FORMS OVER FIELDS AND WITT’S LEMMA

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Abstract. We give an overview of the general framework of forms of Bak, Tits and Wall, when restricting to vector spaces over fields, including its relationship to the classical notions of Hermitian, alternating and quadratic forms. We then prove a version of Witt’s lemma in this context, showing in particular that the action of the group of isometries of a space equipped with a form is transitive on isometric subspaces.

1. Introduction

The “classical groups” over a field \( \mathbb{F} \) are the automorphism groups of different types of forms on vector spaces:

1. A symplectic group is the group of isometries of a vector space equipped with an alternating form, that is a bilinear map \( \omega : E \times E \to \mathbb{F} \) such that
   \[ \omega(v, v) = 0. \]

2. A unitary group is the group of isometries of a vector space equipped with a Hermitian form with respect to a field involution \( \sigma : \mathbb{F} \to \mathbb{F} \), that is a sesquilinear map \( \omega : E \times E \to \mathbb{F} \) such that
   \[ \omega(v, w) = \overline{\omega(w, v)} \]
   where we write \( \bar{c} = \sigma(c) \) in analogy with complex conjugation.

3. An orthogonal group is the group of isometries of a vector space equipped with a quadratic form, that is a set map \( Q : E \to \mathbb{F} \) such that
   \[ Q(cv) = c^2Q(v) \]
   for all \( c \in \mathbb{F}, v \in E \)
   and such that the associated map \( B_Q(v, w) = Q(v + w) - Q(v) - Q(w) \) is bilinear.

These examples of “forms” all fit into a common framework, developed in particular by Bak, Tits, Wall (see eg. [1, 11, 12, 13]) and later by Magurn-Vaserstein-van der Kallen [7, Sec 4]. This framework allows to treat all three cases described above at once, including when the field has characteristic 2, and symmetric and anti-symmetric forms are indistinguishable. The framework makes sense in the context of rings with anti-involutions, and has turned out particularly useful in the context of homological stability, see eg. [4, 8, 9, 10]. In the present paper, we restrict to the case of fields and describe what the theory becomes under this simplifying assumption. We then study the action of the corresponding group of isometries on subspaces, showing that the action is transitive on the set of isometric subspaces, generalizing the classical Witt’s Lemma [14]. Many of the results presented here can be, at least essentially, extracted from the existing literature. They are given here with proofs for the convenience of the reader, as the extraction process is not a totally easy one and, in the case of Witt’s Lemma, because we could not find the exact version of this result that was needed for our companion paper [10]. Our hope is that this note can be of use to others.

We describe our main results in more detail now.
The generalized definition of a form over an \( F \)-vector space \( E \), which we describe now, depends on the choice of a field involution \( \sigma \) on \( F \) (possibly the identity), an element \( \varepsilon \in F \) such that \( \varepsilon \varepsilon = 1 \) (with \( \varepsilon \) short for \( \sigma(\varepsilon) \) as above), and a certain additive subgroup \( \Lambda \subseteq F \). In the present situation, with \( F \) a field, the group \( \Lambda \) is almost always determined by the triple \((F, \sigma, \varepsilon)\) (see Proposition A.1 in the Appendix).

Recall that a map \( f : E \times E \to F \) is called sesquilinear if it is biadditive and satisfies that \( f(av, bw) = a\bar{f}(v, w)b \) for all \( a, b \in F \) and \( v, w \in E \). Let \( \mathrm{Sesq}_\sigma(E) \) denote the vector space of sesquilinear forms on \( E \). The group of all \((\sigma, \varepsilon, \Lambda)\)-quadratic forms on \( E \) is the quotient

\[
\mathrm{Form}(E, \sigma, \varepsilon, \Lambda) := \mathrm{Sesq}_\sigma(E)/\mathrm{X}(E, \sigma, \varepsilon, \Lambda),
\]

of the sesquilinear forms by a certain subgroup \( \mathrm{X}(E, \sigma, \varepsilon, \Lambda) \) (see Definition 2.1). To a form \( q \in \mathrm{Form}(E, \sigma, \varepsilon, \Lambda) \), one associates two new maps \( \omega_q : E \times E \to F \) and \( Q_q : E \to F/\Lambda \) defined by

\[
\omega_q(v, w) = q(v, w) + \varepsilon\bar{q}(w, v) \quad \text{and} \quad Q_q(v) = q(v, v) + \Lambda \in F/\Lambda,
\]

with \( \omega_q \) an \( \varepsilon \)-skew symmetric sesquilinear form" by construction, and \( Q_q \) to be thought of as a “quadratic refinement" of \( \omega_q \). Theorem 2.3 shows that the form \( q \) is always determined by the pair \((\omega_q, Q_q)\) (in fact \( X(E, \sigma, \varepsilon, \Lambda) \) is defined so that this holds), and \( \omega_q \) and \( Q_q \) are related by the equation

\[
Q_q(v + w) - Q_q(v) - Q_q(w) = \omega_q(v, w) \in F/\Lambda.
\]

More than that, Proposition 2.4 shows that only one of \( \omega_q \) or \( Q_q \) is always enough to determine \( q \), but which one of the two depends on the chosen \( \Lambda \). In fact, our main result in the first part of the paper is that, for almost all choices of parameters \((\sigma, \varepsilon, \Lambda)\), the general framework of forms simply specializes to the classical Hermitian, alternating and quadratic forms:

**Theorem A.** (1) When \( \sigma \neq \operatorname{id} \), there is a natural isomorphism

\[
\mathrm{Form}(E, \sigma, \varepsilon, \Lambda) \cong \mathrm{Herm}_\sigma(E).
\]

(2) When \( \sigma = \operatorname{id}, \varepsilon = -1 \) and \( \Lambda = F \), there is a natural isomorphism

\[
\mathrm{Form}(E, \sigma, \varepsilon, \Lambda) \cong \mathrm{Alt}(E).
\]

(3) When \( \sigma = \operatorname{id}, \varepsilon = 1 \) and \( \Lambda = 0 \), there is a natural isomorphism

\[
\mathrm{Form}(E, \sigma, \varepsilon, \Lambda) \cong \mathrm{Quad}(E).
\]

Here \( \mathrm{Herm}_\sigma(E) \), \( \mathrm{Alt}(E) \) and \( \mathrm{Quad}(E) \) are the groups of Hermitian, alternating and quadratic forms of Definition 2.2. When \( F \) is a finite field, there is only one isomorphism class of non-degenerate such form in each dimension, except in the orthogonal case (3) where there are two isomorphism classes in each dimension (see eg. [3, Chap II]). Forms over infinite fields is on the other hand a vast subject, see eg. the book [6] which is concerned with quadratic forms over fields of characteristic not 2.

**Remark 1.1.** (i) When \( \sigma \neq \operatorname{id} \), the result shows that the group of forms depends neither on \( \Lambda \), which is a consequence of the fact that there is only one possible \( \Lambda \) in this case by Proposition A.1, nor on \( \varepsilon \).

