weakly proper over X.

Results obtained so far which involve a proper information triple and a preparation continue to hold under the weaker concept. The doubt one could have concerns the uniqueness assertions of Theorems 10.1 and 10.4. This is, however easy to deal with:

Theorem 11.1. *Let $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ be a weakly proper information triple over the preparation $\mathcal{P}$ and assume that $\hat{\gamma}(\mathcal{P})$ is in equilibrium and that both players have optimal strategies.*

(i) Then there is only one optimal strategy for Observer, say w^*, and all optimal strategies for Nature lie in $\mathrm{ctr}(\mathcal{P})$ and are response-equivalent with w^* as response.

(ii) Further, there is only one Pythagorean control, viz. w^*.

Proof. Regarding (i), we argue as in the proof of Theorem 10.1 and establish the series of inequalities (33). From these, and finiteness of $\mathrm{H}(x^*)$, we conclude, firstly that $\hat{\mathrm{D}}(x^*, w^*) = 0$ and then from $\hat{\mathrm{Ri}}(w^*|\mathcal{P}) = \mathrm{H}(x^*)$ and the weak properness that $w^* = \hat{x^*}$. The proof then continues as before.

Regarding (ii), assume that w is a Pythagorean control. Choose x^* optimal. Then $\mathrm{H}(x^*) + \hat{\mathrm{D}}(x^*, w) \leq \mathrm{H}_{\max}(\mathcal{P}) = \mathrm{H}(x^*) < \infty$, thus $\hat{\mathrm{D}}(x^*, w) = 0$. Assume, for the purpose of an indirect proof, that $w \neq w^*$. Then, by weak properness, $\hat{\mathrm{Ri}}(w|\mathcal{P}) > \mathrm{H}(x^*)$. Therefore, there exists a state $x' \in \mathcal{P}$ such that $\hat{\Phi}(x', w) > \mathrm{H}(x^*)$ and it follows that $\mathrm{H}(x') + \hat{\mathrm{D}}(x', w) > \mathrm{H}(x^*)$. This contradicts the assumption that w is a Pythagorean control since $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$. Thus, $w = w^*$ must hold after all. □

The weak form of properness also applies to effort functions. Thus, for the $\hat{Y}$-domain we say that an effort function $\hat{\Phi}$ is *weakly proper* with respect to $\mathcal{P}$ if, from $x \in \mathcal{P}$, $w \succ \mathcal{P}$, $w \neq \hat{x}$ and $\hat{\Phi}(x, w) = \hat{\Phi}(x, \hat{x}) < \infty$ it can be concluded that $\hat{\mathrm{Ri}}(w|\mathcal{P}) > \mathrm{H}(x)$ with $\hat{\mathrm{Ri}}$ for risk as defined in (28). Note that the effort function associated with a weakly proper information triple $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ is a weakly proper effort function according to this definition.

12 Games based on Utility, Updating

In Section 10 we investigated games related to an effort-based information triple. Similar notions and results apply when we start-out with a utility-based triple. Let us work in the Y-domain and base the first part of our discussion on a proper utility-based information triple $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ over $X \otimes Y$. Then, given a preparation $\mathcal{P}$, the associated game $\gamma(\mathcal{P}) = \gamma(\mathcal{P} \,|\, \mathrm{U})$ has Observer as maximizer and Nature as minimizer [10] and the two values of the game are, for Nature, the *minimax utility* $\mathrm{M}_{\min}(\mathcal{P})$:

$$\mathrm{M}_{\min}(\mathcal{P}) = \inf_{x \in \mathcal{P}} \sup_{y \succ x} \mathrm{U}(x, y) = \inf_{x \in \mathcal{P}} \mathrm{M}(x)\,, \tag{43}$$

and, for Observer, the corresponding *maximin value*

$$\sup_{y \succ \mathcal{P}} \inf_{x \in \mathcal{P}} \mathrm{U}(x, y)\,. \tag{44}$$

For $y \succ \mathcal{P}$, the infimum occurring here is the *guaranteed utility* associated with the strategy y. We denote it $\mathrm{Gtu}(y|\,\mathcal{P})$. The maximin value (44) is also referred to as the *maximal guaranteed utility*. We denote it $\mathrm{Gtu}_{\max}(\mathcal{P})$:

$$\mathrm{Gtu}_{\max}(\mathcal{P}) = \sup_{y \succ \mathcal{P}} \mathrm{Gtu}(y|\,\mathcal{P}) = \sup_{y \succ \mathcal{P}} \inf_{x \in \mathcal{P}} \mathrm{U}(x, y)\,. \tag{45}$$

[10] thus, if T in $\gamma(\mathcal{P}\,|T)$ is declared as a utility function, this convention applies, whereas if T is a declared effort function, the players swop roles with Observer as minimizer and Nature as maximizer as in the previous section.

Notions and results, e.g. related to equilibrium, to optimal or bi-optimal states etc. are developed in an obvious manner, either by following Section 10 in parallel or by applying the results of Section 10 to the effort-based triple $(-\mathrm{U}, -\mathrm{M}, \mathrm{D})$. We leave this for the interested reader to do. However, for the important case of updating, cf. Section 8, we shall be more explicit. Thus for the second part of our discussion, we take as base a general proper divergence function D on $X \otimes Y$, a preparation $\mathcal{P}$ and a prior y_0 with $\mathrm{D}^{y_0} < \infty$ on $\mathcal{P}$. The game associated with the utility-based information triple $(\mathrm{U}_{|y_0}, \mathrm{D}^{y_0}, \mathrm{D})$ we denote $\gamma(\mathcal{P}; y_0)$. According to (43), the value for Nature in this game is $\inf_{x \in \mathcal{P}} \mathrm{D}^{y_0}(x)$, also denoted $\mathrm{D}_{\min}(\mathcal{P}; y_0)$ and referred to as the the *minimum divergence value* or the MinDiv-*value*:

$$\mathrm{D}_{\min}(\mathcal{P}; y_0) = \inf_{x \in \mathcal{P}} \mathrm{D}(x, y_0)\,. \tag{46}$$

An optimal strategy for Nature is here called a D-*projection of* y_0 *on* $\mathcal{P}$. If Nature has a unique optimal strategy, it is *the* D-projection of y_0 on $\mathcal{P}$. Consider an Observer strategy $y \succ \mathcal{P}$, i.e. a possible posterior. We use the same notation as in the general case, "Gtu", to indicate Observers evaluation of the performance of the posterior. Incidentally, the letters can here be taken to stand for "guaranteed updating (gain)". Thus

$$\mathrm{Gtu}(y|\,\mathcal{P}; y_0) = \inf_{x \in \mathcal{P}} \mathrm{U}_{|y_0}(x, y) = \inf_{x \in \mathcal{P}} \Big(\mathrm{D}(x, y_0) - \mathrm{D}(x, y)\Big) \tag{47}$$

is the *guaranteed updating gain* associated with the choice of y as posterior, and

$$\mathrm{Gtu}_{\max}(\mathcal{P}; y_0) = \sup_{y \succ \mathcal{P}} \mathrm{Gtu}(y|\,\mathcal{P}; y_0) \tag{48}$$

is Observers value of the game, the *maximum guaranteed updating gain*, or the MaxGtu-*value* of $\gamma(\mathcal{P}; y_0)$.

The basic results for the updating game may be summarized as follows:

Theorem 12.1. *Let* D *be a general divergence function on* $X \otimes Y$, $\mathcal{P}$ *a preparation and* y_0 *a belief instance with* $\mathrm{D}^{y_0} < \infty$ *on* $\mathcal{P}$. *Consider the updating game* $\gamma = \gamma(\mathcal{P}; y_0)$.

If $x^* \in \mathrm{ctr}(\mathcal{P})$, *then* γ *is in equilibrium with* x^* *as bi-optimal state if and only if the Pythagorean inequality*

$$\mathrm{D}(x, y_0) \geq \mathrm{D}(x, x^*) + \mathrm{D}(x^*, y_0) \tag{49}$$

holds for every $x \in \mathcal{P}$. *And if this condition is satisfied,* x^* *is the* D-*projection of* y_0 *on* $\mathcal{P}$. *Furthermore, the dual Pythagorean inequality*

$$\mathrm{Gtu}(y|\,\mathcal{P}; y_0) + \mathrm{D}(x^*, y) \leq \mathrm{Gtu}(x^*|\,\mathcal{P}; y_0) \tag{50}$$

holds for every $y \succ \mathcal{P}$.

The proof can be carried out by applying Corollary 10.1 and Theorem 10.3 to the effort function $\Phi_{|y_0}$ associated with the updating game considered, cf. (21). Details are left to the reader.

The concept of an attractor, cf. Corollary 10.2, also makes sense for updating games. Then the relevant notion is that of a *relative attractor given* y_0, also referred to as the $\mathrm{D}^{y_0}_{\min}$*-attractor*, which is defined as a state x^* such that, for every sequence (x_n) in $\mathcal{P}$ with $\mathrm{D}(x_n, y_0) \to \mathrm{D}_{\min}(\mathcal{P}; y_0)$ it holds that $x_n \overset{\mathrm{D}}{\to} x^*$. In the situation covered by Theorem 12.1 – assuming also that limit states for convergence in divergence are unique – the relative attractor exists and coincides with the bi-optimal state.

The Pythagorean inequality originated with Chentsov [57] and Csiszár [58] where updating in a probabilistic setting was considered. Further versions, still probabilistic in nature can be found in Csiszár [59] and in Csiszár and Matús [60]. In [61] these authors present a general abstract study, adapting a functional analytical approach building technically on meticulous exploitation of tools of convex analysis, partly developed by the authors. This source may also be consulted for information about the historical development and related works. As a work depending on a *reversed Pythagorean inequality* related to the triple (22), we mention Glonti et al [62].

13 Formulating results with a geometric flavour

The results of Section 10 are formulated analytically. In this section we make a translation to results which have a certain geometric flavour. We shall work entirely in the Y-domain. No mention of controls or response will occur. This corresponds to a model with $\hat{Y} = Y = X$ and where response is the identity map. Throughout the section results are based on a proper effort-based information triple $(\Phi, \mathrm{H}, \mathrm{D})$.

In the previous sections, we had a fixed preparation in mind. Here, we shall also discuss to which extent you can change a preparation without changing an optimal strategy.

Sublevel sets of the form $\{\Phi^y \leq a\}$ play a a key role. These sets appeared before as primitive feasible preparations. Here they have a different role and we prefer to use the bracket notation as above.

Proposition 13.1. *Let x^* be a state with finite entropy $h = \mathrm{H}(x^*)$. Then, given a preparation $\mathcal{P}$, the necessary and sufficient condition that the game $\gamma(\mathcal{P})$ is in equilibrium with x^* as bi-optimal state is that*

$\mathcal{P}$ is squeezed in between $\{x^*\}$ and $\{\Phi^{x^*} \leq h\}$, i.e. that $x^* \in \mathcal{P} \subseteq \{\Phi^{x^*} \leq h\}$. In particular, $\{\Phi^{x^*} \leq h\}$ is the largest such preparation.

This follows directly from Theorem 10.2 and Corollary 10.1.

For a fixed preparation $\mathcal{P}$, we can express the two values of $\gamma(\mathcal{P})$, $\mathrm{H}_{\max}(\mathcal{P})$ and $\mathrm{Ri}_{\min}(\mathcal{P})$, in a geometrically flavoured way. This can be done whether or not the game is in equilibrium and the result can thus be used to check if the game is in fact in equilibrium. It is convenient to introduce some preparatory terminology.

Firstly, a subset of X is an *entropy sublevel set* if it is a (non-empty) set of the form $\{\mathrm{H} \leq a\}$. The *size* of such a set is the smallest number a which can occur in this representation, clearly equal to the MaxEnt-value associated with the preparation $\{\mathrm{H} \leq a\}$. Given a preparation $\mathcal{P}$, the associated *enveloping entropy sublevel set* is the smallest entropy sublevel set containing $\mathcal{P}$.

Secondly, and quite analogously in view of (27) and (28), we introduce the *size* of the Φ^y-sublevel set $\{\Phi^y \leq a\}$ as the smallest number a which can occur in this representation. And we define the *enveloping Φ^y-sublevel set* associated with $\mathcal{P}$ to be the smallest Φ^y-sublevel set containing $\mathcal{P}$.

Proposition 13.2. *Consider the game $\gamma(\mathcal{P})$ associated with a preparation $\mathcal{P}$. Then:*

(i) The MaxEnt-value $\mathrm{H}_{\max}(\mathcal{P})$ is the size of the enveloping entropy sublevel set associated with $\mathcal{P}$;

(ii) For fixed $y \succ \mathcal{P}$, $\mathrm{Ri}(y|\mathcal{P})$ is the size of the enveloping Φ^y-sublevel set associated with $\mathcal{P}$.

(iii) The MinRisk-value $\mathrm{Ri}_{\min}(\mathcal{P})$ is the infimum over $y \succ \mathcal{P}$ of the sizes of the enveloping Φ^y-sublevel sets associated with $\mathcal{P}$.

In view of (27), (28) and (29), this is obvious. Some comments on the result are in order. In (i) it is understood that the size is infinite if no entropy sublevel set exists which contains $\mathcal{P}$. A similar convention applies to (ii). Also note that the result gives rise to a simple geometrically flavoured proof of the minimax inequality (30) by noting that for each $y \succ \mathcal{P}$ and each h, $\{\Phi^y \leq h\} \subseteq \{\mathrm{H} \leq h\}$.

There are two families of sets involved in Proposition 13.2, the entropy sublevel sets and the Φ^y-sublevel sets. As the proposition shows, both families give valuable information about the games we are interested in. From the second family alone, one can in fact obtain rather complete information. Indeed, if $\{\Phi^y \leq a\}$ contains a given preparation for appropriately chosen y and a, the associated game is well behaved:

Proposition 13.3. *Given a preparation $\mathcal{P}$, a necessary and sufficient condition that $\gamma(\mathcal{P})$ is in equilibrium and has a bi-optimal state is that $\{\Phi^y \leq a\} \supseteq \mathcal{P}$ for some (y, a) with $y \in \mathcal{P}$ and $a = \mathrm{H}(y)$. When the condition is fulfilled, a is the value of the game and y the bi-optimal state.*

The simple proof is left to the reader. It is the sufficiency which is most useful in practical applications.

The results above translate without difficulty to results about games associated with a utility-based information triple $(\mathrm{U}, \mathrm{M}, \mathrm{D})$. For this, *superlevel sets* of the form $\{\mathrm{U}^y \geq k\}$ as well as *strict sublevel sets* of the form either $\{\mathrm{M} < a\}$ or $\{\mathrm{U}^y < a\}$ play an important role. The notion of *size* of these latter sets, those defined by strict inequality, is defined as the largest value of a which can occur in the representations given.

We shall consider the largest sets of the form $\{\mathrm{M} < a\}$, respectively $\{\mathrm{U}^y < a\}$, which are contained in the complement $\complement \mathcal{P}$ or, as we shall consistently prefer to say below, which are *external to* $\mathcal{P}$.

Either directly – or as corollaries to Propositions 13.1, 13.2 and 13.3 applied to the effort-based triple $(-\mathrm{U}, -\mathrm{M}, \mathrm{D})$ – one derives the following results:

Proposition 13.4. *Let $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ be a utility-based information triple and consider a state x^* with $k = \mathrm{M}(x^*) > -\infty$. Then, for any preparation $\mathcal{P}$, the game $\gamma(\mathcal{P} \,|\, \mathrm{U})$ is in equilibrium with x^* as bi-optimal state if and only if $x^* \in \mathcal{P} \subseteq \{\mathrm{U}^{x^*} \geq k\}$. In particular, the largest such preparation is the superlevel set $\{\mathrm{U}^{x^*} \geq k\}$.*

Proposition 13.5. *Let $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ be a utility-based information triple and consider a preparation $\mathcal{P}$ and the associated game $\gamma(\mathcal{P} \,|\, \mathrm{U})$. Then:*

(i) The value $\mathrm{M}_{\min}(\mathcal{P})$ is the size of the largest strict sublevel set $\{\mathrm{M} < a\}$ which is external to $\mathcal{P}$.

(ii) For fixed $y \succ \mathcal{P}$, $\mathrm{Gtu}(y|\mathcal{P})$ is the size of the largest strict sublevel set $\{\mathrm{U}^y < a\}$ which is external to $\mathcal{P}$.

(iii) The value $\mathrm{Gtu}_{\max}(\mathcal{P})$, as the supremum of $\mathrm{Gtu}(y|\mathcal{P})$, is the supremum of all sizes of sets of the form $\{\mathrm{U}^y < a\}$ with $y \succ \mathcal{P}$ which are external to $\mathcal{P}$.

Proposition 13.6. *Let $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ be a utility-based information triple and consider a preparation $\mathcal{P}$. Then a necessary and sufficient condition that $\gamma(\mathcal{P} \,|\, \mathrm{U})$ is in equilibrium and has a bi-optimal state is that $\{\mathrm{U}^y < a\}$ is external to $\mathcal{P}$ for some (y, a) with $y \in \mathcal{P}$ and $a = \mathrm{M}(y)$. When the condition is fulfilled, a is the value of the game and y the bi-optimal state.*

We also note that the minimax inequality $\mathrm{Gtu}_{\max}(\mathcal{P}) \leq \mathrm{M}_{\min}(\mathcal{P})$ follows from Proposition 13.5 by applying the fact that, generally, $\{\mathrm{M} < a\} \subseteq \{\mathrm{U}^y < a\}$.

