PRODUCTS AND COPRODUCTS IN STRING TOPOLOGY

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Abstract. Let M be a closed Riemannian manifold. We extend the product of Goresky-Hingston, on the cohomology of the free loop space of M relative to the constant loops, to a non-relative product. It is associative, graded commutative, and compatible with the length filtration on the loop space, like the original product. We prove the following new geometric property of the dual homology coproduct: the non-vanishing of the k–th iterate of the dual coproduct detects the presence of loops with $(k+1)$-fold self-intersections in the image of chain representatives of homology classes in the loop space. For spheres and projective spaces, we show that this is sharp, in the sense that the k–iterated coproduct vanishes precisely when a class has support in the loops with at most k–fold self-intersections. We study the interactions between this cohomology product and the more well-known Chas-Sullivan product, and show that both structures are preserved by degree 1 maps, proving in particular that the Goresky-Hingston product is homotopy invariant. We give explicit integral chain level constructions of these loop products and coproduct, including a new construction of the Chas-Sullivan product, which avoid the technicalities of infinite dimensional tubular neighborhoods and delicate intersections of chains in loop spaces.

Let M be a closed oriented manifold of dimension n, for which we pick a Riemannian metric, and let $\Lambda M = \mathrm{Maps}(S^1, M)$ denote its free loop space. Goresky and the first author defined in [18] a product $\circledast$ on the cohomology $H^*(\Lambda M, M)$, and an associated coproduct $\vee$ on $H_*(\Lambda M, M)$, relative to the constant loops $M \subset \Lambda M$. They showed that this cohomology product behaves as a form of "dual" of the Chas-Sullivan product [8] on the homology $H_*(\Lambda M)$, at least through the eyes of the energy functional. In the present paper, we lift these relative cohomology product and homology coproduct to chain level non-relative product $\widehat{\circledast}\colon C^*(\Lambda M) \otimes C^*(\Lambda M) \to C^*(\Lambda M)$ and coproduct $\widehat{\vee}\colon C_*(\Lambda) \to C_*(\Lambda \times \Lambda)$, on integral singular chains. The operations $\widehat{\circledast}$ and $\widehat{\vee}$ are the "extension by zero" of their relative predecessors $\circledast$ and $\vee$ under the splitting on homology and cohomology induced by the evaluation map $\Lambda M \to M$. We study the algebraic and geometric properties of these two new operations, $\widehat{\vee}$ and $\widehat{\circledast}$, showing in particular that the good properties of the original product and coproduct proved in [18], such as associativity, graded commutativity, and compatibility with the length filtration, remain valid for the extended versions. Our first main result is the following: for $[A] \in H_*(\Lambda M)$, the non-vanishing of the iterated coproduct $\widehat{\vee}^k[A]$ implies the existence of loops with $(k+1)$-fold intersections in the image of any chain representatives of $[A]$; on spheres and projective spaces, we show that this is a complete invariant in the sense that the converse implication also holds. Our second main result is that, in (co)homology, the three operations $\widehat{\vee}, \widehat{\circledast}$ and the Chas-Sullivan product $\wedge\colon H_*(\Lambda M) \otimes H_*(\Lambda M) \to H_*(\Lambda M)$ are preserved by degree 1 maps between manifolds, showing in particular that $\widehat{\vee}$ and $\widehat{\circledast}$, like $\wedge$, are homotopy invariant structures. This improves and extends the earlier known homotopy invariance of the Chas-Sullivan product [13] (see also [14, 19]). The operation $\widehat{\vee}$ is not part of a classical field theory as studied eg. in [17, 10, 9], but rather has a "compactified" flavor: its conjectured algebraic model [1, 25] is associated to a loop diagram, in the sense of [25], and the space of loop diagrams is closely related to Bodigheimer's Harmonic compactification of the moduli space of Riemann surfaces via [31]. (See also [15] for a discussion of field theoretic approaches to string topology.) Our result suggests that "compactified string topology", like the more classical string topology, might be homotopy invariant.

An advantage of having a coproduct defined on $H_*(\Lambda M)$ is that it makes it possible to study its interplay with the Chas-Sullivan product; we show that the equation $\wedge_{Th} \circ \vee = 0$ holds, for

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$\wedge_{Th}$ the "Thom-sign" version of the Chas-Sullivan product (see below). We also exhibit, through computations, the failure of an expected Frobenius equation.