(ii) When \( \sigma = \operatorname{id} \), there are two main possibilities: the alternating case (2), and the quadratic case (3). If the field is not of characteristic 2, cases (2) and (3) are distinguished by whether \( \varepsilon = 1 \) or \( \varepsilon = -1 \) and \( \Lambda \) is determined by this data. If \( F \) has characteristic 2, then \( 1 = -1 \) and cases (2) and (3) in the theorem are distinguished by the choice of \( \Lambda \). This is in fact the one situation where there is a freedom of choice for \( \Lambda \) and cases (2) and (3) correspond to the two possible extremes, the smallest and largest possible \( \Lambda \). Note also that, given \( \Lambda \),
Proposition A.1 shows that \( \varepsilon \) is in fact always redundant information when \( \sigma = \text{id} \). We chose to leave the \( \varepsilon \) in the statement as it is informative.

(iii) The only cases not covered by the theorem are thus in characteristic 2 with \( \sigma = \text{id} \) and 0 < \( \Lambda < F \). Proposition 2.7 shows that under the additional assumption that \( \mathbb{F} \) is a perfect field, also these cases are actually covered by (3). More generally, combining Proposition 2.4 (2) and Lemma 2.5 (2), we get that a “mod \( \Lambda \)” version of (3) holds under the weaker assumption that \( \sigma = \text{id} \) and \( \Lambda < F \), giving the remaining cases.

The above results are all proved in Section 2.

The second part of the paper is concerned with isometries: Given a vector space \( E \) equipped with a form \( q \), called here a formed space \((E,q)\), an isometry of \( E \) is a linear map respecting the form. Restricting the form \( q \) to subspaces of \( E \), we can talk about isometries between subspaces.

Note that in the present paper, we do not assume that forms are non-degenerate, that is they may have a non-trivial kernel, where the kernel \( \mathcal{K}(E) \) of \((E,q)\) is defined to be the kernel of the map

\[ E \to \sigma^* E^*, \; v \mapsto \omega_q(-,v), \]

where \( \sigma^* E^* \) denotes the vector space of \( \sigma \)-skew-linear maps \( E \to \mathbb{F} \). Our main result in the second part of the paper is the following:

**Theorem B** (Witt’s Lemma (Theorem 3.3)). Let \((E,q)\) be a formed space. Suppose that \( f: U \to W \) is a bijective isometry between two subspaces of \( E \) such that \( f(U \cap \mathcal{K}(E)) = W \cap \mathcal{K}(E) \). Then \( f \) can be extended to a bijective isometry of \( E \).

Theorem 3.6 in the paper gives in addition a relative version of Witt’s lemma, where a subspace may be assumed to be fixed by the constructed isometry.

Witt originally considered in [14] the case of non-degenerate symmetric forms in characteristic not 2. His result was generalized by many authors (see eg. [11, Prop 2] and [2, p21,22,36], or [7, Cor 8.3] and the paper [5] for many references to the existing literature). We were however not able to find the result in the above generality, most particularly including the case of degenerate forms.

The proofs of Witt’s lemma and its relative version are given in Section 3.

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## 2. Forms

As above, we fix a field \( \mathbb{F} \), an involution \( \sigma \) of \( \mathbb{F} \) (possibly the identity), and a scalar \( \varepsilon \in \mathbb{F} \) satisfying \( \varepsilon \bar{\varepsilon} = 1 \). Set

\[
\Lambda_{\min} = \{ a - \varepsilon a | \; a \in \mathbb{F} \}
\]

\[
\Lambda_{\max} = \{ a \in \mathbb{F} | \; a + \varepsilon a = 0 \}.
\]

These are additive subgroups of \( \mathbb{F} \). Let \( \Lambda \) be an abelian group such that

\[
\Lambda_{\min} \leq \Lambda \leq \Lambda_{\max}.
\]
Proposition A.1 in the appendix shows that for most triples \((F, \sigma, \varepsilon)\), we actually have \(\Lambda_{\text{min}} = \Lambda_{\text{max}}\) so that there is a unique \(\Lambda\) determined by the triple.

**Definition 2.1.** A \((\sigma, \varepsilon, \Lambda)\)-quadratic form on \(E\) (or just a form for short) is an element of the quotient

\[
\text{Form}(E, \sigma, \varepsilon, \Lambda) := \text{Sesq}_\sigma(E)/X(E, \sigma, \varepsilon, \Lambda),
\]

where \(\text{Sesq}_\sigma(E)\) denotes the vector space of sesquilinear forms on \(E\) and \(X(E, \sigma, \varepsilon, \Lambda)\) is the additive subgroup of all those \(f\) satisfying

\[f(v, v) \in \Lambda \quad \text{and} \quad f(w, v) = -\varepsilon \overline{f(v, w)}\]

for all \(v, w \in E\). A formed space \((E, q)\) is a finite-dimensional vector space \(E\) equipped with a form \(q\).

The set of forms on \(E\) is contravariantly functorial in \(E\): from a linear map \(f : E \to F\) and a form \(q\) on \(F\), we produce a form on \(E\) as

\[f^*q(v, w) = q(f(v), f(w))\]

for \(v, w \in E\). This is indeed well-defined as \(f^*q\) is again sesquilinear, and \(f^*\) takes \(X(F, \sigma, \varepsilon, \Lambda)\) to \(X(E, \sigma, \varepsilon, \Lambda)\).

The goal of this section is to relate forms to the more classical notions of Hermitian, alternating, and quadratic forms.

**Definition 2.2.** Let \(E\) be a vector space over \(F\). Define

\[\text{Herm}_{\sigma, \varepsilon}(E):\] the vector space of sesquilinear forms \(\omega : E \times E \to F\) satisfying that

\[\omega(v, w) = \varepsilon \overline{\omega(w, v)}\]

for all \(v, w \in E\), considered here as a group. We write \(\text{Herm}_\sigma(E) = \text{Herm}_{\sigma, 1}(E)\) for the classical Hermitian forms.

\[\text{Alt}(E) \leq \text{Herm}_{id, -1}(E):\] the subspace of alternating forms, i.e. those such that

\[\omega(v, v) = 0\]

for all \(v \in E\).

\[\text{Set}(E, F/\Lambda):\] the group of set maps \(Q : E \to F/\Lambda\).

\[\text{Quad}(E) \leq \text{Set}(E, F):\] the subgroup of classical quadratic forms, i.e. the maps \(Q : E \to F\) such that

\[Q(av) = a^2 Q(v)\]

for all \(v \in E\), \(a \in F\) and such that \(B_Q : E \times E \to F\) defined by

\[B_Q(v, w) = Q(v + w) - Q(v) - Q(w)\]

is bilinear.

Just like \(\text{Form}(E, \sigma, \varepsilon, \Lambda)\), the groups \(\text{Herm}_{\sigma, \varepsilon}(E), \text{Alt}(E), \text{Set}_\sigma(E, F/\Lambda)\) and \(\text{Quad}(E, F)\) assemble to give contravariant functors from the category of vector spaces and linear maps to the category of abelian groups.

