Proof. Assume that

$$P = \left(p'_1, \cdots, p'_{n'}, p''_1, \cdots, p''_{n''}, 0, \cdots, 0\right)$$

with $p'_1 \geq \cdots \geq p'_{n'} \geq x > p''_1 \geq \cdots \geq p''_{n''}$. Put $s' = \sum p'_i$, $s'' = 1 - s'$. Fix $p'_1, \cdots, p'_{n'}$ and consider $K \subseteq \mathbb{R}^{n''}$ and $G : K \to \mathbb{R}$ defined by

$$K = \{(q_1, \cdots, q_{n''}) \mid \sum q_j = s'', 0 \leq q_j \leq x\},$$
$$G(q) = F(p'_1, \cdots, p'_{n'}, q_1, \cdots, q_{n''}, 0, \cdots, 0)\,.$$

K is compact and convex, $G : K \to \mathbb{R}$ concave and continuous. So G assumes its minimal value at an extremal point of K, say at $q^* = (q^*_1, \cdots, q^*_{n''})$. Assume that $q^*_1 \geq \cdots \geq q^*_{n''}$. Then q^* is of the form

$$q^* = (x, x, \cdots, x, r, 0, \cdots, 0)$$

with $0 < r < x$. May now assume that:

$$P = (p'_1, \cdots, p'_{n'}, r, 0, \cdots, 0)$$

where $0 < r < x$ (or possibly $r = 0$). By convexity of f in $[x, 1]$,

$$F(p'_1, \cdots, p'_{n'}, r, 0, \cdots, 0) \geq F(p_0, \cdots, p_0, r, 0, \cdots, 0)$$

where $p_0 = s'/n'$. With $P_0 = (p_0, \cdots, p_0, r, 0, \cdots, 0)$ we are done. $\square$