Let us look specifically at models of updating, cf. Section 12.

Given is a general divergence function D on $X \otimes Y$ and we consider preparations $\mathcal{P}$ and priors y_0 for which $\mathrm{D}^{y_0} < \infty$ on $\mathcal{P}$. The sets we shall focus on related to the games $\gamma(\mathcal{P}; y_0)$ are of two types, which we associate with, respectively *"balls"* and *"half-spaces"*. Firstly, for $r > 0$, consider the *open divergence ball with radius* r *and centre* y_0, defined as the D^{y_0}-sublevel set

$$\mathrm{B}(r|y_0) = \{\mathrm{D}^{y_0} < r\}\,. \tag{51}$$

In case $r = \mathrm{D}(x^*, y_0)$ for some state x^*, we write this set as $\mathrm{B}(x^*|y_0)$:

$$\mathrm{B}(x^*|y_0) = \mathrm{B}(\mathrm{D}(x^*, y_0)|y_0) = \{x|\, \mathrm{D}(x, y_0) < \mathrm{D}(x^*, y_0)\}\,. \tag{52}$$

And, secondly, we consider sets – all referred to as *half-spaces* – of one of the following forms

$$\sigma^+(y, a|y_0) = \{x|\, \mathrm{U}_{|y_0} < a\} = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) < a\} \tag{53}$$

$$\sigma^-(y, a|y_0) = \{x|\, \mathrm{U}_{|y_0} \geq a\} = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) \geq a\} \tag{54}$$

$$\sigma^+(y|y_0) = \{x|\, \mathrm{U}_{|y_0} < \mathrm{D}(y, y_0)\} = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) < \mathrm{D}(y, y_0\} \tag{55}$$

$$\sigma^-(y|y_0) = \{x|\, \mathrm{U}_{|y_0} \geq \mathrm{D}(y, y_0\} = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) \geq \mathrm{D}(y, y_0)\} \tag{56}$$

Associated with the sets introduced we define certain "boundary sets", respectively *peripheries* and *hyper-spaces*. Notation and definition for the former type of sets is given by

$$\partial\,\mathrm{B}(r|y_0) = \{x|\, \mathrm{D}(x, y_0) = r\} \text{ and}$$
$$\partial\,\mathrm{B}(x^*|y_o) = \{x|\, \mathrm{D}(x, y_0) = \mathrm{D}(x^*, y_0)\}$$

and for the latter type we use

$$\partial\sigma(y, a|y_0) = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) = a\} \text{ and}$$
$$\partial\sigma(y|y_0) = \{x|\, \mathrm{D}(x, y_0) - \mathrm{D}(x, y) = \mathrm{D}(y, y_0)\}\,.$$

When translating basic parts of Propositions 13.4, 13.5 and 13.6 to the setting we are now considering, we find the following result:

Proposition 13.7. *Let* D *be a general divergence function on* $X \otimes Y$ *and consider a belief instance* $y_0 \succ X$ *such that* $\mathrm{D}^{y_0} < \infty$. *Then* *the following results hold for the associated updating games with y_0 as prior:*

(i) For any $x^* \in X$, the largest preparation $\mathcal{P}$ for which $\gamma(\mathcal{P}; y_0)$ is in equilibrium with x^* as bi-optimal state, hence with x^* as the D-*projection of y_0 on $\mathcal{P}$, is the half-space $\sigma^-(x^*|y_o)$.*

(ii) For a fixed updating game $\gamma(\mathcal{P}; y_0)$, the MinDiv-value $\mathrm{D}_{\min}(\mathcal{P}; y_0)$ is the size of the largest strict divergence ball $\mathrm{B}(r|y_0)$ which is external to $\mathcal{P}$, and the maximal guaranteed updating gain $\mathrm{Gtu}_{\max}(\mathcal{P}; y_0)$ is the supremum of a for which there exists $y \succ \mathcal{P}$ such that the half-space $\sigma^+(y, a|y_0)$ is external to $\mathcal{P}$.

(iii) An updating game $\gamma(\mathcal{P}; y_0)$ is in equilibrium and has a bi-optimal state if and only if, for some $y \in \mathcal{P}$, the half-space $\sigma^+(y|y_0)$ is external to $\mathcal{P}$. When this condition holds, y is the bi-optimal state, hence the D-*projection of y_0 on $\mathcal{P}$.*

For illustrations see Section 20, Figures 4a and 4b.

14 Robustness and Core

For this section, $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ is a proper effort-based information triple over $X \otimes \hat{Y}$ and $(\Phi, \mathrm{H}, \mathrm{D})$ the derived triple.

We shall study special circumstances under which the crucial condition (34) holds. Consider a preparation $\mathcal{P}$ and let w^* be a permissible Observer-strategy for the game $\hat{\gamma}(\mathcal{P})$, i.e. $w^* \succ \mathcal{P}$. This strategy is *robust* for $\hat{\gamma}(\mathcal{P})$ if the effort with that strategy for Observer is finite and independent of Natures strategy, i.e. if, for some finite constant h, $\hat{\Phi}(x, w^*) = h$ for all $x \in \mathcal{P}$. The constant h is the *level of robustness*. Similarly, $y^* \succ \mathcal{P}$ is *robust* for $\gamma(\mathcal{P})$ at the *level* h if $\Phi(x, y^*) = h$ for all $x \in \mathcal{P}$.

Theorem 14.1. *[Robustness theorem] Let (x^*, w^*) be an adapted pair of permissible strategies for $\hat{\gamma}(\mathcal{P})$ and assume that w^* is robust with level of robustness h. Then $\hat{\gamma}(\mathcal{P})$ is in equilibrium with h as value and with x^* as bi-optimal state. Furthermore, for any $x \in \mathcal{P}$, the Pythagorean inequality holds with equality:*

$$\mathrm{H}(x) + \hat{\mathrm{D}}(x, w^*) = \mathrm{H}_{\max}(\mathcal{P})\,. \tag{57}$$

Similarly, if (x^*, y^*) are permissible strategies for $\gamma(\mathcal{P})$, if y^* is response-equivalent to x^* (hence $y^* = x^*$ if response is injective) and if y^* is robust for $\gamma(\mathcal{P})$ with level of robustness h, then $\gamma(\mathcal{P})$ is in equilibrium with x^* as bi-optimal state and, for $x \in \mathcal{P}$,

$$\mathrm{H}(x) + \mathrm{D}(x, y^*) = \mathrm{H}_{\max}(\mathcal{P})\,. \tag{58}$$

The result follows directly from Theorem 10.2 and the linking identity. The equality (57) or (58) for $x \in \mathcal{P}$ is the *Pythagorean equality*, here in an abstract version. A more compact geometry flavoured formulation of the first part of Theorem 14.1 à la Corollary 10.1 runs as follows:

Corollary 14.1. *If h is finite and $x^* \in \mathcal{P} \subseteq \mathcal{P}^{\hat{x^*}}(h)$, then $h = \mathrm{H}(x^*)$ and $\hat{\gamma}(\mathcal{P})$ is in equilibrium with x^* as bi-optimal state.*

Whereas Theorem 10.2, Corollary 10.1 and Proposition 13.1 demonstrate the significance of sublevel sets, Theorem 14.1 and Corollary 14.1 does the same but for level sets.

In case response is injective, the second part of Theorem 14.1 really only involves one element, x^*, as the other element, y^*, has to be identical to x^*. The two essential conditions are one on x^* as a strategy for Nature, viz. that it is consistent, and one on x^* as a strategy for Observer, viz. that it is robust. There can only be one such element. If we drop the condition of consistency, there may be many more such elements. They form what we shall call the *core of* $\gamma(\mathcal{P})$.

The core makes sense both for the Y- and for the $\hat{Y}$- domain, and whether or not response is injective. It is defined by the formulas

$$\mathrm{core}(\mathcal{P}) = \{y \in Y | \exists h < \infty : \mathcal{P} \subseteq \mathcal{P}^y(h)\}\,, \tag{59}$$

$$\mathrm{core\hat{}}(\mathcal{P}) = \{w \in \hat{Y} | \exists h < \infty : \mathcal{P} \subseteq \mathcal{P}^w(h)\,. \tag{60}$$

By definition, $y \succ \mathcal{P}$ if $y \in \mathrm{core}(\mathcal{P})$ and $w \succ \mathcal{P}$ if $w \in \mathrm{core\hat{}}(\mathcal{P})$.

If need be, we write $\mathrm{core}(\mathcal{P}|\Phi)$ and $\mathrm{core\hat{}}(\mathcal{P}|\hat{\Phi})$.

For a family $\mathbb{P}$ of preparations the *core* is defined as the intersection of the individual cores:

$$\mathrm{core}(\mathbb{P}) = \bigcap_{\mathcal{P} \in \mathbb{P}} \mathrm{core}(\mathcal{P}) \tag{61}$$

$$\mathrm{core\hat{}}(\mathbb{P}) = \bigcap_{\mathcal{P} \in \mathbb{P}} \mathrm{core\hat{}}(\mathcal{P})\,. \tag{62}$$

The notion of core is closely related to that of *exponential families* (to be elaborated on).

The notion is particularly useful for preparation families consisting of strict feasible preparations. Let us formulate a simple and useful result for the Y-domain. It concerns a typical preparation family $\mathbb{P}^{\mathbf{y}}$, specified by a set $\mathbf{y}= (y_1, \cdots, y_n)$ of elements of Y, as introduced in Section 9. The result we have in mind is the following:

Theorem 14.2. *Consider a preparation family $\mathbb{P}^{\mathbf{y}}$ with $\mathbf{y} = (y_1, \cdots, y_n)$. Let x^* be a state, put $y^* = x^*$ and assume that $y^* \in \mathrm{core}(\mathbb{P}^{\mathbf{y}}|\Phi)$. Further, put $\mathbf{h} = (h_1, \cdots, h_n)$ with $h_i = \Phi(x^*, y_i)$ for $i = 1, \cdots, n$ and assume that these constants are finite. Then $\mathcal{P}^{\mathbf{y}}(\mathbf{h}) \in \mathbb{P}^{\mathbf{y}}$ and $\gamma\big(\mathcal{P}^{\mathbf{y}}(\mathbf{h})\big)$*

is in equilibrium and has x^* as bi-optimal state. In particular, x^* is the MaxEnt strategy for $\mathcal{P}^{\mathbf{y}}(\mathbf{h})$.

This follows directly from the involved definitions and from the robustness theorem. The reader will easily establish the analogous result for the $\hat{Y}$-domain.

The notions robustness and core also make sense for games defined in terms of proper utility-based information triples. If $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ is such a triple, we simply apply the above definitions to the associated effort-based triple $(-\mathrm{U}, -\mathrm{M}, D)$.

The notion of robustness has not received much attention in a game theoretical setting. It is implicit in [58] and in [26] and perhaps first formulated in [24]. Apparently, the existence of suitable robust strategies is a strong assumption. However, for typical models appearing in applications, the assumption is often fulfilled when optimal strategies exist. Results from[27] point in this direction.

15 Adding convexity

It has been recognized since long that notions of convexity play an important role for basic properties of Shannon theory and for optimization theory in general. Deliberately, we have postponed the introduction of this element in our abstract modelling until this late moment. Thereby we demonstrate that a large number of concepts and results, especially those related to games of information, can be formulated quite abstractly and do not require convexity considerations. Also, it becomes clear exactly where these considerations are needed. In this connection note the results starting with Theorem 15.3 below.

The basis in this section is a proper effort-based information triple $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ over $X \otimes \hat{Y}$ with $(\Phi, \mathrm{H}, \mathrm{D})$ the derived information triple over $X \otimes Y$.

Throughout the section we assume that X is a *convex topological space*, i.e. that X is convex and provided with a Hausdorff topology which renders the algebraic operations continuous. The convex hull of a preparation $\mathcal{P}$ is denoted $\mathrm{co}(\mathcal{P})$ and the closed convex hull is denoted $\overline{\mathrm{co}}(\mathcal{P})$. We assume that controllability is adapted to the convex structure in the sense that, firstly, for every $w \in \hat{Y}$, $]w[$ is convex and closed and, secondly, a control w controls a convex combination, say $w \succ \overline{x} = \sum \alpha_i x_i$, if and only if w controls every x_i with $\alpha_i > 0$. In particular, for every convex combination $\overline{x} = \sum \alpha_i x_i$, it holds that $\hat{\overline{x}} \succ x_i$ for all i with $\alpha_i > 0$ and hence, if we switch to the Y-domain, that $\overline{x} \succ x_i$ for every i with $\alpha_i > 0$.

In the foregoing, as in the sequel, a *convex combination* is understood to be a finite convex combination, often written as above without introducing any special notation for the relevant index set. The topology on X is referred to as the *reference topology* and convergence of sequences in this topology is denoted $x_n \to x$ or similar.

Concavity, convexity and *affinity* of real-valued functions f defined on X or on a convex subset of X is defined in the usual way. Thus for concavity, the condition is that if $\sum \alpha_i x_i$ is a convex combination of elements in the domain of definition of f, then $f(\sum \alpha_i x_i) \geq \sum \alpha_i f(x_i)$. For convexity the inequality sign is turned around and for affinity it is replaced by equality. The notions make sense and will also be applied for extended real-valued functions provided they do not assume both values $+\infty$ and $-\infty$.

Emphasis will often be on concavity, convexity or affinity for the w-marginals $\hat{\Phi}^w$ – either all of them or only those with a control of the form $w = \hat{x}$ for some $x \in X$. Note that, say affinity for $\hat{\Phi}^w$ with w of the form $\hat{x}$ for some $x \in X$ amounts to the same as affinity of Φ^x.

Basic properties of entropy and redundancy (hence also divergence) under added conditions about the marginals $\hat{\Phi}^w$ or Φ^y are contained in the following result:

Theorem 15.1. *[Deviation from affinity]*

(i) If the marginals Φ^x with $x \in X$ are concave, then, for every convex combination $\overline{x} = \sum \alpha_i x_i$ of elements in X,

$$\mathrm{H}\Big(\sum \alpha_i x_i\Big) \geq \sum \alpha_i \,\mathrm{H}(x_i) + \sum \alpha_i \,\mathrm{D}(x_i, \overline{x})\,. \qquad (63)$$

In particular, H *is concave and if* $\mathrm{H}(\overline{x}) = \sum \alpha_i \,\mathrm{H}(x_i)$ *and this quantity is finite, then all x_i with $\alpha_i > 0$ are response equivalent, in fact $x_i \hat{\sim} \overline{x}$ for these indices. If response is injective, the entropy function is strictly concave.*

(ii) If the marginals Φ^x with $x \in X$ are even affine, equality holds in (63):

$$\mathrm{H}\Big(\sum \alpha_i x_i\Big) = \sum \alpha_i \,\mathrm{H}(x_i) + \sum \alpha_i \,\mathrm{D}(x_i, \overline{x})\,. \qquad (64)$$

(iii) If the marginals $\hat{\Phi}^w$ with $w \in \hat{Y}$ are affine and if $\mathrm{H}(\overline{x}) < \infty$ for a convex combination $\overline{x} = \sum_i \alpha_i x_i$ then, for every control w with $w \succ \overline{x}$,

$$\sum \alpha_i \hat{\mathrm{D}}(x_i, w) = \hat{\mathrm{D}}\Big(\sum \alpha_i x_i, w\Big) + \sum \alpha_i \,\mathrm{D}(x_i, \overline{x})\,. \qquad (65)$$

(iv) If the marginals $\hat{\Phi}^w$ with $w \in \hat{Y}$ are affine, if $\mathcal{P}$ is a convex preparation with $\mathrm{H}_{\max}(\mathcal{P}) < \infty$ and if $w \in \hat{}[\mathcal{P}]$, then the restriction of

$\hat{\mathrm{D}}^w$ to $\mathcal{P}$ is convex and if $\sum \alpha_i \hat{\mathrm{D}}(x_i, w) = \hat{\mathrm{D}}(\overline{x}, w)$ for a convex combination $\overline{x} = \sum \alpha_i x_i$ of states in $\mathcal{P}$, then all x_i with $\alpha_i > 0$ are response equivalent, in fact $x_i \overset{\frown}{\sim} \overline{x}$ for these indices. If response is injective, the restriction of $\hat{\mathrm{D}}^w$ to $\mathcal{P}$ is strictly convex.

Proof. The result is a natural extension of (parts of) Theorem 1 of [53] and the proof is similar: For (i), apply linking to rewrite the right hand side, then upper bound the expression you get by the assumed concavity and you end up with the desired expression $\Phi(\overline{x}, \overline{x}) = \mathrm{H}(\overline{x})$. The results about concavity of H are easy consequences and property (ii) is proved similarly. For the basic assertion of (iii), add $\sum \alpha_i \hat{\mathrm{D}}(x_i, w)$ to both sides of (64), and use linking to rewrite the right hand side. Then apply the assumed affinity and the term $\hat{\Phi}(\overline{x}, w)$ appears to which you once more apply linking. Finally subtract $\mathrm{H}(\overline{x})$ from both sides. The assertions of (iv) are easy consequences. □

If (65) is only needed for controls of the form $w = \hat{x}$ for some $x \in X$, one need only assume affinity of Φ^x for $x \in X$.