**Theorem 2.3.** (see also [12, Thm 1]) Let \(E\) be a vector space. For \(q \in \text{Form}(E, \sigma, \varepsilon, \Lambda)\), setting

\[\omega_q(v, w) = q(v, w) + \varepsilon \overline{q(w, v)} \quad \text{and} \quad Q_q(v) = q(v, v) + \Lambda \in F/\Lambda,\]

defines a map

\[
\chi = (\omega(-), Q(-)) : \text{Form}(E, \sigma, \varepsilon, \Lambda) \to \text{Herm}_{\sigma, \varepsilon}(E) \times \text{Set}(E, F/\Lambda)
\]
which is an injective homomorphism, natural with respect to linear maps \( E \to E' \). In particular, if \( (E, q) \) and \( (E', q') \) are formed spaces, and \( f : E \to E' \) a linear map, then \( f \) is an isometry if and only if
\[
\omega_q(f(v), f(w)) = \omega_q(v, w) \quad \text{and} \quad Q_q(f(v)) = Q_q(v)
\]
for all \( v, w \in E \). Moreover, for any \( q \in \text{Form}(E, \sigma, \varepsilon, \Lambda) \), the maps \( \omega_q \) and \( Q_q \) are related by the following equation:
\[
Q_q(v + w) - Q_q(v) - Q_q(w) = \omega_q(v, w) \in \mathbb{F}/\Lambda.
\]

**Proof.** The map \( \chi \) is well-defined and injective since \( \omega_f = 0 \) and \( Q_f(v) \in \Lambda \) if and only if \( f \in X(E, \sigma, \varepsilon, \Lambda) \). Also, for any \( q \in \text{Form}(E, \sigma, \varepsilon, \Lambda) \), we have that \( \omega(v, w) = \varepsilon \omega(w, v) \) for all \( v, w \in E \), so the first component of \( \chi \) has image in \( \text{Herm}_{\sigma, \varepsilon}(E) \) as claimed. One checks that the assignment is a group homomorphism that is natural with respect to vector space homomorphisms. The characterization of the isometries follows directly from naturality and injectivity of \( \chi \). Finally, the last equation follows from the fact that \( q(w, v) - \varepsilon q(w, v) \in \Lambda_{\text{min}} \).

\[ \square \]

The group of linear automorphisms \( \text{GL}(E) \) acts on both the domain and codomain of \( \chi \), and naturality implies that \( \chi \) commutes with this action. The isotropy group of an element \( q \in \text{Form}(E, \sigma, \varepsilon, \Lambda) \) is the group \( \mathbf{G}(E, q) \) of bijective isometries of \( (E, q) \). It follows from the above result that
\[
\mathbf{G}(E, q) = \text{Aut}(E, \omega_q) \cap \text{Aut}(E, Q_q)
\]
is exactly the group of maps preserving both \( \omega_q \) and \( Q_q \). The following result show that in fact one of those two conditions is always redundant and the automorphism group \( \mathbf{G}(E, q) \) identifies with either \( \text{Aut}(E, \omega_q) \) or \( \text{Aut}(E, Q_q) \):

**Proposition 2.4.** (1) If \( \Lambda = \Lambda_{\text{max}} \), then \( Q_q \) is determined by \( \omega_q \), in the sense that the assignment
\[
\omega(-) : \text{Form}(E, \sigma, \varepsilon, \Lambda) \to \text{Herm}_{\sigma, \varepsilon}(E)
\]
of Theorem 2.3 is injective. In particular, \( \mathbf{G}(E, q) = \text{Aut}(E, \omega_q) \) as subgroups of \( \text{GL}(E) \).

(2) If \( \Lambda \neq \mathbb{F} \), then \( \omega_q \) is determined by \( Q_q \), in the sense that the assignment
\[
Q(-) : \text{Form}(E, \sigma, \varepsilon, \Lambda) \to \text{Set}(E, \mathbb{F}/\Lambda)
\]
of Theorem 2.3 is injective. In particular, \( \mathbf{G}(E, q) = \text{Aut}(E, Q_q) \) as subgroups of \( \text{GL}(E) \).

**Proof.** We first prove (1). Since we know \( q \mapsto \chi(q) = (\omega_q, Q_q) \) is additive and injective, it suffices to check that \( \omega_q = 0 \) implies \( Q_q = 0 \). Indeed, for \( v \in E \),
\[
0 = \omega_q(v, v) = q(v, v) + \varepsilon q(v, v),
\]
so
\[
q(v, v) \in \{ a \in \mathbb{F} \mid a + \varepsilon a = 0 \} = \Lambda_{\text{max}} = \Lambda,
\]
meaning \( Q_q(v) = q(v, v) + \Lambda = 0 \in \mathbb{F}/\Lambda \), proving (1).

For statement (2), we assume \( \Lambda \neq \mathbb{F} \) is a proper additive subgroup. By the additivity and injectivity of \( \chi \), it suffices now to check that \( Q_q = 0 \) implies \( \omega_q = 0 \). The condition \( Q_q = 0 \) is equivalent to \( q(v, v) \in \Lambda \) for every \( v \in E \). It follows that
\[
q(u, v) = q(u + v, u + v) - q(u, u) - q(v, v) - q(v, u)
= -q(v, u) \mod \Lambda
= -\varepsilon q(v, u) \mod \Lambda,
\]
where the last equality holds as \( q(v, u) + \overline{\varepsilon q(v, u)} \in \Lambda_{\min} \leq \Lambda \). Hence \( \omega(u, v) \in \Lambda \) for every \( u, v \in E \). Scaling \( v \) by any element in \( \mathbb{F} \) defines an ideal of \( \Lambda \) generated by \( \omega(u, v) \). As \( \Lambda \) has no non-trivial ideal, it follows that \( \omega(u, v) = 0 \), which proves (2). \( \square \)

Finally, we prove Theorem A. The proof will use the following two lemmas.

**Lemma 2.5.** (1) The assignment

\[
\omega(-) : \text{Form}(E, \sigma, \varepsilon, \Lambda) \rightarrow \text{Herm}_{\sigma, \varepsilon}(E)
\]

of Theorem 2.3 is surjective unless \( \sigma = id \) and \( \mathbb{F} \) has a field of characteristic 2, in which case the image is the subspace \( \text{Alt}(E) \) of alternating bilinear forms.

(2) If \( \sigma = id \) the assignment

\[
Q(-) : \text{Form}(E, \sigma, \varepsilon, \Lambda) \rightarrow \text{Set}(E, \mathbb{F}/\Lambda)
\]

of Theorem 2.3 has image the \( \text{mod} \Lambda \) quadratic forms, i.e. the set maps \( Q : E \rightarrow \mathbb{F}/\Lambda \) such that \( Q(av) = a^2 Q(v) \) for all \( v \in E, a \in \mathbb{F} \) and \( B_Q(v, w) = Q(v+w) - Q(v) - Q(w) \) is a bilinear function. In particular, if in addition \( \Lambda = 0 \), then \( Q(-) \) has image \( \text{Quad}(E) \).