To state our main results more precisely, we need the following ingredients: Fix a small $\varepsilon > 0$ and let

$$U_M = \{(x,y) \in M^2 \mid |x-y| < \varepsilon\}$$

be the ε–neighborhood of the diagonal ΔM in M^2, with

$$\tau_M \in C^n(M^2, M^2 \backslash U_M)$$

a Thom class for an associated tubular embedding $\nu_M \colon TM \xrightarrow{\simeq} U_M$. (A specific model is given in Section 1.3.) Consider the evaluation maps

$$e \times e : \Lambda M^2 \longrightarrow M^2 \quad \text{and} \quad e_I \colon \Lambda M \times I \longrightarrow M^2$$

taking a pair (γ, δ) to $(\gamma(0), \delta(0))$ and (γ, s) to $(\gamma(0), \gamma(s))$. The tubular embedding ν_M gives rise to a retraction $U_M \to \Delta M$, which we lift to retraction maps

$$\begin{array}{lrcrcl} R_{CS}\colon & (e\times e)^{-1}(U_M) & \longrightarrow & (e\times e)^{-1}(\Delta M) & = & \{(\gamma,\delta) \in \Lambda M^2 \mid \gamma(0) = \delta(0)\} \\ R_{GH}\colon & e_I^{-1}(U_M) & \longrightarrow & e_I^{-1}(\Delta M) & = & \{(\gamma, s) \in \Lambda M \times I \mid \gamma(s) = \gamma(0)\} \end{array}$$

as depicted in Figure 1: both map add a geodesic stick to the loops to make them intersect as required—see Sections 1.4 and 1.5 for precise definitions. Finally, let

$$concat \colon \Lambda M \times_M \Lambda M = (e\times e)^{-1}(\Delta M) \longrightarrow M \quad \text{and} \quad cut \colon e_I^{-1}(\Delta M) \longrightarrow \Lambda M \times \Lambda M$$

be the concatenation and cutting maps, the latter cutting the loop at its self-intersection time s.

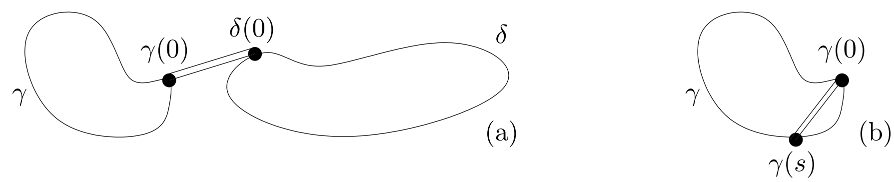

FIGURE 1. The retraction maps R_{CS} and R_{GH}.

Our results take their roots in the following chain-level construction of the Chas-Sullivan product and the Goresky-Hingston coproduct, based on the ideas of Cohen and Jones in [11] but avoiding the technicalities of infinite dimensional tubular neighborhoods, and avoiding the subtle limit arguments of [18].

Theorem A. *The Chas-Sullivan $\wedge$ product of [8] and Goresky-Hingston coproduct $\vee$ of [18] admit the following integral chain level descriptions: for $A \in C_p(\Lambda M)$ and $B \in C_q(\Lambda M)$,*

$$A \wedge B = (-1)^{np+n} concat\left(R_{CS}((e\times e)^*(\tau_M) \cap (A\times B))\right).$$

and for $C \in C_p(\Lambda M, M)$,

$$\vee C = cut(R_{GH}(e_I^*(\tau_M) \cap (C \times I)))$$

with $A \wedge B \in C_{p+q-n}(\Lambda)$ and $\vee C \in C_{p-n+1}(\Lambda M \times \Lambda M)/\big(C_*(M\times \Lambda M) + C_*(\Lambda M \times M)\big)$.

We note that the chain complex $C_*(\Lambda M \times \Lambda M)/\big(C_*(M\times\Lambda M) + C_*(\Lambda M\times M)\big)$ computes the relative homology group $H_*(\Lambda\times\Lambda, M\times\Lambda \cup \Lambda\times M)$, see Remark 1.4 for more details about this. It is technically more convenient for us than the more usual relative complex.

In the case of the Chas-Sullivan product, there exists many approaches to chain level constructions, in particular using more "geometric" chains, applying more directly the original idea of Chas and Sullivan of intersecting chains (see eg. [9, 27, 24]). The idea of using small geodesics to make up for "failed intersections" was already suggested in [34] and can also be found in [15]. In [15] a chain model of the Chas-Sullivan product is given as part of a larger structure, though again not including the above coproduct.