Conditions of affinity are important as is seen from Theorem 15.1. Notions of *affine equivalence* applies in various contexts ($\hat{Y}$-domain, Y-domain, effort-based or utility-based). Some examples will suffice: The effort functions $\hat{\Phi}_1$ and $\hat{\Phi}_2$ over $X \otimes \hat{Y}$ are *affinely equivalent* if there exists a finite-valued affine function f on X such that, for $(x, w) \in X \otimes \hat{Y}$, $\hat{\Phi}_2(x, w) = \hat{\Phi}_1(x, w) + f(x)$. And two effort-based information triples $(\Phi_1, \mathrm{H}_1, \mathrm{D}_1)$ and $(\Phi_2, \mathrm{H}_2, \mathrm{D}_2)$ are *affinely equivalent* if they are equivalent ($\mathrm{D}_2 = \mathrm{D}_1$) and there exists a finite-valued affine function f on X such that, for $(x, y) \in X \otimes Y$, $\mathrm{H}_2(x) = \mathrm{H}_1(x) + f(x)$. Then of course, also $\Phi_2(x, y) = \Phi_1(x, y) + f(x)$.

In the terminology of [63], (65) is the *compensation identity* with the last term as *compensation term*. This term appears as a measure of *deviation from affinity*, both in relation to entropy, cf. (64), and in relation to redundancy (hence also to divergence), cf. (65). The significance of such terms is being more widely recognized. This applies in particular to the case of an even mixture $\overline{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$, for which the term is often, at least in the classical probabilistic setting, called the *Jensen-Shannon divergence* between x_1 and x_2. We shall use the notation

$$\mathrm{JSD}(x_1, x_2) = \frac{1}{2}\,\mathrm{D}(x_1, \overline{x}) + \frac{1}{2}\,\mathrm{D}(x_2, \overline{x}) \text{ with } \overline{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2 \,. \quad (66)$$

In view of its importance, and for reference purposes, let us state the form of the compensation identity for the Y-domain:

$$\sum \alpha_i \,\mathrm{D}(x_i, y) = \mathrm{D}\left(\sum \alpha_i x_i, y\right) + \sum \alpha_i \,\mathrm{D}(x_i, \overline{x}) \,. \quad (67)$$

Here, y is a belief instance. If you want to show that the compensation identity holds for a certain bi-variate function, say D for the Y-domain, a natural procedure is first to check that the function is indeed a general divergence function – i.e. that the properties F, S and P are satisfied – and then to inspect more closely the expression for D. If this expression, apart from pure x-only dependent terms, only contains terms which, for fixed y, are linear terms in x, a suitable entropy can be identified and the compensation identity (67) will hold (when $\mathrm{H}(\overline{x}) < \infty$). The procedure is demonstrated in the following example which, at the same time also illustrates the role of the two assumptions made in part (iii) of the theorem in order for (65) (or (67)).

Example 15.1. *Let $X = Y = \hat{Y}$ be copies of the real line $]-\infty, \infty[$ provided with the standard structure, let response be the identity map and let visibility be the diffuse relation. Further, let α be a positive parameter and consider the bivariate function* D *given by*

$$\mathrm{D}(x, y) = |x - y|^{\alpha} . \tag{68}$$

Clearly, this is a genuine divergence function.

If $\alpha \neq 2$, (67) does not hold. Indeed, if you consider the mixture $\overline{x} = \frac{1}{2}0 + \frac{1}{2}1$ and as y take $y = 1$, then the left hand side of (67) equals $\frac{1}{2}$ whereas the right hand side equals $2^{1-\alpha}$. Thus, when $\alpha \neq 2$, there is no information triple $(\Phi, \mathrm{H}, \mathrm{D})$ equivalent to $(\mathrm{D}, 0, \mathrm{D})$ for which (67) holds generally. So you cannot add a finite entropy function to $(\mathrm{D}, 0, \mathrm{D})$ and obtain an effort function with affine marginals.

If $\alpha = 2$, the matter is quite different. Then $\mathrm{D}(x, y) = x^2 + y^2 - 2xy$ and you can subtract x^2 to obtain a function with linear dependency on x for a given value of y. In other words, if you consider the triple equivalent to $(\mathrm{D}, 0, \mathrm{D})$ for which entropy is given by $\mathrm{H}(x) = -x^2$, all conditions of Theorem 15.1, (iii) are fulfilled, thus (67) must hold. Further material on this and similar examples can be found in Section 18.

We turn to a study of the impact of convexity assumptions for the games of information introduced in Section 10. We start with a rather trivial corollary to (i) of Theorem 15.1:

Corollary 15.1. *Assume that the marginals $\hat{\Phi}^x$ with $x \in X$ are concave and consider the game $\hat{\gamma}(\mathcal{P})$ for a convex preparation $\mathcal{P}$. Then the set of optimal strategies for Nature in this game is convex and reduces to a singleton in case response is injective.*

The purpose of the next result is to indicate that "normally" a bi-optimal state will be risk-stable, i.e. lie in the core of the preparation concerned. This result, in a more concrete set-up goes back to Csiszár, cf. [58]. It depends on the following notion: A state x is an *algebraic inner point* of $\mathcal{P}$ (typically assumed convex) if, for every $x_1 \in \mathcal{P}$ distinct from x, there exists $x_2 \in \mathcal{P}$ such that x is a genuine convex combination of x_1 and x_2.

Theorem 15.2. *Assume that Φ^x is affine for all $x \in X$ and let $\mathcal{P}$ be a convex preparation. If $\gamma(\mathcal{P})$ is in equilibrium and has a bi-optimal state x^* and if this state is algebraic inner in $\mathcal{P}$, then x^* is risk-stable for $\gamma(\mathcal{P})$ at the level $\mathrm{H}_{\max}(\mathcal{P})$. In particular, $x^* \in \mathrm{core}(\mathcal{P})$.*

Proof. With assumptions as stated, consider any $x \in \mathcal{P}$ distinct from x^* and determine $x' \in \mathcal{P}$ such that x^* is a genuine convex combination of x and x', say $x^* = \alpha x + \beta x'$. We find that $\Phi(x, x^*) \leq \mathrm{Ri}(x^*|\mathcal{P}) = \mathrm{H}_{\max}(\mathcal{P})$. Similarly, $\Phi(x', x^*) \leq \mathrm{H}_{\max}(\mathcal{P})$. As the convex combination $\alpha\Phi(x, x^*) + \beta\Phi(x', x^*)$ equals $\Phi(\alpha x + \beta x', x^*) = \Phi(x^*, x^*) = \mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$, we conclude that $\Phi(x, x^*) = \mathrm{H}_{\max}(\mathcal{P})$. As this holds for every $x \in \mathcal{P}$, the result follows. □

As remarked in Section 14 there may be many strategies in the core but only one at the level $\mathrm{H}_{\max}(\mathcal{P})$. A fruitful strategy for the search of a bi-optimal state, in particular of a MaxEnt-strategy is first to determine the core and then to select that element (if any) in the core which is consistent. The reader should note that this route to determine MaxEnt strategies does not involve the infinitesimal calculus, in particular, it does not need the use of Lagrange multipliers.

Certain further results which exploit convexity assumptions depend on topological considerations, especially on notions of *sequential lower semi-continuity*. This applies to functions of the form $\hat{\mathrm{D}}^w$ with $w \in \hat{Y}$ and, for $x \in X$, to functions of the form D_x when restricted to arguments in X. Let (x_n) be a convergent sequence in X, say $x_n \to x_0$. Then, for $\hat{\mathrm{D}}^w$ the semi-continuity condition pointed to is that $\hat{\mathrm{D}}(x_0, w) \leq \liminf_{n\to\infty} \hat{\mathrm{D}}(x_n, w)$ when these expressions make sense. By assumption, it suffices for this that $x_n \in]w[$, eventually. For D_x, the assertion $\mathrm{D}(x, x_0) \leq \liminf_{n\to\infty} \mathrm{D}(x, x_n)$ is relevant. This is to be understood in the sense that if $\liminf_{n\to\infty} \hat{\mathrm{D}}(x, \hat{x_n}) < \infty$ (which presupposes that $x_n \succ x$, eventually), then $x_0 \succ x$ and the stated inequality holds.

From our previous study in Section 10 we have realized the central importance of (34), equivalent to $\mathrm{Ri}(y^*|\mathcal{P}) \leq \mathrm{H}(x^*)$. From that condition, assuming also $x^* \in \mathcal{P}$, you can conclude equilibrium of $\gamma(\mathcal{P})$ and also identify the bi-optimal state. In particular, you can conclude that $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$. Adding conditions of convexity, (34) actually follows from the formally weaker condition $\mathrm{H}(x^*) = \mathrm{H}_{\max}$ as we shall now see:

Theorem 15.3. *Assume that the marginal functions* Φ^y *with* $y \in X$ *are concave and that the marginal functions* D_x *with* $x \in X$ *are sequentially lower semi-continuous on* X*. Let* $\mathcal{P}$ *be a convex preparation and let* $x^* \in \mathcal{P}$ *have finite entropy. Then, the condition* $\mathrm{H}(x^*) = \mathrm{H}_{\max}(\mathcal{P})$ *is not only necessary, but also sufficient for* (34) *to hold, hence for* $\gamma(\mathcal{P})$ *to be in equilibrium with* x^* *as bi-optimal state.*

Proof. In order to establish (34), consider an element $x \in \mathcal{P}$ and apply (63) to a convex combination of the form $y_n = (1-\frac{1}{n})x^* + \frac{1}{n}x$. We find that $\mathrm{H}(x^*) \geq \mathrm{H}(y_n) \geq (1-\frac{1}{n})\,\mathrm{H}(x^*) + \frac{1}{n}\,\mathrm{H}(x) + \frac{1}{n}\,\mathrm{D}(x, y_n)$ from which we conclude that $\mathrm{H}(x) + \mathrm{D}(x, y_n) \leq \mathrm{H}(x^*)$. By sequential lower semi-continuity of D_{x^*}, $x^* \succ x$ and $\mathrm{H}(x) + \mathrm{D}(x, x^*) \leq \mathrm{H}(x^*)$ follows. As $x \in \mathcal{P}$ was arbitrary, (34) holds. The result then follows from Corollary 10.1. □

One may criticize the result as you cannot apply it to feasible preparations in case the marginals Φ^y are strictly concave, since then the feasible preparations will, typically, not be convex. Rather than reacting negatively towards this observation, we take it as a strong indication that really useful modelling requires that the marginals Φ^y are in fact affine.

The kind of reasoning in the above proof can be expanded, roughly speaking by replacing the occurring optimal strategy by an asymptotically optimal sequence, and then leads to results about existence of the maximum entropy attractor, cf. [53] as pointed to before. We shall not go into that for games based on an effort function but will do so below when we turn to games of updating, cf. Theorem 15.5.

Translating Theorem 15.3 to a setting based on utility and formulating it with an assumption of affinity instead of concavity, one finds the following result:

Corollary 15.2. *Let* $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ *be a proper utility-based information triple. Assume that all marginals* U^y *with* $y \in X$ *are affine and that all marginals* D_x *with* $x \in X$ *are sequentially lower semi-continuous on* X*. Let* $\mathcal{P}$ *be a convex preparation and* x^* *a state in* $\mathcal{P}$ *with* $\mathrm{M}(x^*)$ *finite. Then, if* $\mathrm{M}(x^*) = \mathrm{M}_{\min}(\mathcal{P})$*,* $x^* \in \mathrm{ctr}(\mathcal{P})$ *and the game* $\gamma(\mathcal{P}\,|\,\mathrm{U})$ *is in equilibrium and has* x^* *as bi-optimal state. In particular, the Pythagorean inequality* $\mathrm{M}(x) \geq \mathrm{D}(x, x^*) + \mathrm{M}(x^*)$ *holds for every* $x \in \mathcal{P}$.

Finally, we shall investigate updating games under convexity. For this, D is a general divergence function on $X \otimes Y$. We shall say that the compensation identity holds for D if (65) holds for every convex combination $\overline{x} = \sum \alpha_i x_i$ of states and every $y \succ \overline{x}$.

Theorem 15.4. *Assume that the compensation identity holds for the general divergence function* D *on* $X \otimes Y$ *and that the marginal functions*

D_x *with* $x \in X$ *are sequentially lower semi-continuous on* X*. Consider a convex preparation* $\mathcal{P}$ *and an associated prior* y_0 *for which* D^{y_0} *is finite on* $\mathcal{P}$*. Then, if the* D*-projection of* y_0 *on* $\mathcal{P}$ *exists, say equal to* x^**, it holds that* $x^* \succ \mathcal{P}$ *and that the updating game* $\gamma(\mathcal{P}\,|\,\mathrm{U}_{|y_0})$ *is in equilibrium with* x^* *as bi-optimal state. In particular, the Pythagorean inequality* (49) *holds for all* $x \in \mathcal{P}$*.*

Proof. We shall apply Corollary 15.2 to the utility-based triple $(\mathrm{U}_{|y_0}, \mathrm{D}^{y_0}, \mathrm{D})$ on $\mathcal{P} \times [\mathcal{P}]$. Consider any $y \in Y$ and any convex combination $\overline{x} = \sum \alpha_i x_i$ of states in $\mathcal{P}$. As $\mathrm{D}^{y_0} < \infty$ on $\mathcal{P}$, the sum $\sum \alpha_i \mathrm{D}(x_i, y_0)$ is finite. By the compensation identity, so is the sum $\sum \alpha_i \mathrm{D}(x_i, \overline{x})$. For $y \in Y$, we find that

$$\begin{aligned}
\mathrm{U}_{|y_0}(\overline{x}, y) &= \mathrm{D}(\overline{x}, y_0) - \mathrm{D}(\overline{x}, y) \\
&= \Big(\sum \alpha_i \mathrm{D}(x_i, y_0) - \sum \alpha_i \mathrm{D}(x_i, \overline{x})\Big) \\
&\quad - \Big(\sum \alpha_i \mathrm{D}(x_i, y) - \sum \alpha_i \mathrm{D}(x_i, \overline{x})\Big) \\
&= \sum \alpha_i \mathrm{D}(x_i, y_0) - \sum \alpha_i \mathrm{D}(x_i, y) \\
&= \sum \alpha_i \mathrm{U}_{|y_0}(x_i, y)\,.
\end{aligned}$$

Thus the condition of affinity from Corollary 15.2 is fulfilled. The result follows. □

It lies nearby to search for conditions which ensure existence of the D-projection. This requires extra properties. A sequence (x_n) of states is a JSD- *Cauchy sequence* if

$$\lim_{n,m\to\infty} \mathrm{JSD}(x_n, x_m) = 0\,. \tag{69}$$

And X is JSD-*complete* if every JSD-Cauchy sequence converges in the reference topology. This notion is adapted from [53] [11]. Let us collect the key results about updating games in one theorem:

Theorem 15.5. *Assume that the compensation identity holds for* D*, that* X *is* JSD*-complete, and that, for* $x \in X$*, the marginal functions* D_x *as well as* D^x *are sequentially lower semi-continuous on* X*. Consider a preparation* $\mathcal{P}$ *and an associated prior* y_0 *such that* D^{y_0} *is finite on* $\mathcal{P}$*. Then the following holds:*

(i) Observer strategies for $\gamma(\mathrm{co}(\mathcal{P}); y_0)$ *and for* $\gamma(\mathcal{P}; y_0)$ *coincide, i.e.* $[\mathrm{co}(\mathcal{P})] = [\mathcal{P}]$*, and for every such strategy* y*,* $\mathrm{Gtu}(y|\,\mathrm{co}(\mathcal{P}); y_0) =$

[11] In fact, we only need the condition that a JSD-Cauchy sequence has a convergent subsequence; however, this condition is very close to the stated condition which follows from it under an added assumption of joint lower semi-continuity of D.

$\mathrm{Gtu}(y|\,\mathcal{P};y_0)$, *hence*

$$\mathrm{Gtu}_{\max}(\mathrm{co}(\mathcal{P});y_0)=\mathrm{Gtu}_{\max}(\mathcal{P};y_0). \tag{70}$$

(ii) Without adding extra conditions, Observer has a unique optimal strategy, y^*, in the game $\gamma(\mathcal{P};y_0)$.

(iii) If $\mathcal{P}$ is convex, the game $\gamma(\mathcal{P})$ is in equilibrium and the $\mathrm{D}^{y_0}_{\min}$-attractor exists. This attractor, say x^*, is identical to the optimal Observer strategy y^* from (ii); it is the D*-projection of y_0 on $\mathcal{P}$ if and only if $x^*\in\mathcal{P}$.*

(iv) If $\mathcal{P}$ is closed and convex, the D*-projection of y_0 on $\mathcal{P}$ exists.*

(v) The game $\gamma(\mathcal{P}\,|y_0)$ is in equilibrium if and only if

$$\mathrm{D}_{\min}(\mathrm{co}(\mathcal{P});y_0)=\mathrm{D}_{\min}(\mathcal{P};y_0)\,. \tag{71}$$

Proof. The results follow by adapting Theorem 2 and Corollary 1 of [53] to the present setting. Let us indicate the details.