**Proof.** To prove (1), note first that in the exceptional case \( (\sigma = id \text{ in characteristic } 2) \), the form \( \omega_q \) is indeed alternating, since we necessarily have \( \varepsilon = -1 \) in this case, which forces

\[
\omega_q(v, v) = q(v, v) + \varepsilon q(v, v) = 0.
\]

Now, returning to the general case, let \( \omega \) be any element of \( \text{Herm}_{\sigma, \varepsilon}(E) \), assumed alternating in the special case. Arbitrarily pick a basis \( \{x_i\} \) of \( E \), and define

\[
q(x_i, x_j) = \begin{cases} 
\omega(x_i, x_j) & \text{if } i < j, \\
a_i & \text{if } i = j, \\
0 & \text{if } i > j,
\end{cases}
\]

extending sesquilinearly to obtain an element \( q \in \text{Sesq}_{\sigma}(E) \). Here \( a_i \in \mathbb{F} \) is a scalar chosen so that

\[
a_i + \varepsilon a_i = \omega(x_i, x_i).
\]

This is possible in the special case by setting \( a_i = 0 \). In the other cases, applying Proposition A.1 we get

\[
\Lambda_{\min}(\mathbb{F}, \sigma, -\varepsilon) = \{ a + \varepsilon a \mid a \in \mathbb{F} \} = \{ b \in \mathbb{F} \mid b = \varepsilon b \} = \Lambda_{\max}(\mathbb{F}, \sigma, -\varepsilon)
\]

and \( \omega(x_i, x_i) \) lies in the right hand side, so such an \( a_i \) does exists. The form \( q \) satisfies that \( \omega_q = \omega \) in all cases.

For (2), we now assume that \( \sigma = id \). Given a formed space \( (E, q) \), we first check that \( Q_q \) is a mod \( \Lambda \) quadratic form, i.e. that

\[
B_{Q_q}(v, w) := Q_q(v + w) - Q_q(v) - Q_q(w)
\]

is bilinear. But \( B_{Q_q}(x, y) = \omega_q \mod \Lambda \), which is bilinear under the assumption that \( \sigma = id \).

Conversely, assume that \( Q : E \rightarrow \mathbb{F}/\Lambda \) is a quadratic form (mod \( \Lambda \)). We want to show that it is in the image of \( Q(-) \). Arbitrarily pick a basis \( \{x_i\} \) of \( E \), and pick \( b_{ij}, c_i \in \mathbb{F} \) such that \( B_Q(x_i, x_j) = b_{ij} \mod \Lambda \) and \( Q(x_i) = c_i \mod \Lambda \). Then define

\[
q(x_i, x_j) = \begin{cases} 
b_{ij} & \text{if } i < j, \\
c_i & \text{if } i = j, \\
0 & \text{if } i > j,
\end{cases}
\]
extending bilinearly to obtain an element \( q \in \text{Sesq}_{\text{id}}(E) \). For each \( i \), we have \( Q_q(x_i) = Q(x_i) \) by construction. Now
\[
\omega_q(x_i, x_j) = q(x_i, x_j) + \varepsilon q(x_j, x_i) = q(x_i, x_j) + q(x_j, x_i) \mod \Lambda
\]
as \( q(x_j, x_i) = \varepsilon q(x_j, x_i) \mod \Lambda_{\text{min}} \). Hence \( \omega_q(x_i, x_j) = B_Q(x_i, x_j) \mod \Lambda \) if \( i \neq j \), while
\[
\omega_q(x_i, x_i) = 2q(x_i, x_i) = 2Q(x_i) \mod \Lambda
\]
and
\[
B_Q(x_i, x_i) = Q(2x_i) - 2Q(x_i) = 2Q(x_i).
\]
It follows that \( \omega_q = B_Q \), while \( Q_q \) and \( Q \) coincide on a basis. This implies that \( Q_q = Q \) because \( Q(v+w) = Q(v) + Q(w) + B_Q(v, w) \) and likewise \( Q_q(v+w) = Q_q(v) + Q_q(w) + \omega_q(v, w) \) mod \( \Lambda \).

**Lemma 2.6.** If \( \sigma \neq \text{id} \), there exists an \( a = a(\varepsilon) \in F^\times \) such that multiplication by \( a \) induces an isomorphism
\[
\text{Form}(E, \sigma, \varepsilon, \Lambda) \xrightarrow{\cong} \text{Form}(E, \sigma, 1, a\Lambda).
\]

Note that \( G(E, q) = G(E, aq) \) as subgroups of \( GL(E) \).

**Proof.** Multiplication by \( a \in F \) takes \( X(V, \sigma, \varepsilon, \Lambda) \) to \( X(V, \sigma, a\varepsilon^{-1}, a\Lambda) \). So we need to check that there exists an \( a \in F \) such that \( \varepsilon = a^{-1}\bar{a} \). The existence of such an element is given by applying Hilbert’s Theorem 90 to the field extension \( F \) of \( F^\sigma \). \( \square \)

**Proof of Theorem A.** We start by proving (1). So assume \( \sigma \neq \text{id} \). Then by Proposition A.1, we must have \( \Lambda = \Lambda_{\text{max}} \). Now by Lemma 2.4(1), the map \( \omega_{(-)} \) is injective and by Lemma 2.5 it is also surjective. Statement (1) then follows from Lemma 2.6.

To prove (2), we now assume that \( \sigma = \text{id}, \varepsilon = -1 \) and \( \Lambda = F \). In particular, \( \Lambda = \Lambda_{\text{max}} \) in all cases by Proposition A.1. Applying Lemma 2.4(1) again, we have that the map \( \omega_{(-)} \) is injective and by Lemma 2.5 it is also surjective onto \( \text{Herm}_{\text{id}, -1}(E) \) unless \( F \) has characteristic 2 in which case the image is the subgroup of alternating forms \( \text{Alt}(E) \). Now statement (2) follows from the fact that \( \text{Herm}_{\text{id}, -1}(E) \) and \( \text{Alt}(E) \) are actually isomorphic if the field is not of characteristic 2, as \( \omega \in \text{Herm}_{\text{id}, -1}(E) \) satisfies that \( \omega(v, v) = -\omega(v, v) \).

For (3), we now assume that \( \sigma = \text{id}, \varepsilon = 1 \) and \( \Lambda = 0 \). By Proposition 2.4(2), the map \( Q_{(-)} \) is injective, and by Lemma 2.5 it surjects onto the quadratic forms \( \text{Quad}(E) \), which proves the result. \( \square \)

Theorem A gives a description of almost all the possible formed space in the better-known terms of Hermitian, alternating or quadratic forms. As we saw in Remark 1.1, left is the case \( \sigma = \text{id}, p = 2, \) and \( 0 < \Lambda < F \). We show now that at least for \( F \) finite, this case is in fact redundant, producing the same isometry groups as if \( \Lambda = 0 \):

**Proposition 2.7.** Assume \( F \) is a perfect field of characteristic 2. Let \( \sigma = \text{id} \) and \( \Lambda < F \) be proper. Then
\[
X(E, \text{id}, 1, \Lambda) = X(E, \text{id}, 1, 0).
\]

**Proof.** What we must show is that, if \( q(v, v) \in \Lambda \) for all \( v \in V \), then in fact \( q(v, v) = 0 \) for all \( v \). For any \( c \in F^\times \), we have
\[
q(c^{-1}v, c^{-1}v) = c^{-2}q(v, v) \in \Lambda.
\]
Now, if \( q(v, v) \) is nonzero, then every element of \( F \) is of that form for some \( c \), because the squaring function on \( F \) is bijective. That contradicts \( \Lambda \) being a proper subset of \( F \). \( \square \)
Remark 2.8. The proposition is false for imperfect fields of characteristic two. For instance, for \( \mathbb{F} = \mathbb{F}_2(t) \), we have proper inclusions
\[
X(\mathbb{F}, id, 1, 0) < X(\mathbb{F}, id, 1, \mathbb{F}_2(t^2)) < X(\mathbb{F}, id, 1, \mathbb{F}_2(t)).
\]
Indeed, the forms \( q(a, b) = ab \) and \( q'(a, b) = tab \) demonstrate properness of the two inclusions.