We emphasize the similarity between the definitions of $\wedge$ and $\vee$ given above. The Chas-Sullivan product is corrected by a sign for better algebraic properties; see Appendix B for more details about this. We will also correct the dual of the coproduct by a sign. We denote by $\wedge_{Th}$ the uncorrected, *Thom-signed*, Chas-Sullivan product, i.e. the operation defined by $A \wedge_{Th} B = concat\,(R_{CS}((e\times e)^*(\tau_M)\cap(A\times B)))$.

The splitting of $H_*(\Lambda M)\cong H_*(\Lambda M,M)\oplus H_*(M)$ by the evaluation map makes it possible to define an "extension by 0" of the Goresky-Hingston coproduct, setting it to be trivial on the constant loops. The same holds for chains:

Theorem B. *There is a unique lift*

$$\widehat{\vee}\colon C_*(\Lambda M)\longrightarrow C_*(\Lambda M\times\Lambda M)$$

of the coproduct $\vee$ *of Theorem A satisfying that*

$$\langle x\times y,\widehat{\vee}Z\rangle = 0 \qquad \text{if}\ \ x\in e^*C^*(M), y\in e^*C^*(M),\ \text{ or } Z\in C_*(M)$$

for $e\colon \Lambda M\to M$ *the evaluation at* 0*. For* $A\in C_*(\Lambda M)$*, with* $p_*A\in C_*(\Lambda M,M)$ *its projection, it is defined by*

$$\widehat{\vee}A = (1-e_*)\times(1-e_*)\vee(p_*A).$$

The induced map in homology $\widehat{\vee}\colon H_*(\Lambda)\to H_{*-n+1}(\Lambda\times\Lambda)$ *has the following properties:*

(i) (Vanishing) For $\wedge_{Th}\colon H_*(\Lambda\times\Lambda)\to H_{*-n}(\Lambda)$ *the Thom-signed Chas-Sullivan product (as above),*

$$\wedge_{Th}\circ\widehat{\vee} = 0.$$

(ii) (Support) If $Z\in C_*(\Lambda)$ *has the property that every nonconstant loop in its image has at most* k*-fold intersections, then*

$$\widehat{\vee}^k[Z] = \widehat{\vee}^k(\Delta[Z]) = 0,$$

for $\Delta\colon H_*(\Lambda M)\to H_{*+1}(\Lambda M)$ *the map induced by the circle action on* ΛM*.*

We define in Section 4.1 the *intersection multiplicity* $\mathrm{int}([Z])$ of a homology class $[Z]\in H_*(\Lambda)$; it has the property that $\mathrm{int}([Z])\le k$ whenever there exists a representative of $[Z]$ with image in the loops with at most k–fold intersections (in the sense of Definition 4.1). The above result shows that if $\mathrm{int}([Z])\le k$, then $\widehat{\vee}^k[Z]=0$, or equivalently that when $\widehat{\vee}^k[Z]\neq 0$, then *any* chain representative of $[Z]$ must have at least one loop with a $k+1$–fold intersection in its image. For spheres and projective spaces, we get the following stronger statement:

Theorem C. *If* $M=S^n$*,* $\mathbb{R}P^n$*,* $\mathbb{C}P^n$ *or* $\mathbb{H}P^n$*, then for any* $[Z]\in H_*(\Lambda)$*, for any coefficients, and* $k\ge 1$*,*

$$\mathrm{int}([Z])\le k \iff \widehat{\vee}^k[Z]=0.$$

In Section 3.3, we give a complete computation of the coproduct $\widehat{\vee}$ on the homology $H_*(\Lambda S^n)$ with S^n an odd sphere. We use this computation to show that the "Frobenius formula" fails:

$$\widehat{\vee}\circ\wedge_{(Th)}\neq(1\times\wedge_{(Th)})\circ\widehat{\vee}+(\wedge_{(Th)}\times 1)\circ\widehat{\vee}$$

for $\wedge_{(Th)}=\wedge$ or $\wedge_{Th}$, and give in Remark 3.20 geometric reasons for this. We also show that, in contrast to $\wedge_{Th}\circ\widehat{\vee}$, the composition $\wedge\circ\widehat{\vee}$ is non-zero in general.

In the next result, we relate the dual cohomology product to the product of [18] and show that it still has the same properties as the unlifted product. In the statement, $\widehat{\vee}^*$ denotes the dual of the map $\widehat{\vee}$ of Theorem B.