The proof of (i) is trivial. Now assume that $\mathcal{P}$ is convex and let us analyze the game $\gamma=\gamma(\mathcal{P};y_0)$. Consider a sequence (x_n) of states in $\mathcal{P}$ such that $\mathrm{D}(x_n,y_0)\to\mathrm{D}$ with $\mathrm{D}=D_{\min}(\mathcal{P};y_0)$. Put $\delta_n=\mathrm{D}(x_n,y_0)-\mathrm{D}$ and assume that $\delta_n<1$ for all n. Appealing to convexity of $\mathcal{P}$ and to the compensation identity, it is seen that (x_n) is a JSD-Cauchy sequence. Then $x_n\to x^*$, say. We may assume that all δ_n are positive (otherwise we are in a situation which can be covered by Theorem 15.4). Given $x\in\mathcal{P}$, consider (y_n) given by $y_n=\sqrt{\delta_n}x+(1-\sqrt{\delta_n})x^*$. Apply the fact that $\mathrm{D}\le\mathrm{D}(y_n,y_0)$, use the compensation identity, throw away one term and divide by $\sqrt{\delta_n}$. This shows that $\mathrm{D}(x,y_n)+\mathrm{D}(x_n,y_0)\le\mathrm{D}(x,y_0)+\sqrt{\delta_n}$. Going to the limit and exploiting the semi-continuity property of D_x, we find that $x^*\succ x$ and that $\mathrm{D}\le\mathrm{U}_{|y_0}(x,x^*)$. As this holds for every $x\in\mathcal{P}$, we conclude that $y^*=x^*$ is in $[\mathcal{P}]$ and that $\mathrm{D}\le\mathrm{Gtu}(y^*|\,\mathcal{P};y_0)$. Thus y^* is an optimal Observer strategy and also, we see that x^* is the $\mathrm{D}^{y_0}_{\min}$-attractor. From these observations and from (i), the remaining properties claimed are easily derived. □

Part II
Examples, towards Applications

16 Protection against misinformation

We present a very general type of application of the foregoing theory which depends on an interpretation different from what has been indicated before. This involves a theme which has been important for the development of the notion of proper score functions.

In a sense, what we shall discuss here is what happens if Nature can communicate. Then we speak instead about *Expert*. And Observer becomes *Customer*. Expert holds the truth, x, or rather, x represents Experts best evaluation of what the truth is. Customer wants to know what Expert thinks about a certain situation and asks Expert for advice – against payment, to be agreed upon. For despicable reasons, Expert may be tempted to advice against better knowing, i.e. to give as advice y, instead of the honest advice x. Misinformation could either be due to the difficulty Expert may have in reaching a true expert opinion or it could be out of self-interest, with Expert taking advantage of false information given to Customer. Or Expert may try to mislead Customer in order to hide a business secret.

We assume that truth will be revealed to both Expert and Customer soon after Expert has given advice to Customer and further, that a proper effort function $\Phi = \Phi(x, y)$ is known to both Expert and Customer. We shall device a payment scheme which will protect Customer against misinformation. The idea is simple. At the time of signing a contract – before advice is given – Customer pays a flat sum to Expert and further, Expert and Customer agree on an insurance scheme stipulating a penalty to be payed by Expert to Customer proportional to $\Phi(x^*, y)$ where x^* represents what really happened and y is the advice given. If Expert is confident that he knows what will happen, he will assume that $x^* = x$ will hold and it will be in his own interest to give to Customer the honest advice $y = x$.

In the literature this scheme is mainly considered based on a *proper score function*, the same as a proper utility function. This gives an obvious variation of the payment scheme with the score function determining payment from Customer to Expert. The most often treated situation is probably that of weather forecasting with Brier [41] the first and Weijs and Giesen [64] the presently most recent contribution.

But also situations from economy and statistics have been studied frequently. Apart from sources just cited we refer to the sources pointed to in Section 6 and to McCarthy [65] as well as to the recent contribution [66] by Chambers. As a final reference we point to Hilden [67] where applications to diagnostics is discussed.

Works cited and their references will reveal a rich literature. With access to our abstract modelling, further meaningful applications, not necessarily tied to probabilistic modelling may emerge.

17 Cause and Effect

We continue with one more section where the basic interpretations are changed. For this we assume that $Y = X$ and define $\hat{X} = \hat{Y}$, equivalently, $\hat{X} = W$. Elements of X are now interpreted as *causes* and response, considered as a map defined on X, as the transformation of a cause into its associated consequence. This change moves the focus from Observers thoughts as discussed in Section 3 to a reflection of causality in Nature, a basic mechanism of the world. The set-up is in this way conceived as a model of *cause and effect*.

Previously we considered possible choices of Observer (for γ- or $\hat{\gamma}$-type games). Now it is more pertinent to focus on consequences – elements of W – as possible observations by Observer of the effect of the actual cause. For $x \in X$ and $w \in W$, $\hat{\Phi}(x, w)$ may now be interpreted as the cost to Observer if he has observed (or believes to have observed) the effect w when the actual cause is x.

Consider the game $\hat{\gamma}$, say with preparation $\mathcal{P} = X$. With the new interpretation in mind it appears particularly pertinent to consider Observers risk associated with the various possible observations.

Concrete situations where the change of interpretation makes sense, involve information theoretical problems of capacity.

18 Primitive triples and generation by integration

Natural building blocks for information triples will be defined. We shall here concentrate on a simple, important and easy-to-apply approach. Response or controls will not be involved. In the following section we present a substantial expansion, mainly of theoretical interest. The key applications and concrete examples can be handled based on the material in the present section.

Let I be a subinterval of $]-\infty, \infty[$ with endpoints a and b $(-\infty \leq$

$a < b \leq \infty$). Either none, one or both endpoints belong to I but neither $+\infty$ nor $-\infty$ are members of I. Provide I with its usual algebraic and topological structure. We take I as state space as well as belief reservoir. Thus $X = Y = I$. Visibility is taken to be the diffuse relation so that any state $s \in I$ is visible from any belief instance. However, at times a more restricted notion of visibility is relevant, especially for $I = [0, 1]$ or $I = [0, \infty[$. Then

$$I \otimes I = I^2 \setminus \{(s, u) | s > 0, u = 0\} \tag{72}$$

may well be a better choice. An example of this is given in Section 22.

An effort-based information triple over $I \times I$ is said to be *primitive*. The “primitivity” lies in the fact that the state space and belief reservoir appear to be as simple as one can think of – if you do not want to enter into discrete structures with a finite state space. We use lower case letters as in $(\phi, \mathrm{h}, \mathrm{d})$ for such triples. We are especially interested in proper primitive triples. The conditions they must satisfy are as follows (linking, fundamental inequality, soundness and properness):

$$\phi(s, u) = \mathrm{h}(s) + \mathrm{d}(s, u)\,, \tag{73}$$

$$\mathrm{d}(s, u) \geq 0\,, \tag{74}$$

$$\mathrm{d}(s, s) = 0\,, \tag{75}$$

$$\mathrm{d}(s, u) = 0 \Rightarrow u = s\,. \tag{76}$$

It is understood here and later on that such requirements are to hold for $s \in I$ or for $(s, u) \in I \times I$. From Section 15 we know that it is desirable for the effort function to have affine marginals ϕ^u. For this to be the case, there must exist functions on I, η and ξ, such that

$$\phi(s, u) = s\eta(u) + \xi(u)\,. \tag{77}$$

There is a simple way to generate a multitude of such information triples. The method is inspired by Bregman, [68] who used the construction for other purposes. Given is a *Bregman generator* h which is here understood to be a continuous, real-valued, strictly concave function on I which is sufficiently smooth on the interior of the interval, say continuously differentiable. We take this function as the entropy function, h. Defining effort and divergence by

$$\phi(s, u) = \mathrm{h}(u) + (s - u)\,\mathrm{h}'(u)\,, \tag{78}$$

$$\mathrm{d}(s, u) = \mathrm{h}(u) - \mathrm{h}(s) + (s - u)\,\mathrm{h}'(u)\,, \tag{79}$$

the triple $(\phi, \mathrm{h}, \mathrm{d})$ is indeed a proper primitive information triple with affine marginals, ϕ^u. Figure 1 illustrates what is involved.

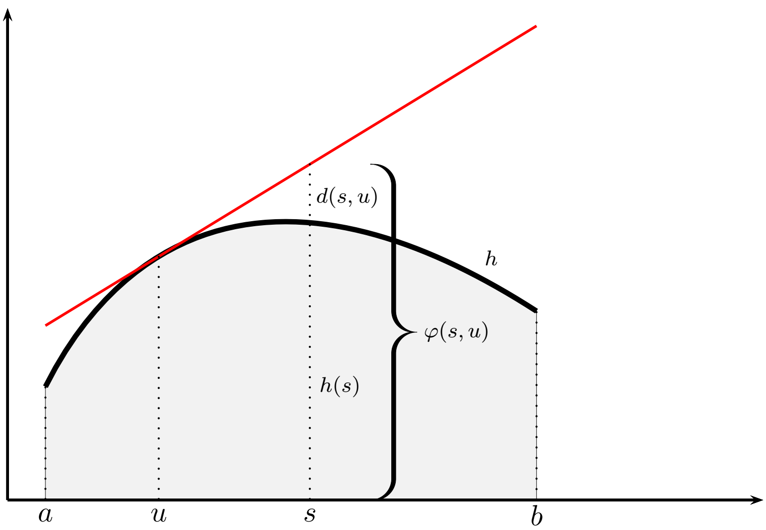


Figure 1: Bregman generator and primitive effort-based information triple

The construction also makes sense if we relax the requirement of strict concavity to one of plain concavity. Then, the information triple you obtain need not be proper, in fact, it may even be degenerate. As seen from (79) this happens if and only if the generator is affine, i.e., with standard terminology for real functions of a real variable, if and only if the generator is a linear function on I. Such generators are said to be *degenerate*.

The utility-based analogues of notions introduced are defined in an obvious manner (see also examples below). We shall use $(\mathrm{u}, \mathrm{m}, \mathrm{d})$ as generic notation for primitive utility-based triples.

As two examples of effort-based Bregman generated primitive triples, we point to the *standard algebraic triple* given by

$$\phi(s,u) = u^2 - 2su\,, \tag{80}$$

$$\mathrm{h}(s) = -s^2\,, \tag{81}$$

$$\mathrm{d}(s,u) = (s-u)^2 \tag{82}$$

over $]-\infty,+\infty[$ and to the *standard logarithmic* triple

$$\phi(s,u) = u - s + s\ln\frac{1}{u}\,, \tag{83}$$

$$\mathrm{h}(s) = s\ln\frac{1}{s}\,, \tag{84}$$

$$\mathrm{d}(s,u) = u - s + s\ln\frac{s}{u}\,. \tag{85}$$

over $[0, \infty]$. Both triples are given in their effort-based versions. If need be, we refer to these triples as standard primitive effort-based triples.

The first triple is equivalent to a triple we met in Example 15.1. It leads to basic concepts of real Hilbert space theory by a natural process of summation or, more generally, integration. By a similar process, the second triple leads to basic concepts of Shannon information theory. Before elaborating on that, we shall generalize both examples by the introduction of a parameter q. In fact, we shall see that, modulo affine equivalence, both examples can be conceived as belonging to the same family of triples.

In order to modify the standard algebraic triple, it lies nearby to consider generators of the form

$$\mathrm{h}_q(s) = \alpha(q)s^q + \beta(q)s + \gamma(q) \tag{86}$$

with α, β and γ functions depending on a real parameter q. Let us agree to work mainly with $I = [0, \infty]$ as state space. Then q could in principle be any real parameter. For each fixed q, h_q is either strictly concave – an effort-based Bregman generator – strictly convex – a utility-based Bregman generator – or degenerate. Applications of (78) and (79) give the formulas

$$\phi_q(s, u) = \alpha(q)(1-q)u^q + \alpha(q)qsu^{q-1} + \beta(q)s + \gamma(q) \tag{87}$$
$$\mathrm{d}_q(s, u) = \alpha(q)(1-q)u^q + \alpha(q)qsu^{q-1} - \alpha(q)s^q\,. \tag{88}$$

When $\alpha(q)q(q-1)$ is negative, h_q is a genuine effort-based Bregman generator and the triple $(\phi_q, \mathrm{h}_q, \mathrm{d}_q)$ is a proper primitive effort-based information triple. When $\alpha(q)q(q-1)$ is positive, h_q is strictly convex and the triple $(\phi_q, \mathrm{h}_q, -\mathrm{d}_q)$ is a proper primitive utility-based information triple (which should then rather be denoted $(\mathrm{u}_q, \mathrm{m}_q, \mathrm{d}_q)$). Thus, if you consider the triple $(\phi_q, \mathrm{h}_q, |\,\mathrm{d}_q\,|)$ you are certain to obtain a primitive triple, either effort-based or utility-based (or degenerate). It also follows from (86) – (88) that modulo affine equivalence, the triples you obtain from different choices of α, β and γ are scalarly equivalent. For some choices you may prefer to restrict the parameter so that only effort-based triples emerge, for others you may find it interesting to focus on triples where there is a smooth variation from effort-based to utility-based triples. In applications – purely speculative at the moment – this could reflect situations in economic or physical or chemical systems where e.g. a change from positive to negative rent or from exothermic to endothermic reaction can take place.

If you choose $\alpha = 1$ and $\beta = \gamma = 0$, then $(\phi_q, \mathrm{h}_q, |\,\mathrm{d}_q\,|)$ equals

$$\big((1-q)u^q + qsu^{q-1}, s^q, |(1-q)u^q + qsu^{q-1} - s^q|\big)\,. \tag{89}$$

As you go from large to small values of q this primitive triple starts out as utility-based, then, for $q = 1$, becomes degenerate, after which it switches to the effort-based mode until, for $q = 0$, it again becomes degenerate, after which it switches back to the utility-based mode. For $q = 2$, the triple is the *utility-based standard algebraic triple*, the utility-based version of the triple given in (80) – (82). That triple is most naturally considered over $I \times I$ with $I =]-\infty, \infty[$.

We can remove the "singularity" of the system at $q = 1$ by blowing up the generator near $q = 1$. Let us choose α, β and γ as follows:

$$\alpha(q) = \frac{1}{1-q}, \; \beta(q) = \frac{-1}{1-q}, \; \gamma(q) = \gamma_0 \,. \tag{90}$$

Here, the constant γ_0 represents an eventual *overhead* (touched upon towards the end of Section 6). With choices as specified, we obtain the triples $(\phi_q, \mathrm{h}_q, \mathrm{d}_q)$ with

$$\phi_q(s,u) = u^q + \frac{1}{1-q}\big(qu^{q-1} - 1\big)s + \gamma_0 \,, \tag{91}$$

$$\mathrm{h}_q(s) = s\frac{s^{q-1} - 1}{1-q} + \gamma_0 \,, \tag{92}$$

$$\mathrm{d}_q(s,u) = u^q + \frac{q}{1-q}u^{q-1}s - \frac{1}{1-q}s^q \,. \tag{93}$$

The equation (91) gives you *gross effort* with *net effort* obtained by putting $\gamma_0 = 0$. Similarly, (92) is *gross entropy* and the same formula with $\gamma_0 = 0$ gives you *net entropy*.

The family of triples (91)-(93) is well defined for all $q \geq 0$ if we allow for an interpretation by continuity for $q = 1$. For $q = 0$ the triple is degenerate, for $q > 0$ it determines a proper primitive effort-based information triples. For $q = 1$ continuity considerations show that $(\phi_1, \mathrm{h}_1, \mathrm{d}_1)$ is identical to the standard logarithmic triple given in (83)-(85) (assuming that the overhead is neglected, $\gamma_0 = 0$).

The triples we have identified may all be conceived to be of the same structure as the standard logarithmic triple. What is meant by this, is that if we introduce the *deformed logarithms*, $\ln_q$, defined by the formula

$$\ln_q t = \begin{cases} \ln t \text{ if } q = 1 \\ \frac{1}{1-q}\big(t^{1-q} - 1\big) \text{ otherwise} \,, \end{cases} \tag{94}$$

then the formulas (91) - (93) may be expressed as follows in terms of the deformed logarithms:

$$\phi_q(s,u) = u^q - s + qs\ln_q\Big(\frac{1}{u}\Big) + \gamma_0\,, \tag{95}$$

$$\mathrm{h}_q(s) = s\ln_q\Big(\frac{1}{s}\Big) + \gamma_0\,, \tag{96}$$

$$\mathrm{d}_q(s,u) = u^q - s + qs\ln_q\Big(\frac{1}{u}\Big) - s\ln_q\Big(\frac{1}{s}\Big)\,. \tag{97}$$

These formulas are used for $s, u \geq 0$ and $q \geq 0$ (for negative q you do not obtain effort-based quantities). Note that if $q \leq 1$, then $\ln_q \frac{1}{t} = \infty$ for $t = 0$. The formulas indicate that it is not so much the logarithmic function $t \mapsto \ln_q t$ which is of importance but more so the function $t \mapsto \ln_q \frac{1}{t}$. This is no surprise to information theorists as the latter expression has a well known interpretation in terms of coding when $q = 1$ provided t represents a probability. No convincing interpretation of $\ln_q \frac{1}{t}$ appears to be known for other values of q. For $q = 1$, (95) – (97) reduce to (83) – (85) pertaining to the standard logarithmic triple,.