3. Witt’s Lemma

In this section, we give a proof of Witt’s lemma in the context of formed spaces, stated as Theorem 3.3, and its relative version, stated as Theorem 3.6.

Given a vector space \( E \), recall that \( \sigma E^* \) denotes the vector space of \( \sigma \)-skew-linear maps \( E \to \mathbb{F} \), that is, additive maps \( f : E \to \mathbb{F} \) such that \( f(av) = \hat{a}f(v) \), with vector space structure defined pointwise from that of \( \mathbb{F} \). Recall also the maps \( \omega_q \) and \( Q_q \) of Theorem 2.3 associated to a form \( q \).

Definition 3.1. Let \( E = (E, q) \) be a formed space.
(1) The kernel of \( E \) is defined to be the kernel \( K(E) = \ker(b_q) \) of the associated linear map
\[
b_q : E \to \sigma E^*, \ v \mapsto \omega_q(-, v).
\]
(2) The orthogonal complement \( U^\perp \) of a subspace \( U \leq E \) is the subspace consisting of all \( v \in E \) such that \( \omega(v, U) = 0 \). That is, \( U^\perp \) is the kernel of the composition
\[
E \xrightarrow{b_q} \sigma E^* \xrightarrow{\text{incl}^*} \sigma U^*.
\]
(3) The radical of \( E \) is the set
\[
\mathcal{R}(E) = \{ v \in K(E) \mid Q_q(v) = 0 \}.
\]
(4) A subspace \( U \) of \( E \) is called isotropic if \( q|_U = 0 \) (or equivalently, \( \omega_q|_U = 0 \) and \( Q_q|_U = 0 \)).

Remark 3.2. (1) The radical \( \mathcal{R}(E) \) is in fact a subspace of \( E \), because \( Q_q \) is additive on \( K(E) \), and satisfies \( Q_q(cv) = c^2Q_q(v) \) for \( c \in \mathbb{F}, v \in E \). The radical is isotropic. If the characteristic of \( \mathbb{F} \) is not 2, then \( \mathcal{R}(E) = K(E) \), since equation (2.1) in that case gives that \( \omega_q|_{K(E)} = 0 \) implies \( Q_q|_{K(E)} = 0 \).
(2) Note that orthogonality is a symmetric relation: \( \omega_q(v, w) = 0 \) if and only if \( \omega_q(w, v) = 0 \), but note also that the orthogonal complement \( U^\perp \) defined above will usually not be a complement in the sense of vector spaces. In fact, if \( U \) is isotropic, we actually have \( U \subset U^\perp \! \perp \).

The following result is a version of Witt’s Lemma. Witt originally considered non-degenerate symmetric forms in characteristic not 2 [14]. His result was generalize by eg., Tits [11, Prop 2], Dieudonné [2, p21,22 and 35], and Magurn-Vaserstein-van der Kallen [7, Cor 8.3], though all keeping a non-degeneracy assumption.

Theorem 3.3 (Witt’s Lemma). Let \( (E, q) \) be a formed space. Suppose that \( f : U \to W \) is a bijective isometry between two subspaces of \( E \) such that \( f(U \cap K(E)) = W \cap K(E) \). Then \( f \) can be extended to a bijective isometry of \( E \). In particular, the group \( G(E, q) \) of bijective isometries of \( (E, q) \) acts transitively on the set of isotropic subspaces \( U \) with given values of \( \dim U \) and \( \dim \mathcal{R}(E) \).

Regarding the last statement, observe that when \( U \) is isotropic, \( U \cap \mathcal{R}(E) = U \cap K(E) \). Note also that for general \( U, W \) and \( f \) as in the statement, we have that \( f(U \cap \mathcal{R}(E)) = W \cap \mathcal{R}(E) \) as \( f \) is an isometry that takes \( U \cap K(E) \) to \( W \cap K(E) \).
The result implies that the group \( \mathcal{G}(E, q) \) acts transitively on the elements of any fixed rank of the building of isotropic subspaces
\[
\mathcal{P}(E) = \{ W < E \mid W \text{ isotropic, } \mathcal{R}(E) < W \}.
\]
This action is used in [10] to study the homology of the group \( \mathcal{G}(E, q) \).

The proof of the theorem will use the following basic result about vector spaces:

**Lemma 3.4.** Let \( A, B \leq V \) be any two subspaces of the same dimension. Then there exists a subspace \( L \) which is simultaneously complementary to both \( A \) and \( B \).

Notice this would be false with three subspaces instead of two in e.g. \( V = \mathbb{F}_2^2 \).

**Proof.** By modding out \( A \cap B \) from \( V \), we may assume without loss of generality that \( A \cap B = 0 \). By restricting to \( A + B < V \), we may assume that \( A + B = V \). Pick any linear isomorphism \( f : A \to B \), and define \( g : A \to V \) by setting \( g(u) = u + f(u) \). We claim that \( L = \text{Im}(g) \leq V \) is the desired complement. Now, \( A \cap L = 0 \) because \( f \) is injective and \( A, B \) are disjoint. Similarly \( B \cap L = 0 \). Also \( B + L = V \) since it contains \( A \), and \( A + L = V \) since it contains \( \text{Im}(f) = B \). □

In the proof of Witt’s Lemma, we will repeatedly use the following construction:

**Lemma 3.5.** Suppose \( A, B, C \leq E \) are subspaces of a formed space \((E, q)\) such that \( C \leq A^\perp \cap B^\perp \) and \( A \cap C = 0 = B \cap C \). Then for any isometry \( f : A \to B \), the map \( f \oplus id : A \oplus C \to B \oplus C \) is also an isometry. If \( A = \langle a \rangle \), \( B = \langle b \rangle \) and \( f(a) = b \), the same holds under the weaker assumption that \( C \leq (a - b)^\perp \).