Theorem D. *The dual product*

$$\widehat{\circledast}\colon C^p(\Lambda M)\otimes C^q(\Lambda M)\xrightarrow{(-1)^{q(n-1)}\times} C^{p+q}(\Lambda M\times\Lambda M)\xrightarrow{\widehat{\vee}^*} C^{p+q+n-1}(\Lambda M)$$

is a chain-level definition of the canonical extension by 0 of the relative cohomology Goresky-Hingston product [18]. The induced map in cohomology has the following properties:

(i) It is associative and satisfies the following graded commutativity relation:

$$[x]\widehat{\circledast}[y] = (-1)^{(p+n-1)(q+n-1)}[y]\widehat{\circledast}[x]$$

for all $[x] \in H^p(\Lambda)$ *and* $[y] \in H^q(\Lambda)$.

(ii) (Morse theoretic inequality) Let $[x], [y] \in H^*(\Lambda)$ *with* $[x]\widehat{\circledast}[y] \neq 0$, *then*

$$Cr([x]\widehat{\circledast}[y]) \geq Cr([x]) + Cr([y])$$

where $Cr([a]) = \sup\{l \in \mathbb{R} \,|\, [a]$ *is supported on* $\Lambda^{\geq \ell}\}$ *for* $\Lambda^{\geq \ell}$ *the subspace of loops of energy at least* ℓ^2.

Note that by definition, we have the following relation between the operations $\widehat{\vee}$ and $\widehat{\circledast}$: for any $x \in C^p(\Lambda)$, $y \in C^q(\Lambda)$ and $Z \in C_{p+q+n-1}(\Lambda)$,

$$\langle x\widehat{\circledast}y, Z\rangle = (-1)^{q(n-1)}\langle x \times y, \widehat{\vee}Z\rangle.$$

Our construction of the product and coproduct uses a Riemannian metric on M. However, we show that it is independent of the Riemannian structure, and in fact only depends on the homotopy type of the manifold:

Theorem E. *A degree 1 map* $f\colon M_1 \to M_2$ *induces a map of algebra and coalgebra*

$$(\Lambda f)_*\colon H_*(\Lambda M_1) \longrightarrow H_*(\Lambda M_2)$$

with respect to the Chas-Sullivan product $\wedge$ *and the extended coproduct* $\widehat{\vee}$. *In particular, a homotopy equivalence respecting the orientation induces an isomorphism of these algebra and coalgebra structures.*

More generally, if f has degree d, it takes d times the (co)product on M_1 to the (co)product on M_2, see Remark 5.9. Note also that dually, $\Lambda f^*\colon H^*(\Lambda M_2) \to H^*(\Lambda M_1)$ preserves their $\widehat{\circledast}$–algebra structure if f has degree 1. This result extends and generalises a result of Cohen-Klein-Sullivan who showed the homotopy invariance of the Chas-Sullivan product [13] (see also [19, 14]), and a result of the first author in [21], who treated the case of the coproduct on the based loop space, assuming that M_2 is simply-connected.

Organization of the paper. In Section 1 we give a precise construction of chain representatives $\wedge, \vee$ and $\circledast$ for the Chas-Sullivan and Goresky-Hingston products and coproduct. In Section 2, we prove that our definitions do indeed model the earlier defined products and coproducts. This involves the explicit construction of a tubular neighborhood of $\Lambda M \times_M \Lambda M$ inside Λ^2, see Proposition 2.2, and of $F_{[0,1]} := e_I^{-1}(\Delta M)$ inside $\Lambda M \times I$, see Proposition 2.9. Theorem A is proved by combining Propositions 2.4 and 2.12. In Section 3, we define the lifted product $\widehat{\vee}$ and coproduct $\widehat{\circledast}$, and prove Theorem D (as Theorem 3.1) and the first part of Theorem B (Theorems 3.3 and 3.7)—The vanishing statement in Theorem B is shown in Section 3.2 to be closely related to the almost complete vanishing of the so-called "trivial coproduct", as pointed out by Tamanoi in [34]. Section 4 is then concerned with the proof of the support statement, statement (ii) in Theorem B (given as Theorem 4.2), as well as its stronger form in the case of spheres and projective spaces, Theorem C (given as Theorem 4.11). Finally, in Section 5 we prove the invariance under degree 1 maps (Theorem E, given as Theorem 5.1).

In the present paper, all homology and cohomology is taken with integral coefficients, and the chains and cochains are the singular chains and cochains, unless explicitly otherwise specified.