The family of triples (95)-(97), $q \geq 0$, is referred to as the family of deformed primitive triples – adding a qualifying "effort-based" if need be. The analogous utility-based primitive is the family of triples $(\mathrm{u}_q, \mathrm{m}_q, \mathrm{d}_q) = (-\phi_q, -\mathrm{h}_q, \mathrm{d}_q)$, i.e., for $q \geq 0$,

$$\mathrm{u}_q(s,u) = -u^q + s - qs\ln_q\Big(\frac{1}{u}\Big) - \gamma_0\,, \tag{98}$$

$$\mathrm{m}_q(s) = -s\ln_q\Big(\frac{1}{s}\Big) - \gamma_0\,, \tag{99}$$

$$\mathrm{d}_q(s,u) = u^q - s + qs\ln_q\Big(\frac{1}{u}\Big) - s\ln_q\Big(\frac{1}{s}\Big)\,. \tag{100}$$

The families (95)-(97) – and, analogously, (98)-(100) – are expressed in a convenient way, using the deformed logarithms. These logarithms were introduced by Tsallis [69]. They constitute a natural family of approximations to the the natural logarithm, and inserting them in the formulas pertaining to Shannon entropy, you have another route to the deformed triples, than the route we have taken.

Let us return to the process of integration hinted at in the beginning of the section. A substantial amount of concrete triples which illustrate the theory developed can be constructed by combining the Bregman construction with a process of integration.

Integration may be applied to any family of information triples and gives us new triples to work with. Note that by linearity of integration, the important property of affinity of marginals is preserved.

We comment mainly on integration of effort-based triples with a view towards applications in information theory and in statistical physics. Consider integration of one and the same primitive triple $(\phi, \mathrm{h}, \mathrm{d})$ over $I \times I$ with Bregman generator h. Partly for technical convenience we assume that h is non-negative. Then effort, entropy and divergence, will all be non-negative, also in the integrated version. Considering the intended applications, this is only natural.

Let T be a set provided with a Borel structure and with an associated measure μ. Let $X = Y$ be the function space consisting of all measurable functions $x : T \mapsto I$. To be precise, functions in X are identified if they agree μ-almost everywhere. Note that X is a convex cone. Consider the *integrated triple*

$$(\Phi, \mathrm{H}, \mathrm{D}) = \int_T (\phi, \mathrm{h}, \mathrm{d}) d\mu(t) \tag{101}$$

by which we express that the following equations hold:

$$\Phi(x, y) = \int_T \phi\big(x(t), y(t)\big) d\mu(t)\,, \tag{102}$$

$$\mathrm{H}(x) = \int_T \mathrm{h}\,\big(x(t)\big) d\mu(t)\,, \tag{103}$$

$$\mathrm{D}(x, y) = \int_T \mathrm{d}\,\big(x(t), y(t)\big) d\mu(t)\,. \tag{104}$$

As $\mathrm{h} \geq 0$, H is well defined and $0 \leq \mathrm{H}(x) \leq \infty$ for all $x \in X$.

As $t \mapsto (x(t), y(t))$ is measurable and as $(s, u) \mapsto \mathrm{d}(s, u)$ is non-negative and measurable, cf. (79), D is well-defined by (104). By linking, also Φ is well defined. Thus, $(\Phi, \mathrm{H}, \mathrm{D})$ is a well defined triple over $X \times Y$. We leave it to the reader to verify that $(\Phi, \mathrm{H}, \mathrm{D})$ is a proper information triple. And if ϕ has affine marginals ϕ^u for all $u \in I$, then Φ has affine marginals Φ^y for all $y \in Y$. The divergence functions which can be obtained in this way are *Bregman divergences*. Note that with this construction, the essential fundamental inequality $\mathrm{D} \geq 0$ even holds pointwise as $\mathrm{d} \geq 0$. For this reason, when we discuss the integrated triple, we refer to (74) as the *pointwise fundamental inequality*.

Bregman divergence may be used to modify visibility by taking $X \otimes Y$ to consist of all pairs $(x, y) \in X \times Y$ with $\mathrm{D}(x, y) < \infty$.

For the standard logarithmic triple (83) – (85), creation of new triples by integration has a few useful variants. Classically, we may take $I = [0, 1]$ and for T consider a countable set, referred to as the *alphabet*, and provide T with counting measure. Further, as state space, we may take the set of probability distributions over T. Then Φ becomes discrete *Kerridge inaccuracy*, H classical *Shannon entropy* and D discrete *Kullback-Leibler divergence*. If we generalize to cover non-discrete settings, entropy can only be finite for distributions with countable support, whereas the generalization of divergence makes sense more generally. For instance, we may consider the generator $s \mapsto s \ln \frac{1}{s}$ on the entire half-line $I = [0, \infty[$ and for (T, μ) take an arbitrary measure space, provided with some measure μ. As state space we can then, as one possibility, take the set of measures absolutely continuous with respect to μ and with finite-valued Radon-Nikodym derivatives with respect to μ. For two such measures, say $P = pd\mu$ and $Q = qd\mu$ we find that

$$\mathrm{D}(P, Q) = \int \Big(p(t) \ln \frac{p(t)}{q(t)} + q(t) - p(t) \Big) d\mu(t) \,. \tag{105}$$

This may be called *generalized Kullback-Leibler divergence*. It is the more natural divergence to consider. For one thing, the integrand is non-negative by the pointwise fundamental inequality. If we restrict attention to finite measures P and Q with the same total mass, this reduces to the standard expression $\int p \ln \frac{p}{q} d\mu$. The standard expression also gives a divergence measure if the two measures are finite and $Q(T) \leq P(T)$ and, moreover, the important compensation identity also holds in this case since the additional terms (stemming from $u - s$ in (85)) are integrable and affine.

Now consider extensions to cover also integration of the family $(\phi_q, \mathrm{h}_q, \mathrm{d}_q)$. It is natural to consider these triples over $I \times I$ with $I = [0, 1]$ in order to ensure that $\mathrm{h}_q \geq 0$. By integration we obtain the triples

$$(\Phi_q, \mathrm{H}_q, \mathrm{D}_q) \tag{106}$$

defined over appropriate function spaces, typically representing probability distributions. For $q > 0$ these triples are proper effort-based information triples. For $q = 0$ you obtain degenerate triples. The quantity H_q, is meaningful in discrete cases with T finite or countably infinite, and defines *Tsallis entropy*. For the continuous case, Tsallis entropy does not make much sense, but the divergence function D_q does.

So far, we have discussed integration of primitive triples. This concerns a process where the original state space (the interval I) is changed to a new state space and then, an information triple over the new state space is constructed. A similar process applies if we start out with a family $\big((\Phi_t, \mathrm{H}_t, \mathrm{D}_t)\big)_{t \in T}$ of proper information triples over the same state space X (formally, over $X \times Y$ or $X \otimes Y$ with structures as usual and, typically, $Y = X$). Then we may consider the integrated triple

$$(\Phi, \mathrm{H}, \mathrm{D}) = \int_T (\Phi_t, \mathrm{H}_t, \mathrm{D}_t) d\mu(t) \tag{107}$$

defined by

$$\Phi(x,y) = \int_T \Phi_t(x,y) d\mu(t)\,, \tag{108}$$

$$\mathrm{H}(x) = \int_T \mathrm{H}_t(x,y) d\mu(t)\,, \tag{109}$$

$$\mathrm{D}(x,y) = \int_T \mathrm{D}_t(x,y) d\mu(t)\,. \tag{110}$$

With suitable measurability conditions, $(\Phi, \mathrm{H}, \mathrm{D})$ is a well-defined proper information triple. Also, the standard restriction of affinity is preserved by this process. As a useful but trivial remark, we note that properness of the integrated triple only needs properness of $(\Phi_t, \mathrm{H}_t, \mathrm{D}_t)$ for a set of positive μ-measure. An instance of this feature with T a two-element set was already discussed in Section 7.

The most obvious application of the process of integration probably is to integrate the utility-based standard algebraic triple $(\mathrm{u}, \mathrm{m}, \mathrm{d}) = \big(-u^2 + 2su, s^2, (s-u)^2\big)$, cf. (89). This triple is considered over $I \times I$ with $I =]-\infty, \infty[$. Integrating over a measure space (T, μ), you are led to take as state space the L^2-space over (T, μ). In standard notation, the integrated triple $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ is given by

$$\mathrm{U}(x,y) = -\|y\|^2 + 2\langle x, y\rangle\,, \tag{111}$$

$$\mathrm{M}(x) = \|x\|^2 \tag{112}$$

$$\mathrm{D}(x,y) = \|x-y\|^2\,. \tag{113}$$

We collect in Section 20 comments on these classical concepts, seen in the light of the theory here developed.

Some comments on the generation of information triples by the method inspired by Bregman [68] are in order. The focus of Bregman's method has often been on the divergence measures it generates. Before Bregman's work one mainly studied *f-divergences*, introduced independently by Csiszár [70], Morimoto [71] and by Ali and Silvey [72]. We find that often, Bregman divergences occur more naturally and have more convincing interpretations.

As we have seen, the widely studied entropies bearing Tsallis' name can be derived via a Bregman-type construction. In Section 22 we shall have a closer look at these entropies. They have received a good deal of attention, especially within statistical physics. Some comments on the origin of these measures of entropy are in place. Tsallis' trend-setting paper [2] is from 1988 but, originally, the entropies go back to Havrda and Charvát [73], to Daróczy [74] and to Lindhard and Nielsen [75], [76] who all, independently of each other, found the notion of interest. Characterizations via functional equations were derived in Aczél and Daróczy [77], see also the reference work [78] as well as [40]. Regarding the physical literature, there is a casual reference to Lindhard's work in one of Jaynes' papers, [79]. But only after the publication of Tsallis 1988-paper mathematicians and, especially, physicists took an interest in the "new" entropy measures. We refer to the database maintained by Tsallis with more than 2000 references. From the recent literature we only point to Naudts, [80] who also emphasized the convenient approach via Bregman generators.

19 Control and Support with Minimal Risk

Three considerations underlie the refinements of this section.

Firstly, MaxEnt problems can often be tackled without recourse to the technique involving Lagrange multipliers, cf. the remark following Theorem 15.2. So no differentiation is needed for certain problems involving side conditions [12] . What if you have no side conditions? Then you seek identification of maxima or minima for real functions of a real variable. Can this also be done without differentiation?

Secondly, have a look at Figure 1. Really, what is dominating is the curve h and the straight line. The belief instance u is not that prominent. More so the line it determines. True, this is the tangent at $(u, \mathrm{h}(u))$, determined by differentiation but what is essential is that it dominates h, a feature ensured by concavity. Domination by control appears as the right focus..

Thirdly, also non-concave functions can of course have maxima. Therefore, avoiding differentiation, there may be no need for the convenience of the assumed concavity of the generator.

Motivated by these considerations we embark on the intended refinement. It will be more geometric than the approach in the previous section. We shall work in the subset $I \times \mathbb{R}$ of the plane $\mathbb{R}^2$ with I an open interval. The reason why I is taken to be open is that this will save us some comments on end-point behaviour. For linear functions

[12] This is not a new observation, see e.g. Csiszár [58], Topsøe [26], Campenhout and Cover [81] and the recent work by Pavon and Ferrante, [82]. In contrast to this, the many examples contained in Kapur [83] builds excessively on differentiation techniques.

on I we use the bracket notation as in

$$\langle s, w \rangle = \alpha + \beta s;\ s \in I\,. \tag{114}$$

A linear function w is identified with its graph, which could be any non-vertical line. For a point $Q \in I \times \mathbb{R}$ we talk about points *to the left of* Q (*to the right of* Q) as points left of (right of) the vertical line through Q.

We shall work with some special sets, called *butterfly sets*. Such a set is characterized by two linear functions w^- and w^+, the *boundary lines*, and a point $Q \in w^- \cap w^+$, the *crossing point*. This terminology is also applied if w^- and w^+ coincide. The butterfly set determined by (w^-, w^+) and with crossing point$Q$ is the set $B(w^-, w^+|Q)$ of points $(s,t) \in I \times \mathbb{R}$, "squeezed in" between the boundary lines, in short

$$B(w^-, w^+|Q) = \{\min(\langle s, w^- \rangle, \langle s, w^+ \rangle) \le t \le \max(\langle s, w^- \rangle, \langle s, w^+ \rangle)\}\,. \tag{115}$$

In the notation for butterfly sets it is assumed that either $w^- = w^+$ or else w^- is below w^+ to the left of Q and above w^+ to the right of Q. If $w^- = w^+$, the butterfly set is *thin*. Otherwise it is *fat*.

We shall consider a *generalized generator* which is just any real-valued function h defined on I. For our standard modelling, I will be the state space as well as the belief reservoir: $X = Y = I$. And a *control*, here also called a *control line*, is any linear function w which dominates h, i.e. $\mathrm{h}(s) \le \langle s, w \rangle$ for $s \in I$. As usual, the set of controls is denoted W or $\hat{Y}$. We assume that $W \neq \emptyset$. Visibility and controllability are the diffuse relations on $X \times Y$, respectively $X \times \hat{Y}$.

The key lemma is the following geometry-based result. We shall not write out all details of the proof. This is standard routine. You should observe that both parts of the result are existence statements which do not have purely constructive proofs. The proof is based only on the most basic elements of the infinitesimal calculus via appeal to statements about existence of suprema and infima of sets of real numbers. The lemma replaces the approach via differentiation of the previous section, cf. (78)-(79).

Lemma 19.1. *(i) With assumptions as stated (I open, $W \neq \emptyset$), there exists a function $\overline{\mathrm{h}}$ on I such that every point on the graph of $\overline{\mathrm{h}}$ lies on some control line and such that this property applies to no point below this graph.*

(ii) Further, for every $u \in I$ there exists a butterfly set $B_u = B(w_u^-, w_u^+|Q_u)$ with $Q_u = (u, \overline{\mathrm{h}}(u))$ as crossing point such that the set of control lines which passes through Q_u is identical with the set of control lines contained in B_u.

Proof. Property (i) is trivial. One simply defines $\overline{\mathrm{h}}$ by

$$\overline{\mathrm{h}}(s) = \inf\{t|\exists w \in W : \langle s, w\rangle = t\} \tag{116}$$

for $s \in I$[13].

As to (ii), we shall outline one way to the proof. Let $u \in I$ and have a look at Figure 2. There, $P = (u, \mathrm{h}(u))$ and $Q = (u, \overline{\mathrm{h}}(u))$. For every pair of points (P^-, P^+) on the graph of h, with P^- to the left and P^+ to the right of Q, the set T, understood to be open, which lies above the butterfly set in the figure, does not contain any point from the graph of h. Clearly, the union of all sets T which can be constructed in this way, call it T_u, is the set above two, possibly coinciding control lines w_u^- and w_+^+ which constitute the boundary of T_u. The set $B(w_u^-, w_+^+|Q)$ is the butterfly set B_u we were looking for. $\square$

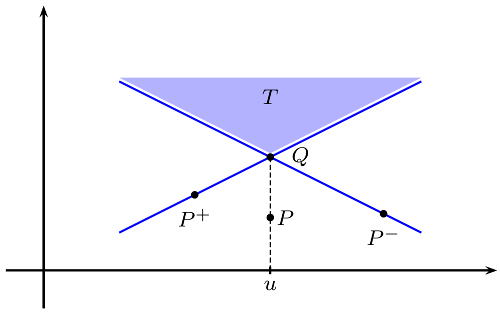


Figure 2: For the proof of Lemma 19.1.

With the lemma in place, we can define response as a point map from I into W. The map will not be surjective and, depending on h, possibly not injective either. To define the map, let $u \in Y$ be a belief instance and consider the butterfly set $B_u = B(w_u^-, w_u^+|Q_u)$. As response of u we take $w_u^+ = w_u^-$ if these control lines coincide. If the horizontal line through Q_u is contained in B_u, we take this control line as response. In the remaining cases we take as response that control line $\hat{u}$ among w_u^+ and w_u^- which, numerically, has the smallest slope. In other words, $\hat{u}$ is that control line contained in B_u which, numerically, has the smallest slope.

The above construction defines $\hat{u} \in W$ uniquely. When B_u is thin, there is only one control line to choose from, whereas when B_u is fat, we made a specific choice so as to minimize the risk. The control lines constructed this way are called *minimal-risk controls*. As to the nature

[13]As you will realize, this is the *concave envelope* of h. Automatically, this function is upper semi-continuous.

of the result, one may note that it involves *global* rather than *local* considerations as would be involved in an approach via differentiation. This is only proper as the result is designed to be an aid in global optimization problems.