**Proof.** In both cases, it is easy to see that \( f \oplus id \) preserves \( \omega_q \). By Theorem 2.3, we thus just have to check that \( f \) preserves \( Q_q \). In the first case, we have \( Q_q(f(a) + c) = Q_q(f(a)) + Q_q(c) + \omega_q(f(a), c) = Q_q(a) + Q_q(c) \). Similarly, \( Q_q(a + c) = Q_q(a) + Q_q(c) \). In the second case, we have instead \( Q_q(f(a) + c) = Q_q(b) + Q_q(c) + \omega_q(b, c) = Q_q(a) + Q_q(c) + \omega_q(a, c) = Q_q(a) + Q_q(c) \). This proves the result in both cases. □

**Proof of Theorem 3.3.** We will prove the theorem by induction on the dimension of \( U \), starting with the case
\[
U \leq K := \mathcal{K}(E).
\]
By our hypothesis, it follows that \( W \leq K \) as well. By Lemma 3.4, we can find a subspace \( L \leq E \) such that
\[
U \oplus L = E = W \oplus L.
\]
We extend \( f \) to a bijection of \( E \) by setting \( f \) to be identity on \( L \). As \( U^\perp = W^\perp = E \), Lemma 3.5 shows that this is an isometry.

Now, we may assume that
\[
U \not\leq K.
\]
Then we can pick a codimension-one subspace \( H < U \), such that
\[
U \cap K \leq H.
\]
By induction hypothesis we may find an isometry of \( E \) which coincides with \( f \) on \( H \); by composing \( f \) with the inverse of such an isometry, we reduce to the case where \( f|_H = id_H \).

Write
\[
U = H \oplus \langle a \rangle \quad \text{and} \quad W = H \oplus \langle b \rangle \quad \text{with} \quad b = f(a),
\]
noting that \( a, b \not\in K \) by our choice of \( H \). The subspace \( (b - a)^\perp \leq E \) has codimension at most 1 since it is the kernel of a linear map with 1-dimensional codomain.
Case 1: \((b - a)^\perp = E\). Then, by applying Lemma 3.4 again to the nonzero vectors \(a\) and \(b\) in the vector space \(E/H\), we obtain a hyperplane \(M\) with \(H \leq M < E\), which is complementary to both \(\langle a \rangle\) and \(\langle b \rangle\). By Lemma 3.5, \(f\) extends to a bijective isometry on all of \(E\) by declaring it to be the identity on \(M\) as \(M \leq (b - a)^\perp\).

Case 2: \((b - a)^\perp < E\) is a hyperplane. Note that \(H \leq (b - a)^\perp\), since \(\omega_q(h, a) = \omega_q(h, b)\) for all \(h \in H\). Also, we have that \(a \in (b-a)^\perp\) if and only if \(b \in (b-a)^\perp\) because \(\omega_q(b, b) = \omega_q(a, a)\), so
\[
\omega_q(b - a) = \omega_q(b, b) - \omega_q(b, a) = \omega_q(a, a) - \omega_q(b, a) = \omega_q(a - b, a).
\]

Case 2.1: \(a, b \notin (b - a)^\perp\). This means that
\[
(b - a)^\perp \oplus \langle a \rangle = E = (b - a)^\perp \oplus \langle b \rangle.
\]
As in case one, we can extend \(f\) by defining it to be the identity on \((b - a)^\perp\) using again Lemma 3.5 as this space is by definition orthogonal to \(b - a\).

Case 2.2: \(a, b \in (b - a)^\perp\). We first check that this implies that
\[
Q_q(b - a) = 0.
\]
Indeed, if the characteristic of \(\mathbb{F}\) is two, this is true because
\[
Q_q(b - a) = Q_q(b) + Q_q(a) - \omega_q(b, a) = 2Q_q(b) - \omega_q(a, a) = q(a, a) - \varepsilon q(a, a) = 0 \mod \Lambda.
\]
If the characteristic of \(\mathbb{F}\) is not 2, instead we observe that
\[
\omega_q(b - a, b - a) = 0,
\]
and then recall that, since \(\Lambda_{\min} = \Lambda = \Lambda_{\max}\) in this case by Proposition A.1, the vanishing of \(\omega_q(b - a, b - a)\) implies that of \(Q_q(b - a)\), as in the proof of Lemma 2.4(1).

We have \(H, K \leq (b - a)^\perp\) and by the additional assumption also \(U = H \oplus \langle a \rangle, W = H \oplus \langle b \rangle\) are also subspaces of \((b - a)^\perp\). Recall also that \(K \cap U = K \cap H = K \cap W\) by our choice of \(H\). Let \(L \leq (b - a)^\perp\) by a common complement of \(U\) and \(W\) inside \((b - a)^\perp\) that contains \(K\). (Such a complement can be constructed by first picking a complement \(N\) of \(H \cap K\) inside \(K\) and then using Lemma 3.4 on \((b - a)^\perp/N\).) Setting \(M = H \oplus L \leq (b - a)^\perp\), we have
\[
(b - a)^\perp = M \oplus a = M \oplus b,
\]
and by construction \(K, H \leq M\).

As \(M\) is orthogonal to \(b - a\), we can again extend \(f\) to an isometry of \((b - a)^\perp\) by setting it to be the identity map on \(M\). We need to extend \(f\) to the whole of \(E\).

Recall that \(a, b \notin K\), so both
\[
a^\perp, b^\perp < E
\]
are hyperplanes. We claim that neither of them contains \(M^\perp\). Assume to the contrary that
\[
M^\perp \subset a^\perp.
\]
That implies that \(a \in a^{\perp \perp} \leq M^{\perp \perp} = M + K = M\), a contradiction since \(a\) is not in \(M\). Similar logic applies to \(b\). So \(M^\perp\) is contained in neither \(a^\perp\) nor \(b^\perp\). So,
\[
M^\perp \cap a^\perp, M^\perp \cap b^\perp < M^\perp
\]
have codimension one. Using Lemma 3.4 again, there is some vector
\[
v \in M^\perp
\]
which is in neither of them. That is, \( v \) not orthogonal to either \( a \) or \( b \). We break into two cases depending on whether or not \( v \) is orthogonal to \( b - a \).

**Case 2.2.1: \( v \not\in (b-a)\perp \).** We intend to extend \( f \) to all of \( E \) by setting

\[
f(v) = c(b-a) + dv
\]

for some scalars \( c, d \in \mathbb{F} \) with \( d \neq 0 \). This will be bijective, so we are done if we can choose scalars yielding an isometry. Note that \( f(v) - v \in M^\perp \) so

\[
\omega_q(m, v) = \omega_q(m, f(v))
\]

for all \( m \in M \). Now,

\[
\omega_q(b, f(v)) = d\omega_q(b, v).
\]

Since \( v \) is not orthogonal to \( a \) or \( b \),

\[
\omega_q(b, v), \omega_q(a, v) \neq 0,
\]

so we can set

\[
d = \frac{\omega_q(a, v)}{\omega_q(b, v)} \neq 0 \quad \text{to get} \quad \omega_q(b, f(v)) = \omega_q(a, v) \quad \text{as needed.}
\]

Lastly,

\[
Q_q(f(v)) = ccQ_q(b-a) + ddQ_q(v) + cd\omega_q(b-a, a).
\]

The first term is zero. In the last term, both \( d \) and \( \omega_q(v, b-a) \) are nonzero (since we have assumed that \( v \) is not orthogonal to \( b-a \)). So we can pick

\[
c = \frac{(1 - dd)\omega_q(v)}{\omega_q(v, b-a)} \quad \text{to get} \quad Q_q(f(v)) = Q_q(v).
\]

This shows that \( f \) is an isometry of \( E \).