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Contents

1. Chain level definitions of the loop products

Assume that M is a closed, compact, oriented Riemannian manifold of dimension n. Fix $0<\varepsilon<\frac{\rho}{14}$ where ρ is the injectivity radius of M. In this section, we use an explicit tubular neighborhood of the diagonal embedding $\Delta\colon M\hookrightarrow M\times M$ to give chain-level definitions for the loop products. We start by introducing our model of the free loop space.

1.1. H^1 loops.

A path $\gamma\colon[0,1]\to M$ is H^1 if it is absolutely continuous, which in particular implies that the derivative $\gamma'(t)$ is defined almost everywhere, and if the energy

$$E(\gamma)=\int_0^1|\gamma'(t)|^2dt$$

is finite. The *length*

$$\mathcal{L}(\gamma)=\int_0^1|\gamma'(t)|dt$$

is also defined and finite for H^1 paths, and

$$E(\gamma)\geq(\mathcal{L}(\gamma))^2,$$

with equality if and only if γ has constant speed. For technical reasons the best function on path spaces is the square root of the energy, which has the units of length. Like the length, it adds well, but (like energy) it extracts a penalty for bad parameterization. The reader who thinks of $\sqrt{(}E)$ as the length will not go far wrong.

Our model for the loop space of M will be the *closed* H^1 paths, also called H^1 *loops*:

$$\Lambda=\Lambda M=\{\gamma\colon[0,1]\to M\mid\gamma\text{ is }H^1\text{ and }\gamma(0)=\gamma(1)\}.$$

Note that the inclusions

$$C^\infty\text{ paths }\subset\text{ piecewise }C^\infty\text{ paths }\subset H^1\text{ paths }\subset C^0\text{ paths}$$

and

$$C^\infty \text{ loops } \subset \text{ piecewise } C^\infty \text{ loops } \subset H^1 \text{ loops } \subset C^0 \text{ loops}$$

are continuous and induce homotopy equivalences (see [26, Thm 1.2.10] for the last inclusion, and eg. [32, Thm 4.6]). The space Λ of H^1–loops on M is a Hilbert manifold [26, Thm 1.2.9], on which concatenation of loops is defined, and such that the energy is defined, and satisfies Condition C of Palais and Smale [30] (see [16]). Thus Λ is appropriate for infinite dimensional Morse theory.

A *constant speed path*, or path *parametrized proportional to arc length* is an H^1 path with $|\gamma'(t)|$ constant (where it is defined). In this case $|\gamma'(t)| = \mathcal{L}(\gamma)/(b-a)$ for all t where the derivative is defined. We denote by $\overline{\Lambda} \subset \Lambda$ the space of *constant speed loops.* Anosov [3, Thm 2] has proved that there is a continuous map

$$\Psi \colon \Lambda \to \overline{\Lambda}$$

that takes an H^1 loop $\gamma \in \Lambda M$ to the constant speed reparametrization of γ, exhibiting $\overline{\Lambda}$ as a deformation retract of Λ. (See Theorem 3 in the same paper.) This model of the loop space of M will be used in Section 4.

1.2. **Concatenation of paths.** If $\gamma, \delta \colon [0,1] \to M$ are paths with $\gamma(1) = \delta(0)$, then the *optimal concatenation*

$$\gamma \star \delta \colon [0,1] \to M$$

is the concatenation of the affine reparametrization of γ by $\frac{\mathcal{L}(\gamma)}{\mathcal{L}(\gamma)+\mathcal{L}(\delta)}$ and of δ by $\frac{\mathcal{L}(\delta)}{\mathcal{L}(\gamma)+\mathcal{L}(\delta)}$. (So the concatenation is happening at time $t = \frac{\mathcal{L}(\gamma)}{\mathcal{L}(\gamma)+\mathcal{L}(\delta)}$.) This concatenation has the property of being strictly associative, and has minimal energy.

We will also use the *concatenation at time* t: for γ and δ as above, we define

$$\gamma *_t \delta : [0,1] \to M$$

as the concatenation of the affine reparametrization of γ by a factor t and δ by a factor $(1-t)$, so the concatenation time in this case is t.