The following obvious corollary is a replacement of a classical basic result on maxima of functions based on differentiation. Though not difficult to establish, the result – properly extended to higher dimensions, also incorporating boundary behaviour – could be taken as a basis parts of global optimization theory. As such, one wonders if it has received any attention previously.

Corollary 19.1. *Let $I \subseteq \mathbb{R}$ be an open interval and h a real function defined on I which is dominated by a real line.*

A necessary and sufficient condition that h has a maximum in I is that for some point $u \in I$, the butterfly set B_u contains a horizontal line, necessarily w_u, and that $\mathrm{h}(u) = \overline{\mathrm{h}}(u)$. Assume that these conditions are fulfilled for some point $u \in I$. Then u is a maximum point of h and a necessary and sufficient that u is the unique maximum point of h in I is that $w = \hat{u}$ intersects no other point on the graph of h than the point $Q_u = \big(u, \mathrm{h}(u)\big)$.

As to the various possibilities for the type of B_u – fat or thin – and for $\overline{\mathrm{h}}$ in relation to h, we note the following:

Lemma 19.2. *Let $u \in I$.*

(i) If B_u is fat, then $\overline{\mathrm{h}}(u) = \limsup_{v \to u} \mathrm{h}(v)$.

(ii) If h is upper semi-continuous, in particular if h is continuous, and if $\overline{\mathrm{h}}(u) > \mathrm{h}(u)$, then B_u is thin.

Proof. (i) follows by noting that if $w_u^- \neq w_u^+$, then no line segment connecting a point on w_u^- to the left of Q_u with a point on w_u^+ to the right of Q_u can dominate the relevant part of h since then the prolongation of the line segment would dominate h for all arguments in I, clearly contradicting the definition of $\overline{\mathrm{h}}(u)$.

Part (ii) is an easy consequence. □

Figures 3a - 3c illustrate some possibilities for the location of the possible butterfly sets in relation to $\overline{\mathrm{h}}$.

Our construction allows us to define a pretty natural information triple associated with any generalized generator. We simply define $\hat{\phi}$ and $\hat{\mathrm{d}}$ for $(s, w) \in I \times W$ by

$$\hat{\phi}(s, w) = \langle s, w \rangle \,, \tag{117}$$

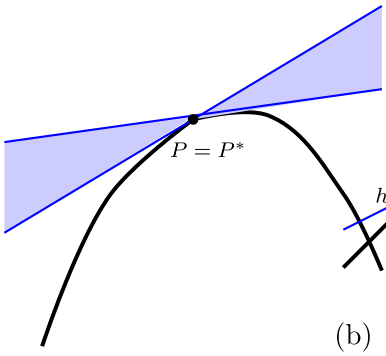
$$\hat{\mathrm{d}}(s, w) = \langle s, w \rangle - \overline{\mathrm{h}}(s) \tag{118}$$

and can then assert as follows:

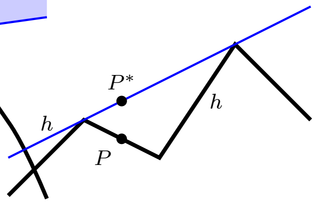
Theorem 19.1. *With the definitions* (117) *and* (118)*,* $(\hat{\phi}, \overline{\mathrm{h}}, \hat{\mathrm{d}})$ *is a weakly proper effort-based information triple over* $I \times W$*. The triple has affine marginals* $\hat{\phi}^w$*.*

With the thorough preparations, this is evident.

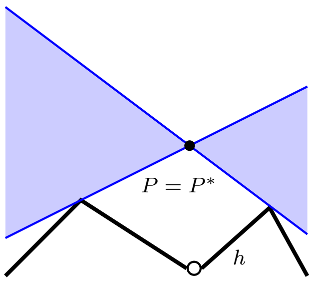
If all butterfly sets B_u, $u \in I$ are thin, e.g. if h is a genuine Bregman generator as in the previous section, the information triple constructed is proper. If $\overline{\mathrm{h}} = \mathrm{h}$, i.e. if h is concave, our construction has some merits over the standard Bregman construction as smoothness is not required.



(a) Non-smooth generator



(b) Mixed concave/convex



(c) Upper semi-continuous generator with a discontinuity point

Figure 3: Possible types of generators.

Regarding the assumption that I is open, this can be dispensed with at the cost of some comments on *degenerate control lines*, lines which really only give control at one of the endpoints. This may be formulated by allowing infinite values for the controls or one may focus on decompositions of $I \times \mathbb{R}$ into two convex sets. We leave it to the reader to work this out (and to modify the proof of Lemma 19.1 accordingly, working separately to the left of Q and to the right of Q).

Many other issues triggered by the construction in this section deserves a closer study. Clearly, one may "change sign" and discuss utility-based systems. This involves notion of *support lines* and *minimal-risk supports*. Then just as with standard Bregman constructions, one should deal with the more involved geometric complications when functions over (convex) areas in finite dimensional Euclidean spaces are involved. Further one may replace h, first with its graph (in fact done) but then further with any subset of $I \times \mathbb{R}$. More generally, you may consider subsets G of a separable Hilbert space provided with a hyperplane π and a choice of direction orthogonal to π. The hyperplane is a replacement for the x-axes of our discussion and the direction a replacement for the y-axes. For such systems, height over the hyperplane will be a replacement for function values.

Many possibilities appear attractive for further study.

20 A geometric model

Let us return to the model $(\mathrm{U}, \mathrm{M}, \mathrm{D})$ given by (111) - (113) of Section 18. This is the utility-based information triple $\big(-\|y\|^2+2\langle x, y\rangle, \|x\|^2, \|x-y\|^2\big)$ pertaining to the Hilbert space $X = Y = \mathrm{L}^2(T, \mu)$. The triple is proper and has affine marginals U^y given y.

In this case, the linking identity (after rearrangement of terms) is identical to the cosine relations. Other well-known basic facts of inner-product spaces can be derived by combining the linear structure of such spaces with the basic properties of information triples. We leave this to the interested reader, except for pointing to the identity you obtain from the compensation identity (65) applied to D. The identity you get is of central importance for classical least squares analysis[14]

Among games we can study are *absolute games* which are games directly associated with the information triple $(\mathrm{U}, \mathrm{M}, \mathrm{D})$. They involve minimization of M over various preparations, in other terms, the search for elements closest to the origin subject to certain restrictions. Rather than providing concrete results about the absolute games, we choose to concentrate on a study of *relative games*, which are games depending on the specification of a preparation and a prior $y_0 \in Y$, cf. Section 8. If the preparation $\mathcal{P}$ is convex and closed, the D-projection x^* of y_0 on $\mathcal{P}$ exists; it is the unique point in $\mathcal{P}$ which is closest in norm to y_0[15]. As standard convexity- and continuity assumptions are also in place, Theorem 15.4 applies. It follows that the game $\gamma(\mathcal{P}; y_0)$ is in equilibrium with the D-projection x^* as bi-optimal state. The updating gain for this game is given by (18), i.e.

$$\mathrm{U}_{|y_0}(x, y) = \|x - y_0\|^2 - \|x - y\|^2 \,. \tag{119}$$

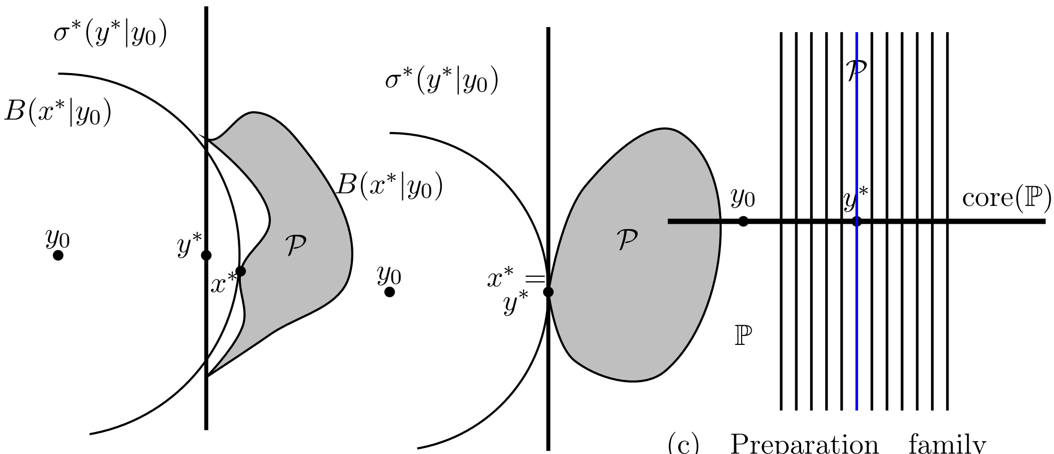
In this case the Pythagorean inequality reduces to the classical inequality

$$\|x - y_0\|^2 \geq \|x - x^*\|^2 + \|x^* - y_0\|^2 \,, \tag{120}$$

valid for every $x \in \mathcal{P}$.

[14]apparently, the identity has no special name in this setting – it goes back at least to Gauss.

[15]though classical, the reader may appreciate to note that this existence result is derived with ease from the compensation identity and completeness of Hilbert space.



(a) Game not in equilibrium

(b) Game in equilibrium

(c) Preparation family and its core

Figure 4: Optimal strategies, typical equilibrium via core.

Combining Proposition 13.7 and Theorem 15.5 we obtain rather complete information about the updating games, also for preparations which are not necessarily convex. For instance, Figure 4a illustrates a case with unique optimal strategies for both players and yet, the game is not in equilibrium. Figure 4b illustrates a typical case with a game in equilibrium. For both figures, x^* denotes the optimal strategy for Nature and y^* the optimal strategy for Observer. Indicated on the figures you also find the largest strict divergence ball $\mathrm{B}(x^*|y_0)$ and the largest half-space $\sigma^+(y^*|y_0)$ which is external to $\mathcal{P}$. The two values of the game can then be determined from the figures, $\|x^* - y_0\|^2$ for Nature, respectively $\|y^* - y_0\|^2$ for Observer.

Lastly some words on the typical preparations you meet in practice. In consistency with the philosophy expressed in Section 9 these are the feasible preparations. The strict ones are affine subspaces and the slack ones are convex polyhedral subsets. We shall determine the core of families of strict preparations:

Proposition 20.1. *Consider a family $\mathbb{P} = \mathbb{P}^{\mathbf{y}}$ of strict feasible preparations determined by finitely many points $\mathbf{y} = (y_1, \cdots, y_n)$ in X. The core of this family consists of all points in the affine subspace through y_0 generated by the vectors $y_i - y_0$; $i = 1, \cdots, n$, i.e.*

$$\mathrm{core}(\mathbb{P}) = \left\{ y_0 + \sum \alpha_i (y_i - y_0) \middle| (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \right\}. \qquad (121)$$

Proof. An individual member $\mathcal{P}$ of $\mathbb{P}$ is determined by considering all $x \in X$ for which the values of $\mathrm{U}_{|y_0}(x, y_i)$; $i = 1, \cdots, n$ have been fixed. Note that fixing these values is the same as fixing the inner products $\langle x-y_0, y_i-y_0\rangle$ or, equivalently, the inner products $\langle x, y_i-y_0\rangle$. If y^* is of the form given by (121), $y^* = y_0 + \sum \alpha_i(y_i - y_0)$, then $\langle x, y^* - y_0\rangle = \sum \alpha_i \langle x, y_i - y_0\rangle$ and we realize that this is independent of x if x is restricted to run over some preparation in $\mathbb{P}$. Then also $\mathrm{U}_{|y_0}(x, y^*)$ is independent of x when x is so restricted. We conclude that $y^* \in \mathrm{core}(\mathbb{P})$. This proves the inclusion "$\supseteq$" of (121).

To prove the other inclusion, assume, as we may, that $y_0 = 0$ and that the y_i forms an orthonormal system. Consider a point $y^* \in \mathrm{core}(\mathbb{P})$. Determine $\mathcal{P} \in \mathbb{P}$ such that $y^* \in \mathcal{P}$. By Theorem 14.1, y^* is the bi-optimal state of $\gamma(\mathcal{P}; y_0)$. Let c_i; $i = 1, \cdots, n$ denote the common values of $\langle x, y_i\rangle$ for $x \in \mathcal{P}$. Then $x^* = \sum c_i y_i$ is the orthogonal projection of $y_0 = 0$ on $\mathcal{P}$, hence $y^* = x^*$. This argument shows that the core is contained in the subspace generated by the y_i. This is the result we want as we assumed that $y_0 = 0$. □

In order to determine the projection of y_0 on a specific preparation $\mathcal{P} = \mathcal{P}^{\mathbf{y}}(\mathbf{h}) \in \mathbb{P}$, we simply intersect $\mathrm{core}(\mathbb{P})$ with $\mathcal{P}$. If you do this analytically, one may avoid trivial cases and assume that $y_i - y_0$; $i = 1, \cdots, n$ are linearly independent. In Figure 4c we have illustrated the situation in the simple case when $n = 1$.

21 Randomization, Sylvester's problem, capacity problems

In this section we focus on an important general method of generating new triples from old ones. The key word is *randomize!* As starting point we take a simple Y-domain model with $Y = X$, a convex set. For visibility we take the diffuse relation $X \times Y$. [16] Given is a finite-valued general divergence function over $X \times Y$ for which the compensation identity (65) holds.

As concrete example, you may have in mind, take that of a real Hilbert space X provided with norm-squared distance, $\mathrm{D}(x, y) = \|x - y\|^2$. And as motivating example, consider *Sylvester's problem, to determine the point with the least maximal distance to a given finite set $\mathcal{P}$ of points in X*, cf. [84] or the monograph [85]. For the original problem, X was the Euclidean plane. But the problem makes good sense in the general setting with X any convex set provided with a suitable replacement for classical squared distance.

[16] later we comment on a more general $\hat{Y}$-domain model with $\hat{Y}$ any set and response as a map $x \mapsto \hat{x}$ from X to $\hat{Y}$; the necessary modifications of the construction to follow are quite straight forward and will be left to the interested reader.

The problem is a minimax problem and may formally be conceived as related to the special information triple $(\mathrm{D}, 0, \mathrm{D})$. Indeed, the problem is to find optimal Observer strategies for the associated game $\gamma(\mathcal{P})$ and to calculate Observers value of the game, the MinRisk-value $\mathrm{Ri}_{\min}(\mathcal{P})$. However, this game is rather trivial as Natures value in the game is 0. Thus no equilibrium-type results are available.

To find a remedy, we apply a process of randomization. For that, we embed the state space X in the convex space $\tilde{X} = \mathrm{MOL}(X)$ of *molecular probability measures*. An element $\alpha \in \tilde{X}$ is represented as a family $\alpha = (\alpha_x)_{x \in X}$ of non-negative numbers such that $\sum_{x \in X} \alpha_x = 1$ and such that the *support* of α, i.e. the set $\mathrm{supp}(\alpha) = \{x | \alpha_x > 0\}$, is finite.

The new model we shall construct is conceived as a $\hat{Y}$-type model. As state space we take $\tilde{X}$. As X, this is a convex set. There is no belief reservoir – or you may artificially stick to $Y = X$. The emphasis will be on control. A bit confusing, perhaps, we take Y as control space. So now we could also put $W = Y$! We shall not do so, but it is a fact – so we claim – that the many possible roles for central objects of study have often contributed to a failure to realize that the same basic underlying structure can be used for seemingly different problems. In the present case, the good sense in considering elements of Y as controls is the idea, really an element of location theory, that from a point $y \in Y$ you should try to control the given points in the set $\mathcal{P}$ as best you can. Response must now be defined as a map from $\tilde{X}$ into Y. We do this in what appears to be the natural way by taking as response the map $\alpha \mapsto \mathrm{b}(\alpha)$ with $\mathrm{b}(\alpha)$ the *barycenter* of α given by

$$\mathrm{b}(\alpha) = \sum_{x \in X} \alpha_x x \,. \tag{122}$$

The new triple we shall consider is the triple $(\tilde{\Phi}, \tilde{\mathrm{H}}, \tilde{\mathrm{D}})$ over $\tilde{X} \times Y$ given by

$$\tilde{\Phi}(\alpha, y) = \sum_{x \in X} \alpha_x \, \mathrm{D}(x, y) \,, \tag{123}$$

$$\tilde{\mathrm{H}}(\alpha) = \sum_{x \in X} \alpha_x \, \mathrm{D}(x, \mathrm{b}(\alpha)) \,, \tag{124}$$

$$\tilde{\mathrm{D}}(\alpha, y) = \mathrm{D}(\mathrm{b}(\alpha), y) \,. \tag{125}$$

For $\mathcal{P} \subseteq X$, denote by $\tilde{\mathcal{P}}$ the set of $\alpha \in \tilde{X}$ which are supported by $\mathcal{P}$, i.e. $\sum_{x \in \mathcal{P}} \alpha_x = 1$. By $\tilde{\gamma}(\mathcal{P})$ we denote the game corresponding to the triple $(\tilde{\Phi}, \tilde{\mathrm{H}}, \tilde{\mathrm{D}})$ with $\tilde{P}$ as preparation. A basic fact which contributes to the significance of games of this type is that, as easily seen, risk does not increase when you replace the game $\gamma(\mathcal{P})$ with $\tilde{\gamma}(\mathcal{P})$, in particular, with self-explanatory notation,

$$\tilde{\mathrm{Ri}}_{\min}(\mathcal{P}) = \mathrm{Ri}_{\min}(\mathcal{P})\,. \tag{126}$$

This fact relies on the affinity of the marginals of $\tilde{\Phi}$ for fixed y.