**Case 2.2.2: \( v \in (b-a)^\perp \).** Choose a complement

\[
z \notin (b-a)^\perp.
\]

This time we intend to extend \( f \) to all of \( V \) by setting

\[
f(z) = z + c(b-a) + dv
\]

for some scalars \( c, d \in \mathbb{F} \). This will be bijective, so we are done if we can choose scalars yielding an isometry. Again, \( f(z) - z \in M^\perp \) so

\[
\omega_q(m, z) = \omega_q(m, f(z))
\]

for all \( m \in M \). Now,

\[
\omega_q(b, f(z)) = \omega_q(b, z) + d\omega_q(b, v).
\]

Since \( v \) is not orthogonal to \( b \), we can set

\[
d = \frac{\omega_q(a, z) - \omega_q(b, z)}{\omega_q(b, v)} \quad \text{to get} \quad \omega_q(b, f(z)) = \omega_q(a, z) \quad \text{as needed.}
\]

Lastly,

\[
Q_q(f(z)) = Q_q(z) + ccQ_q(b-a) + ddQ_q(v) + cw_\omega(z, b-a) + d\omega_q(z, v) + cd\omega_q(b-a, v)
\]

\[
= Q_q(z) + ddQ_q(v) + cw_\omega(z, b-a) + d\omega_q(z, v),
\]

since we have assumed that \( v \) is orthogonal to \( b-a \). But \( z \) is not orthogonal to \( b-a \), so we can set

\[
c = -\frac{ddQ_q(v) + d\omega_q(z, v)}{\omega_q(z, b-a)} \quad \text{to get} \quad Q_q(f(z)) = Q_q(z).
\]

Using Theorem 2.3, this shows that \( f \) is an isometry of \( E \). \( \square \)
The following result is a relative variant of Witt’s Lemma, which tells us when we can extend an isometry between two subspaces to an isometry of the ambient space which is the identity on a fixed subspace $A$:

**Theorem 3.6** (Relative Witt Lemma). Let $(E, q)$ be a formed space and $A \leq E$ a subspace. Suppose that $f : U \to W$ is a bijective isometry between two subspaces of $E$ such that

1. $U \cap A = W \cap A$ and $f|_{U \cap A} = id$;
2. $f = id \mod A^\perp$, i.e. $f(u) - u \in A^\perp$ for any $u \in U$;
3. $f(U \cap K(E)) = W \cap K(E)$.

Then $f$ can be extended to a bijective isometry of $E$ that restricts to the identity on $A$.

The following result shows that the assumption that $f$ is the identity modulo $A^\perp$ is actually necessary.

**Lemma 3.7.** Let $(E, q)$ be a formed space and $A \leq E$ any subspace. Any isometry of $E$ fixing $A$ pointwise preserves $A^\perp$ and is the identity modulo $A^\perp$.

**Proof.** Let $f : E \to E$ be an isometry fixing $A$ pointwise. If $v \in E$ and $a \in A$, then

$$\omega_q(f(v), a) = \omega_q(f(v), f(a)) = \omega_q(v, a),$$

so $f(v) - v \in A^\perp$, showing both that $f$ preserves $A^\perp$ and that it is the identity modulo $A^\perp$. □

**Proof of Theorem 3.6.** Let $A_0$ be a complement of $U \cap A = W \cap A$ inside $A$. We claim that the map $\bar{f} = f \oplus id : U \oplus A_0 \to W \oplus A_0$ is an isometry. Indeed, $\bar{f}$ restricts to an isometry on both $U$ and $A_0$. Let $u \in U$ and $a \in A_0$. By assumption, we have $f(u) = u + v$ with $v \in A^\perp$. Hence $\omega_q(f(u), f(a)) = \omega_q(u + v, a) = \omega_q(u, a)$, so $\bar{f}$ preserves $\omega_q$. Also, $Q_q(\bar{f}(u + a)) = Q_q((u + v) + a) = Q_q(f(v)) + Q_q(a) + \omega_q(u + v, a) = Q_q(v) + Q_q(a) + \omega_q(u, a) = Q_q(u + a)$ where we used that $f$ is an isometry. Hence, by Theorem 2.3, $\bar{f}$ is an isometry. As $\bar{f}((U \oplus A_0) \cap K(E)) = f(U \cap K(E)) \oplus (A_0 \cap K(E)) = (W \oplus A_0) \cap K(E)$, we can apply Theorem 3.3 to extend $\bar{f}$ to a bijective isometry $\hat{f}$ of $E$ that takes $U \oplus A_0$ to $W \oplus A_0$. By construction, $\hat{f}$ extends $f$ and restricts to the identity on $A$. □

The following corollary of the relative Witt’s Lemma was our motivation for writing the result:

**Corollary 3.8.** Let $(E, q)$ be a formed space, and $U,W_1,W_2$ isotropic subspaces containing $R(E)$. Suppose $\dim W_1 = \dim W_2$ and $W_1 + U^\perp = W_2 + U^\perp = E$. Then there is a bijective isometry of $E$ sending $W_1$ to $W_2$, and restricting to the identity on $U$. Furthermore, an isomorphism $f : W_1 \to W_2$ restricting to the identity on $R(E)$ can be extended to an isometry of $E$ fixing $U$ pointwise if and only if $f$ is the identity map modulo $U^\perp$.

For an isotropic subspace $U$ containing the radical $R(E)$, define the relative building to be

$$P(E, U) = \left\{ W < E \mid W \text{ isotropic, } R(E) < W, W + U^\perp = E \right\}.$$

The corollary implies in particular that the group

$$A(E, U) := \{ g \in G(E, q) \mid g|_{U} = id_{U} \}$$

of the bijective isometries of $(E, q)$ fixing $U$ pointwise acts transitively on the elements of any given fixed rank. This is used in [10] to study the homology of the groups $A(E, U)$.

The proof of the corollary will use the following:

**Lemma 3.9.** Let $(E, q)$ be a formed space and let $U,W \leq E$ be subspaces. If $U$ is isotropic and $W + U^\perp = E$, then $U \cap W^\perp \leq R(E)$. In particular, if $W$ is also isotropic, then $U \cap W \leq R(E)$. 

Proof of Lemma 3.9. Suppose that \( U \cap W^\perp = A \). As \( A \subset W^\perp \), we also have \( W \subset A^\perp \). And \( A \subset U \) implies that \( U^\perp \subset A^\perp \). So \( E = W + U^\perp \subset A^\perp \). This means that \( A \subset \mathcal{K}(E) \). But \( A \subset U \) is isotropic, so \( Q_{q|A} = 0 \) and we conclude \( A \leq \mathcal{R}(E) \) as claimed. The last claim follows from the fact that \( W \subset W^\perp \) when \( W \) is isotropic. \( \square \)

Proof of Corollary 3.8. We want to apply Theorem 3.6. For this, we need to construct a bijective isometry \( f : W_1 \to W_2 \) satisfying the assumptions of the theorem. As \( W_1 \) and \( W_2 \) are isotropic, any isomorphism will give a bijective isometry. Note now that, by Lemma 3.9, \( W_i \cap U \leq \mathcal{R}(E) \), and hence \( W_i \cap U = \mathcal{R}(E) \) as \( \mathcal{R}(E) \) is assumed to be contained in \( U, W_1 \) and \( W_2 \). This gives condition (1) in the theorem. As \( W_1 + U^\perp = E = W_2 + U^\perp \), we can pick a isomorphism \( f \) which is the identity modulo \( U^\perp \), and hence satisfies condition (2). As \( W_1 \cap \mathcal{R}(E) \subset W_2 \), we can choose such an \( f \) which is the identity on \( \mathcal{R}(E) \). This shows that condition (3) is also satisfied as \( W_i \cap \mathcal{K}(E) = W_i \cap \mathcal{R}(E) = \mathcal{R}(E) \) as each \( W_i \) is isotropic. Hence there exists a bijective isometry of \( E \) taking \( W_1 \) to \( W_2 \) and restricting to the identity on \( U \).

For the last statement, Lemma 3.7 shows that any isometry of \( E \) that fixes \( U \) necessarily is the identity modulo \( U^\perp \), which gives one direction. On the other hand, if \( f \) is the identity on \( \mathcal{R}(E) \) and the identity modulo \( U^\perp \), we see that it satisfies the conditions of Theorem 3.6 as it is automatically an isometry, satisfies conditions (2) and (3) by assumption, and condition (1) is satisfied by the above argument, which does not depend on \( f \). \( \square \)

Appendix A. The possible values of \( \Lambda \)

Let \((F, \sigma)\) be as above a field with chosen involution, and write as before \( \bar{c} = \sigma(c) \). If \( \sigma \) is not the identity, then

\[
F \supset F^\sigma
\]

is a field extension of degree 2, where \( F^\sigma \) denotes the fixed points of the involution. We fix as before a scalar \( \varepsilon \in F \) satisfying \( \varepsilon \bar{\varepsilon} = 1 \).

Recall that

\[
\Lambda_{\text{min}} = \{ a - \varepsilon \bar{a} \mid a \in F \} \quad \text{and} \quad \Lambda_{\text{max}} = \{ a \in F \mid a + \varepsilon \bar{a} = 0 \}.
\]

Note first that \( \Lambda_{\text{min}} \leq \Lambda_{\text{max}} \), because \( (a - \varepsilon \bar{a}) + \varepsilon(a - \varepsilon \bar{a}) = 0 \).

The following result gives the values of \( \Lambda_{\text{min}} \) and \( \Lambda_{\text{max}} \) in the case of fields, depending on whether the involution is trivial or not, the characteristic of the field is even or not, and whether \( \varepsilon \) is equal to 1, \(-1\) or something else.

Proposition A.1. Let \( F \) be a field of characteristic \( p \), \( \sigma \) an involution on \( F \) and \( \varepsilon \in F \) such that \( \varepsilon \bar{\varepsilon} = 1 \). Then the values of \( \Lambda_{\text{min}} \) and \( \Lambda_{\text{max}} \) are given by the following table:

<table>
<thead>
<tr>
<th>( \sigma = \text{id?} )</th>
<th>( p = 2? )</th>
<th>( \varepsilon )</th>
<th>( \Lambda_{\text{min}} )</th>
<th>( \Lambda_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>no</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
<td>1</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>1</td>
<td>0</td>
<td>( F^\sigma )</td>
</tr>
<tr>
<td>no</td>
<td></td>
<td>( \neq -1 )</td>
<td>( (1 + \varepsilon)F^\sigma )</td>
<td>( (1 + \varepsilon)F^\sigma )</td>
</tr>
</tbody>
</table>

Note in particular that in most cases \( \Lambda_{\text{min}} = \Lambda_{\text{max}} \) in which case there is only one possible choice for \( \Lambda_{\text{min}} \leq \Lambda \leq \Lambda_{\text{max}} \). The only situation where \( \Lambda_{\text{min}} \neq \Lambda_{\text{max}} \) is when \( \sigma \) is the identity and the characteristic is equal to \( 2 \), in which case \( \Lambda \) can be any additive subgroup of \( F \).

Proof. Suppose first that \( \sigma = \text{id} \). This implies that \( \varepsilon = \pm 1 \).
(In characteristic 2, this gives only one possibility.) If $\varepsilon = 1$, then $\Lambda_{\min} = 0$. Otherwise, $(1 - \varepsilon)$ is invertible and $\Lambda_{\min} = \mathbb{F}$. This gives $\Lambda_{\min}$ in the first three cases in the table. If $\varepsilon = -1$, we have that $\Lambda_{\max} = \mathbb{F}$. Otherwise, $(1 + \varepsilon)$ is invertible and $\Lambda_{\max} = 0$. This completes the proof of the first three cases in the table.

Now consider the case when $\sigma$ is nontrivial. The set $\Lambda_{\min}$ is the image of the map $\mathbb{F} \to \mathbb{F}$, $a \mapsto a - \varepsilon \bar{a}$.

This map is $\mathbb{F}^\sigma$-linear, so its image is an $\mathbb{F}^\sigma$-subspace of the two-dimensional vector space $\mathbb{F}$ (over $\mathbb{F}^\sigma$). We claim that this map has rank 1.

First assume that $\varepsilon \neq -1$. Then its kernel contains the element $(1 + \varepsilon)$, so it is not injective. However, it cannot be identically zero, because (by applying the map to $1 \in \mathbb{F}^\sigma$) that would imply $\varepsilon = 1$ and $\sigma = id$, contradiction our assumption. Next assume that $\varepsilon = -1$. Then our map $a \mapsto a + \bar{a}$ is the trace map of the field extension. This map is well-known to be nonzero from character theory. It cannot be surjective since its image lies in $\mathbb{F}^\sigma$. In either case, we conclude that $\Lambda_{\min}$ is a one-dimensional subspace of $\mathbb{F}$ over $\mathbb{F}^\sigma$.

We will compute $\Lambda_{\max}$ to identify $\Lambda_{\min}$. By definition, $\Lambda_{\max}$ is the kernel of the map $\mathbb{F} \to \mathbb{F}^\sigma$, $a \mapsto a + \varepsilon \bar{a}$.

Since this is the same map as before but with $\varepsilon$ replaced by $-\varepsilon$, we have already showed that it has rank 1 as an $\mathbb{F}^\sigma$-linear endomorphism of $\mathbb{F}$. So its kernel is a one-dimensional subspace. Because it must contain $\Lambda_{\min}$, we have

$$\Lambda_{\max} = \Lambda_{\min} = \begin{cases} \mathbb{F}^\sigma & \text{if } \varepsilon = -1, \\ (1 + \varepsilon)\mathbb{F}^\sigma & \text{if } \varepsilon \neq -1. \end{cases}$$

as subsets of $\mathbb{F}$.  

\begin{flushright} \ding{51} \end{flushright}

References