Theorem 21.1. *The triple $(\tilde{\Phi}, \tilde{\mathrm{H}}, \tilde{\mathrm{D}})$ over $\tilde{X} \times Y$ is a proper information triple over $\tilde{X} \times Y$ and the triple has affine marginals.*

Let $\mathcal{P}$ be a subset of X and consider the game $\tilde{\gamma}(\mathcal{P})$. Consider a pair $(\alpha^*, y^*) \in \tilde{\mathcal{P}} \times Y$ of strategies in the game $\tilde{\gamma}(\mathcal{P})$ with y^* adapted to x^*, i.e. $y^* = \mathrm{b}(\alpha)$. Then, if y^* is "sub-robust" in the sense that, for some constant R,

$$\forall x \in X : \mathrm{D}(x, y^*) \leq R\,, \tag{127}$$

$$\forall x \in \mathrm{supp}(\alpha) : \mathrm{D}(x, y^*) = R\,, \tag{128}$$

y^* is the unique optimal strategy for Observer in $\tilde{\gamma}(\mathcal{P})$ as well as in $\gamma(\mathcal{P})$. Further, $\mathrm{Ri}_{\min}(\mathcal{P}) = R$ and x^* is a bi-optimal strategy for $\tilde{\gamma}(\mathcal{P})$.

Proof. With preparations done, the first part is trivial, and the second is also so, obtainable as an application of Corollary 10.1. □

Note that the linking identity is nothing but another way of formulating the compensation identity.

With Theorem 21.1 we have a solution to Sylvester's problem for an abstract model *provided* you can somehow point to a possible solution. It can be shown, modulo technical assumptions to ensure existence of optimal strategies, that the sought optimal Observer strategy must be of the form as stated in the theorem.

Though we shall not go through the technical details of the necessity result indicated, we remark that it depends on a closer study of the new entropy function $\tilde{\mathrm{H}}$. This quantity pops up many places. We called it the *compensation term* in Section 15 and in (66) it appeared as the *Jensen-Shannon divergence*. Here we shall think about it in yet another way, as related to *information transmission rate*[17]. This way of thinking refers to the map $x \mapsto y$ with $y = x$ (more generally, $y = \hat{x}$ of an extended model) as a map from an *input letter* to an *output letter*. Then an element $\alpha \in \tilde{X}$ represents a distribution over the input letters, a *source*, and response tells you what is happening on the output side. What appears important to study is how the rate behaves under mixtures. Thus we have a need to study elements in $\tilde{\tilde{X}} = \mathrm{MOL}(\tilde{X})$.

[17]then also related to the notion of *mutual information* which will not be investigated in the present study.

This is quite easy and as the result we need illuminates the flexibility of our modelling, we shall provide the details.

First, define *information transmission rate* related to $\alpha \in \tilde{X}$ simply as

$$\mathrm{I}(\alpha) = \tilde{\mathrm{H}}(\alpha)\,. \tag{129}$$

The lemma we need is the following:

Lemma 21.1. *With the setting as above, consider any $w = (w_\alpha)_{\alpha \in \tilde{X}} \in \tilde{\tilde{X}}$ and put $\alpha_0 = \sum_{\alpha \in \tilde{X}} w_\alpha \alpha$. Then, for every $w \in \tilde{\tilde{X}}$,*

$$\mathrm{I}\Big(\sum_{\alpha \in \tilde{X}} w_\alpha \alpha\Big) = \sum_{\alpha \in \tilde{X}} w_\alpha\,\mathrm{I}(\alpha) + \sum_{\alpha \in \tilde{X}} w_\alpha\,\mathrm{D}\big(\mathrm{b}(\alpha), \mathrm{b}(\alpha_0)\big)\,. \tag{130}$$

Proof. If you write $\tilde{\mathrm{H}}$ in place of I, this follows from the identity (63) of Theorem 15.1 with $\tilde{\mathrm{H}}$ in place of H. □

With the technical lemma in place, and with an extension to the $\hat{Y}$-domain as hinted at above, a study of abstract models of information transmission systems runs smoothly and you can derive operational necessary and sufficient conditions for the requirements of optimal strategies. On Natures side, an optimal strategy is an input distribution for which the transmission rate reaches the maximum, the *capacity* of the system. The result is a *Kuhn-Tucker type* result, well known from general convexity theory and from Information theory. Though a bit sketchy, it should not be difficult to transform the proof of Topsøe [86] to provide a precise statement and proof in the abstract setting.

22 Tsallis worlds

Recall the introduction in Section 18 of the family of *Tsallis entropies*. In this section we present arguments which may help to appreciate the significance of these measures of entropy.

The main result, Theorem 22.1 was presented in a different form in [36] and, less formally, in [35]. Here we present detailed proofs which were not provided in these sources.

The introduction in Section 18 of the Bregman generators h_q and thereby, via a process of integration, of Tsallis entropy, cf. (106), does not in itself constitute an acceptable interpretation. Via coding considerations, the significance of the Bregman generator h_1, leading to the notion of Shannon entropy is well understood. Despite some attempts to extend this to more general entropy measures, cf. [87], [88] and [89], a general approach via coding has not yet been fully convincing. In [90] you find a previous attempt of the author centred on a certain property of factorization.

The results presented here indicate that possibly, convincing and generally acceptable physical justifications of Tsallis entropy can be provided by involving interaction between the physical system studied and the physicist as a central element. Previous endeavours to find physical justification for Tsallis entropy are discussed in detail in Tsallis, [91], a paper which, quite conveniently, just appeared. We share the view that though the "Tsallis-q" can be viewed just as a parameter introduced simply to fit data, this is not satisfactory and operational justification is needed. Interaction as here emphasized in combination with a notion of *description* may offer a common denominator on the way to more insight.

To set the scene for our study, introduce the *alphabet* $\mathbb{A}$, a discrete set of *basic events* which are identified by an *index*, typically denoted by i. Sensible indexing is often of importance and depends on the concrete physical application. The semiotic assignment of indices shall facilitate technical handling and catalyze semantic awareness. As we have no concrete application in mind, no extra structure is introduced which could justify a specific choice of indices.

The state space X is taken to be identical to the belief reservoir Y and equal to $M_+^1(\mathbb{A})$, the set of probability distributions over $\mathbb{A}$. Generically, $x = (x_i)_{i\in\mathbb{A}}$ will denote a state and $y = (y_i)_{i\in\mathbb{A}}$ a belief instance. Thus x and y are characterized by their point probabilities. As Y_{det}, the set of certain belief instances, we take the set of deterministic distributions over $\mathbb{A}$. Visibility $y \succ x$ shall mean that x is absolutely continuous w.r.t. y. Thus $X \otimes Y$ consists of all pairs $(x, y) \in M_+^1(\mathbb{A}) \times M_+^1(\mathbb{A})$ with $\text{supp}(x) \subseteq \text{supp}(y)$. We shall not need a control space or a response function[18].

A knowledge instance will be a family $z = (z_i)_{i\in\mathbb{A}}$ over $\mathbb{A}$ of real numbers, not necessarily a probability distribution. The interpretation of z_i is as the *intensity* with which the basic event indexed by i is presented to Observer. For this reason, z is referred to as the *intensity function*. The individual elements z_i are *local intensities*.

The interaction between x, y and z is given by an interactor Π, cf. Section 5. We assume that Π acts *locally*, i.e. that there exists a real-valued function π, the *local interactor*, defined on $[0, 1]^2 = [0, 1] \times [0, 1]$ such that, when $z = \Pi(x, y)$, then $z_i = \pi(x_i, y_i)$ for all $i \in \mathbb{A}$. The world defined in this way by a local interactor is denoted Ω_π or, if need be, $\Omega_\pi(\mathbb{A})$. From now on, when we talk about an "interactor", we have a local interactor in mind.

[18] for the classical case, these concepts are introduced in Section 23.

Regarding regularity conditions, we assume that π is finite on $[0,1]\times]0,1]$, continuous on $[0,1]^2 \setminus \{(0,0)\}$ and continuously differentiable on $]0,1[\times]0,1[$. The interactor is *weakly consistent* if $\sum_{i\in\mathbb{A}} z_i = 1$ whenever $(x,y) \in X \otimes Y$ and $(z_i)_{i\in\mathbb{A}} = \Pi(x,y)$. If you can even conclude that $z = (z_i)_{i\in\mathbb{A}}$ is a probability distribution, π is *strongly consistent*. The interactor π is *sound* if $\pi(s,s) = s$ for every $s \in [0,1]$.

For $q \in \mathbb{R}$, the *algebraic interactor* π_q is given on $[0,1]^2$ by

$$\pi_q(s,t) = qs + (1-q)t\,. \tag{131}$$

These interactors are all sound and weakly consistent and, for $0 \le q \le 1$, even strongly consistent. The corresponding worlds are denoted $\Omega_q = \Omega_q(\mathbb{A})$. The notation is consistent with the notation introduced in Section 5. The significance of the algebraic interactors is derived from the following result.

Lemma 22.1. *Assume that the alphabet $\mathbb{A}$ is countably infinite. Then only the algebraic interactors are weakly consistent.*

Proof. Let π be weakly consistent and put $q = \pi(1,0)$. Consider a deterministic distribution δ over $\mathbb{A}$ and apply weak consistency with $x = y = \delta$ to find that $\pi(0,0) = 0$. Thus, if x and y both have support in a subset $\mathbb{A}_0 \subseteq \mathbb{A}$, you can neglect contributions stemming from (x_i, y_i) with $i \notin \mathbb{A}_0$ and conclude consistency over $\mathbb{A}_0$, i.e. that $\sum_{i\in\mathbb{A}_0} \pi(x_i, y_i) = 1$. By weak consistency (in the extended form just established), $\pi(s,t) + \pi(1-s, 1-t) = 1$ for all $(s,t) \in [0,1] \times [0,1]$, in particular, $\pi(0,1) = 1-q$. Consider $(x_0, y_0) = (0,1)$ and $(x_i, y_i) = (\frac{1}{n}, 0)$ for $i = 1, \cdots, n$, apply weak consistency and conclude that $\pi(\frac{1}{n}, 0) = \frac{1}{n}q$. Then, for $p \in \mathbb{N}$, consider vectors (x_i, y_i) of the form $(0,1), (\frac{1}{n}, 0), \cdots, (\frac{1}{n}, 0), (\frac{p}{n}, 0)$. By weak consistency and previous findings, conclude that $\pi(s,0) = sq$ for all rational $s \in [0,1]$. By continuity, this formula holds for all $s \in [0,1]$. Quite analogously, $\pi(0,t) = t(1-q)$ for all $t \in [0,1]$. Finally, $\pi = \pi_q$ follows by weak consistency applied to $(s,t), (1-s, 0), (0, 1-t)$. □

In particular, if $\mathbb{A}$ is infinite then, automatically, a weakly consistent interactor is sound. In fact, all concrete interactors we shall deal with will be sound.

Instead of searching only for a suitable entropy function for the world Ω_π, we find it more rewarding to search for a suitable full information triple for this world. Let us analyze what such a triple, say $(\Phi, \mathrm{H}, \mathrm{D})$, could be. A natural demand is that Φ, H and D should all act locally. Therefore, according to Section 18 what we are really searching for is a primitive information triple $(\phi, \mathrm{h}, \mathrm{d})$ over $[0,1] \times [0,1] \setminus \{(s,u) | s > 0, u = 0\}$, cf. (72), such that $(\Phi, \mathrm{H}, \mathrm{D})$ is obtained from this triple by integration over $\mathbb{A}$ equipped with counting measure. In particular, the requirements (73) - (76) must be satisfied. Obvious names for the sought functions ϕ, h and d are, respectively, *local effort*, *local entropy* and *local divergence*.

Let us suggest a suitable form of local effort. It will depend on the notion of a *descriptor*, defined as any continuous, strictly decreasing function on $[0, 1]$ which is finite-valued and continuously differentiable on $]0, 1]$, vanishes at $t = 1$ and satisfies the condition that

$$\kappa'(1) = -1\,. \tag{132}$$

The value $\kappa(u)$ is conceived as the effort you have to allocate to any basic event in which you have a belief expressed by u. The condition $\kappa(1) = 0$ reflects the fact that if you feel certain that a basic event will occur, there is no reason why you should allocate any effort at all to that event. Also, it is to be expected that events you do not have much belief in are more difficult to describe than those you believe in with higher degree of confidence. Therefore, we may just as well assume from the outset that κ is decreasing. The norming requirement (132) will enable comparisons of effort, entropy and divergence across different descriptors or even different worlds. The unit defined implicitly by (132) is the *natural information unit*, the "nat".

An important class of descriptors is the class $(\kappa_q)_{q\geq 0}$ given on $[0, 1]$ by

$$\kappa_q(s) = \ln_q \frac{1}{s}\,. \tag{133}$$

With access to a descriptor you may suggest to assign the effort $\kappa(u)$ to an event with belief instance u, but you should multiply this effort with the intensity with which the event is presented to you. This gives the suggestion $\phi(s, u) = \pi(s, u)\kappa(u)$ for local effort. Then local divergence should be the function $\mathrm{d}(s, u) = \pi(s, u)\kappa(u) - \pi(s, s)\kappa(s)$. However, this is not going to work as the fundamental inequality (74) is bound to fail (consider (s, u) with u close to 1). Fortunately, insight gained in Section 18 indicates how one may modify the suggestion in order to have a chance that the fundamental inequality could hold, viz. by adding an *overhead* term. Therefore, given a descriptor, we now suggest to define the local functions as follows:

$$\phi_\pi(s, u|\kappa) = \pi(s, u)\kappa(u) + u \tag{134}$$

$$\mathrm{h}_\pi(s|\kappa) = \pi(s, s)\kappa(s) + s \tag{135}$$

$$\mathrm{d}_\pi(s, u)|\kappa) = \big(\pi(s, u)\kappa(u) + u\big) - \big(\pi(s, s)\kappa(s) + s\big)\,. \tag{136}$$

One may study modifications with more general overhead terms, but we shall not do so. The important thing is to realize that something has to be done. And inspired by the fact that for the important cases with descriptors of the form κ_q, adding a simple linear overhead as suggested above works. This is stated explicitly in the corollary below.

Lemma 22.2. *Let π be an interactor and κ a descriptor. Assume that $\mathrm{d}_\pi(\cdot,\cdot|\kappa)$ given by (136) is a genuine primitive divergence function, i.e. that (74) (the pointwise fundamental inequality) and (76) (pointwise properness) hold. Then $\big(\Phi_\pi(\cdot,\cdot|\kappa), \mathrm{H}_\pi(\cdot|\kappa), \mathrm{D}_\pi(\cdot,\cdot|\kappa)\big)$ obtained by integration of the local quantities given in (134) - (136) over $\mathbb{A}$ is a proper information triple over $X \otimes Y$.*

The proof follows directly from the discussion in Section 18.

Note that for sound interactors, the measures of entropy constructed this way only depend on the descriptor, not on the interactor.

Also note that the quantities defined really give *gross effort* and *gross entropy*. In particular, minimal entropy is not 0 as usual, but 1. This may appear odd but, on the other hand, the way to these quantities was very natural and one may ask if it is not advantageous in many situations to incorporate an overhead.

We also remark that if we allow *incomplete probability measures* Q as belief instances, then this change of the space $X \otimes Y$ will not change the conclusion above. But sticking to probability measures also for belief instances, we may subtract the number 1 from gross effort and from gross entropy and obtain the more familiar net-quantities without overhead.

Corollary 22.1. *For $0 < q \leq \infty$ the interactor π_q and the descriptor κ_q satisfy the conditions of Lemma 22.2. Accordingly, the information triple generated by integration over $\mathbb{A}$ is a proper information triple. Furthermore, the effort function has affine marginals.*

The obtained effort- and entropy functions are gross-quantities. The corresponding net-quantities give the information triple $(\Phi_q, \mathrm{H}_q, \mathrm{D}_q)$ in (106) of Section 18. In particular, H_q is standard Tsallis entropy with q as parameter.

The simple checking is left to the reader.

We turn to problems of another nature, viz. if, given an interactor, one can find an appropriate descriptor such that the generated global description effort is proper.

Lemma 22.3. *Assume that the alphabet $\mathbb{A}$ has at least three elements. Let π be a sound interactor and denote by χ the function on $]0,1[$ defined by*

$$.\chi(t) = \frac{\partial \pi}{\partial t}(t,t)\,. \tag{137}$$

Under the assumption that χ is bounded in the vicinity of $t = 1$, there can only exist one descriptor κ such that the net-effort function generated by π and κ, i.e. the function Φ given by

$$\Phi(x, y) = \sum_{i \in \mathbb{A}} \pi(x_i, y_i)\kappa(y_i) \tag{138}$$

is a proper effort function over $X \otimes Y$. Indeed, κ must be the unique solution in $]0, 1[$ to the differential equation

$$\chi(t)\kappa(t) + t\kappa'(t) = -1 \tag{139}$$

for which $\kappa(1) = \lim_{t\to1} \kappa(t) = 0$.

Proof. Assume that κ exists with $\Phi_\pi(\cdot, \cdot|\kappa)$ proper. For $0 < t < 1$ put

$$f(t) = \chi(t)\kappa(t) + t\kappa'(t)\,.$$

Consider a, for the time fixed probability vector $x = (x_1, x_2, x_3)$ with positive point probabilities. Then the function F given by

$$F(y) = F(y_1, y_2, y_3) = \sum_1^3 \pi(x_i, y_i)\kappa(y_i)$$

on $]0, 1[\times]0, 1[\times]0, 1[$ assumes its minimal value at the interior point $y = x$ when restricted to probability distributions. As standard regularity conditions are fulfilled, there exists a Lagrange multiplier λ such that, for $i = 1, 2, 3$,

$$\frac{\partial}{\partial y_i}\big(F(y) - \lambda \sum_1^3 y_i\big) = 0$$

when $y = x$. This shows that $f(x_1) = f(x_2) = f(x_3)$.

Using this with $(x_1, x_2, x_3) = (\frac{1}{2}, x, \frac{1}{2} - x)$ for a value of x in $]0, \frac{1}{2}[$, we conclude that f is constant on $]0, \frac{1}{2}]$. Then consider a value $x \in]\frac{1}{2}, 1[$ and the probability vector $\big(x, \frac{1}{2}(1-x), \frac{1}{2}(1-x)\big)$ and conclude from the first part of the proof that $f(x) = f\big(\frac{1}{2}(1 - x)\big)$. As $0 < \frac{1}{2}(1 - x) < \frac{1}{2}$, we conclude that $f(x) = f(\frac{1}{2})$. Thus f is constant on $]0, 1[$. By letting $t \to 1$ in (139) and appealing to the technical boundedness assumption, we conclude that the value of the constant is -1. □

Theorem 22.1. *Assume that the alphabet has at least three elements.*

(i) If $q \le 0$, there is no descriptor which, together with π_q, generates a proper effort function.

(ii) If $q > 0$ there exists a unique descriptor, κ_q defined by (133), such that π_q and κ_q generates a proper effort function. The generated information triple $(\Phi_q, \mathrm{H}_q, \mathrm{D}_q)$ is proper.

Proof. By Lemma 22.3 we see that κ_q given by (133) is the only descriptor which, together with π_q, could possibly generate a proper effort function. That it does so for $q > 0$, follows by Lemma 22.2. For $q \leq 0$, this is not the case as the reader can verify by considering atomic situations with $x = (1-\varepsilon, \varepsilon)$ and $y = (\frac{1}{2}, \frac{1}{2})$ and letting ε tend to 0. □

We may add that for the case of a black hole, $q = 0$, the descriptor is given by $\kappa_0(s) = \frac{1}{s} - 1$ and, using $|\cdot|$ for "number of elements in $\cdots$", the generated information triple $(\Phi_0, \mathrm{H}_0, \mathrm{D}_0)$ is given by

$$\Phi_0(x, y) = |\operatorname{supp}(y)|\,, \tag{140}$$
$$\mathrm{H}_0(x) = |\operatorname{supp}(x)|\,, \tag{141}$$
$$\mathrm{D}_0(x, y) = |\operatorname{supp}(y) \setminus \operatorname{supp}(x)| \tag{142}$$

for all $y \succ x$. Note that if terms of the form $\pi(x_i, y_i)\kappa(y_i)$ were to be interpreted by continuity, the resulting triple would be discrete.

We have noted that the descriptor is uniquely determined from the interactor. Therefore, in principle, only the interactor needs to be known. Examples will show that different interactors may well determine the same descriptor. For instance, interaction defined as a geometric average rather than an arithmetic average as in the definition of π_q will lead to the same descriptor. Thus, knowing only the descriptor, you cannot know which world you operate in, in particular, you cannot determine divergence or description effort. But you *can* determine the entropy function. This emphasizes again the general thesis, that *entropy should never be considered alone*.

Finally a comment on the descriptors κ_q. A focus on their inverses is also in order. They may be interpreted as *probability checkers*: Indeed, if, in a Tsallis world with parameter q, you have access to a nats and ask how complex an event this will allow you to describe, the appropriate answer is "you can describe any event with a probability as low as $\kappa^{-1}(a)$". Thus, when $q \leq 1$, however large your resources to nats are, there are events so complex that you cannot describe them, whereas, if $q > 1$ you can describe any event if you have access to K nats if only K is sufficiently large ($K \geq \frac{1}{q-1}$).

23 Maximum Entropy Problems of classical Shannon Theory

Terminology and results of e.g. Sections 7, 10 and 14 are evidently inspired by maximum entropy problems of classical information theory. The classical problems concern inference of probability distributions over some finite or countably infinite *alphabet* $\mathbb{A}$, typically with preparations given in terms of certain constraints, often interpreted as *"moment constraints"* related to random variables of interest. Such preparations will, modulo technical conditions, be feasible in the sense as defined in Section 9. Examples are numerous, from information theory proper, from statistics, from statistical physics or elsewhere. The variety of possibilities may be grasped from the collection of examples in Kapur's monograph [83]. The abstract results developed in Part I can favorably be applied to all such examples. This then has a unifying effect. However, for many concrete examples, it may involve a considerable amount of effort actually to verify the requirements needed for the abstract results to apply. This may involve the verification of Nash's inequality (34) or the determination of the core of models under study, cf. 14.1 and 14.2. No detailed work for specific examples will be carried out here.

When available, we have chosen to use classical probabilistic notation for this section. The material is, in a sense, not new, in fact a very large number of researchers have worked with these problems. The related publications of the present author comprises [26] and [92]. We focus on application of the general theory with special emphasis on the role of the new concept of feasible preparations.

The basic model we shall discuss is the same as in Section 22 based on a finite or countably infinite alphabet $\mathbb{A}$. Note that, in principle, discrete alphabets with more than enumerably many elements could be allowed. However, that would contradict the sensible requirement (3).

The relevant information triple is the proper information triple considered previously, composed of Kerridge inaccuracy, Shannon entropy and Kullback-Leibler divergence:

$$\Phi(P,Q) = \sum_{a\in\mathbb{A}} P(a)\ln\frac{1}{Q(a)}; \tag{143}$$

$$\mathrm{H}(P) = \sum_{a\in\mathbb{A}} P(a)\ln\frac{1}{P(a)}; \tag{144}$$

$$\mathrm{D}(P,Q) = \sum_{a\in\mathbb{A}} P(a)\ln\frac{P(a)}{Q(a)}. \tag{145}$$

We find it illuminating also to implicate the $\hat{Y}$-domain and introduce the action space $\hat{Y} = K(\mathbb{A})$ consisting of all *code length sequences* κ, in short *codes*. These are functions $\kappa : \mathbb{A} \mapsto [0,\infty]$ satisfying *Kraft's equality*

$$\sum_{a\in\mathbb{A}} \exp(-\kappa(a)) = 1\,. \tag{146}$$

Response shall be the bijection $Q \mapsto \hat{Q}$ from Y to $\hat{Y}$ given, for $a \in \mathbb{A}$, by

$$\hat{Q}(a) = \ln \frac{1}{Q(a)} \,. \tag{147}$$

And controllability is the relation for which control $\kappa \succ P$ means that $P(a) = 0$ whenever $\kappa(a) = \infty$.

The interpretation of code length sequences is well known from information theory. We have merely replaced binary logarithms with natural ones and allowed values which are not necessarily integers. The information triple to work with in the $\hat{Y}$-domain is $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ with entropy as above and with

$$\hat{\Phi}(P, \kappa) = \sum_{a \in \mathbb{A}} P(a)\kappa(a) \,, \tag{148}$$

$$\hat{\mathrm{D}}(P, \kappa) = \sum_{a \in \mathbb{A}} P(a)\big(\kappa(a) - \hat{P}(a)\big) \,. \tag{149}$$

As we already know from Section 18, $(\Phi, \mathrm{H}, \mathrm{D})$ and $(\hat{\Phi}, \mathrm{H}, \hat{\mathrm{D}})$ are genuine proper information triples with affine marginals. Thus all parts of the abstract results developed are available and ready to apply. However, we limit the discussion by focusing only on the role of the feasible preparations, leaving elaborations in concrete examples to those interested.

Thinking of states P as determining the distribution of a random element ξ over $\mathbb{A}$, it is often desirable to consider preparations corresponding to the prescription of one or more mean values of ξ. A typical preparation consists of all $P \in X$ such that

$$\sum_{a \in \mathbb{A}} P(a)\lambda(a) = c \tag{150}$$

with c a given constant and $\lambda = (\lambda(a))_{a \in \mathbb{A}}$ a given function on $\mathbb{A}$. This is a strict feasible preparation if and only if the *partition function* (a special *Dirichlet series*),

$$Z(\beta) = \sum_{a \in \mathbb{A}} \exp\big(-\beta\lambda(a)\big) \tag{151}$$

has a finite abscissa of convergence, i.e. converges for some finite constant β, cf. [26] (or monographs on Dirichlet series). However, for the most important part, having concrete applications in mind, viz. the "if"-part, this is clear. Indeed, if the condition is fulfilled, there exist constants α_0 and β_0 such that the function κ_0 given for $a \in \mathbb{A}$ by

$$\kappa_0(a) = \alpha_0 + \beta_0\lambda(a) \tag{152}$$

defines a code. Then $\mathcal{P} = \mathcal{P}^{\kappa_0}(k)$ for some constant k, hence it is a strict feasible preparation of genus 1. It is a member of the preparation family $\mathbb{P} = \mathbb{P}^{\kappa_0}$. Consider, for any β with $Z(\beta) < \infty$, the code κ_β given for $a \in \mathbb{A}$ by

$$\kappa_\beta(a) = \ln Z(\beta) + \beta\lambda(a)\,. \tag{153}$$

Then this code is a member of $\mathrm{core}\hat{\ }(\mathbb{P}^{\kappa_0})$ as is easily seen. In fact all members of the core are of this form [19]. If we can adjust the parameter β such that the corresponding distribution P_β given by

$$P_\beta(a) = \frac{\exp\big(-\beta\lambda(a)\big)}{Z(\beta)} \text{ for } a \in \mathbb{A} \tag{154}$$

is a member of the original preparation $\mathcal{P}$, this must be the maximum entropy distribution of $\mathcal{P}$, as follows from Theorem 14.2, translated to the $\hat{Y}$-domain.

Schematically then: In searching for the MaxEnt distribution of a given preparation, first identify the preparation as a feasible preparation (of genus 1 or higher), then calculate if possible the appropriate partition function and finally adjust parameters to fit the original constraint(s). This gives you the MaxEnt distribution searched for. If calculations are prohibitive, you may resort to numerical and/or graphical methods instead.

As already mentioned, the literature very often solves MaxEnt-problems by the introduction of *Lagrange multipliers*. As shown, this is not necessary. An intrinsic approach building on the abstract theory of Part I appears preferable. For one thing, the fact that you obtain a maximum for the entropy function (and not just a stationary point) is automatic – it is all hidden in the fundamental inequality. And, for another, the quantities you work with when appealing to the abstract theory, have natural interpretations. The Lagrange multipliers in the standard approach, especially within statistical physics, are often of significance considering the nature of the problem at hand. However, these multipliers also come up as natural quantities to consider if you tackle MaxEnt-problems as here suggested.

24 Determining D-projections

The setting is basically the same as in the previous section, especially we again consider a preparation $\mathcal{P}$ given by (150). The problem we shall consider is how to update a given prior $Q_0 \in M^1_+(\mathbb{A})$. Then,

[19] This fact can be proved as a kind of exercise in linear algebra, but more elegant proofs using the structure of the problem should be possible.

the triple $(\Phi, \mathrm{H}, \mathrm{D})$ given by (143) is no longer relevant but should be replaced by the triple $(\mathrm{U}_{|Q_0}, \mathrm{D}^{Q_0}, \mathrm{D})$ as defined in Section 8, cf. (18). This makes good sense if D^{Q_0} is finite on $\mathcal{P}$. The update we seek is the D-projection of Q_0 on $\mathcal{P}$ as defined in Section 12 in connection with (46).

We shall apply much the same strategy as in the previous section. However, we choose not to introduce response and an action space in this setting [20]. Instead, we work directly in the Y-domain and seek a representation of $\mathcal{P}$ as a strict feasible preparation of genus 1, now to be understood with respect to $\mathrm{U}_{|Q_0}$. Analyzing what this amounts to, we find that if the partition function, now defined by

$$Z(\beta) = \sum_{a \in \mathbb{A}} Q_0(a) \exp(-\beta\lambda(a) \,, \tag{155}$$

converges for some $\beta < \infty$, a representation as required is indeed possible. Assuming that this is the case we realize that for each β with $Z(\beta) < \infty$, the distribution Q_β defined by

$$Q_\beta(a) = \frac{Q_0(a) \exp(-\beta\lambda(a))}{Z(\beta)} \text{ for } a \in \mathbb{A} \tag{156}$$

is a member of the core of $\mathcal{P}$. Then it is a matter of adjusting β such that Q_β is consistent, and we have found the sought update.

The cancellation that takes place from (17) to (18) allows an extension of the discussion of updating from the discrete setting to a setting based on a general measurable space. For instance, one may consider a measurable space provided with a σ-finite *reference measure* μ and then work with distributions that have densities with respect to μ. As is well known, cf. also Section 18, the definition of Kullback-Leibler divergence makes good sense in the more general setting. Thus updating problems can be formulated quite generally. If the prior has density q_0, the partition function one should work with is given by $Z(\beta) = \int \exp(-\beta\lambda) q_0 d\mu$. Strategies for updating may be formulated much in analogy with the strategies of Section 23. Further details and consideration of concrete examples are left to the interested reader.

25 Concluding Remarks

The theory presented provides a general abstract framework for the treatment of a wide range of optimization problems within geometry,

[20]this can be done with controls consisting of *code improvements* which are code length functions measured relative to the code κ_0 associated with Q_0. This may appear less convincing, especially for extensions beyond the discrete case.

statistics, statistical physics and other disciplines. Looking back, considering the methods applied and the demonstrated wide applicability, two factors seem to be essential, the *type of modelling* and *affinity*. Regarding the modelling, the key focus was on our information triples involving three interrelated quantities, *effort, entropy* and *divergence* – dually, *utility, max-utility* and *divergence* – each one being in itself of great significance and seen together playing different well-defined roles.

The other factor we want to emphasize is the focus on affinity. True, for the basic theoretical results this is not necessary – recall the provocatively late introduction in Section 15 of this aspect – but, for almost every successful concrete application, affinity seems to pop up and appears both as a necessity and as a guarantee of success. There is something fundamental about this – possibly rooted in deep facts concerning the essential nature of observation, description or measuring ...

As to our modelling, the game theoretical approach expressing the "man/system" or, as here, the "Observer/Nature" interface has been pronounced. In convex optimization theory it is an empirical fact that interesting and tractable problems of real optimization concern either a "minimax"- or a "maximin" problem for which the first optimization is easy to solve. This aspect is also present in our modelling through the linking identity and the fundamental inequality. Thus, for fixed second argument, minimal effort is entropy.

We find that one more element of our approach is essential, viz. the extensive appeal to loose, sometimes speculative philosophical considerations. This exposes us to criticism but also should serve as an aid in catalyzing meaningful applications to look into.

Other attempts to build a "theory of everything" in this area of science includes Jaynes [9], Csiszár and Matús [61], Amari and Nagaoka [10] and then the recent work of Pavon and Ferrante [82]. In the latter we find a focus on the same kind of issues as we have promoted, simplicity of modelling and affinity. With simplicity also, as here, pointing to the unnecessary appeal to techniques involving Lagrange multipliers. The base for the modelling of Pavon and Ferrante is geometry via a lemma of *geometric orthogonality*. So, as "models of the world" these authors, as well as Amari and Nagaoka (and followers) take geometry which – since the times of Euclid – has been a respected approach, whereas we take a more "social" approach via game theory, emphasizing man's role in the world – likewise a respected approach since the times of von Neumann and Morgenstern.

It may not be that difficult to build a tie between the different types of modelling. We believe that the approach as here presented is technically the more elementary one.

Along the way, our approach gave rise to a few points worth emphasizing. A modelling of what can be known (Section 9) appears simple, useful and new. The weakening of the notion of properness in Section 11 is new and possibly the related generalization in Section 19, which serves to justify the new notion of properness, is also new.

The notion of interaction introduced in Section 5 and its role in the discussion of Tsallis entropy in Section 22 has been announced before but is here given a more full treatment, also incorporating a Bregman construction in Section 18. Regarding the discussion of Tsallis entropy also note the emphasis on the descriptors κ_q.

Many other issues are left for further discussion and consolidation of the theory. Some of the possibilities are indicated in the text. Others concern a look at sufficiency, duality, mutual information, learning theory and more. Much of this appears feasible. However, there is an important area where we do not see that our approach and results provide any clue, viz. quantum information theory. Let this challenge to the reader be the last word for now.

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