1.3. **Constructions in M.** The normal bundle of the diagonal $\Delta : M \hookrightarrow M \times M$ is isomorphic to the tangent bundle TM of M. We will throughout the paper identify TM with its open disc bundle of radius ε. We make here the following explicit choice of tubular neighborhood for Δ: let $TM \equiv \{(x,V) \mid x \in M, V \in T_xM, |V| < \varepsilon\}$ and

$$\nu_M \colon TM \longrightarrow M^2 \tag{1.1}$$

be the map taking (x,V) to $(x, \exp_x(V))$, where $\exp_x : T_xM \to M$ denotes the exponential map at the point $x \in M$. Let $U_M = U_{M,\varepsilon} \subset M^2$ be the neighborhood of the diagonal defined by

$$U_M := \{(x,y) \in M^2 \mid |x-y| < \varepsilon\}.$$

Then the map ν_M has image U_M and is a homeomorphism onto its image. Let

$$Th(TM) = DTM/STM$$

denote the Thom space of TM, its ε_0–disc bundle modulo its ε_0–sphere bundle, where $0 < \varepsilon_0 < \varepsilon$. The *Thom collapse map* associated to the tubular neighborhood ν_M is the map

$$\kappa_M : M^2 \longrightarrow Th(TM)$$

defined by

$$\kappa_M(x,y) = \begin{cases} (x, \exp_x^{-1}(y)) & |x-y| < \varepsilon_0 \\ * & |x-y| \geq \varepsilon_0 \end{cases}$$

where $*$ is the basepoint of $Th(TM)$. Note that

$$\nu_M \colon TM \longrightarrow M^2 \qquad \text{and} \qquad \kappa_M \colon M^2 \longrightarrow Th(TM)$$

are inverses of one another when restricted to the open ε_0–disc bundle $T_{\varepsilon_0}M$ and the ε_0–neighborhood U_{M,ε_0} of the diagonal in M^2. We pick a cochain representative

$$u_M \in C^n(TM, T_{\varepsilon_0}M^c)$$

of the Thom class of the tangent bundle TM, so that $\kappa_M^* u_M = 0$ on chains supported on the complement of U_{M,ε_0}. We will in the paper write

$$\tau_M := \kappa_M^* u_M \in C^n(M^2, U_{M,\varepsilon_0}^c).$$

By abuse of notations, we will also call τ_M its restriction to $C^n(U_{M,\varepsilon}, U_{M,\varepsilon_0})$.

The cap product plays a crucial role in our construction of the loop (co)products. We make precise in Appendix A which maps and associated cap products we use, and give their relevant naturality properties. For now we will just point out that one can compute the cap product

$$\cap\tau_M \colon C_*(M^2) \longrightarrow C_*(U_{M,\varepsilon}$$

using the usual formula for the cap product and the tubular neighborhood of ΔM in M^2.

1.4. **The homology product.** Recall from Section 1.1 that $\Lambda = \Lambda M$ denotes the space of H^1-loops in M. We denote by

$$e : \Lambda \to M$$

the evaluation at 0, that is the map taking a loop γ to its evaluation $e(\gamma) = \gamma(0)$.

Let $\Lambda \times_M \Lambda \subset \Lambda^2$ be the "figure-eight space", the subspace

$$\Lambda \times_M \Lambda = \{(\gamma, \lambda) \in \Lambda^2 \mid \gamma_0 = \lambda_0\},$$

where $\gamma_0 = \gamma(0)$ and $\lambda_0 = \lambda(0)$ are the basepoints of the loops, and consider its neighborhood inside Λ^2 defined by

$$U_{CS} = U_{CS,\varepsilon} := \{(\gamma, \lambda) \in \Lambda^2 \mid |\gamma_0 - \lambda_0| < \varepsilon\} = (e \times e)^{-1}(U_M).$$

We define the retraction

$$R_{CS} \colon U_{CS} \longrightarrow \Lambda \times_M \Lambda$$

by

$$R_{CS}(\gamma, \lambda) = (\gamma\ ,\ \overline{\gamma_0, \lambda_0}\ \star\ \lambda\ \star\ \overline{\lambda_0, \gamma_0})$$

where, for $x, y \in M$ with $|x - y| < \rho$, $\overline{x, y}$ denotes the minimal geodesic path $[0, 1] \to M$ from x to y, and $\star$ is the optimal concatenation of paths defined above. That is, we add small "sticks" $\overline{\gamma_0, \lambda_0}$ and $\overline{\lambda_0, \gamma_0}$ to λ to make it have the same basepoint as γ. (See Figure 1(a).)

Define

$$\tau_{CS} := (e \times e)^* \tau_M \in C^n(\Lambda \times \Lambda, U_{CS\varepsilon_0}^c)$$

as the pulled-back Thom class τ_M and note that capping with this class defines a map

$$\tau_{CS} \cap\, : C_*(\Lambda \times \Lambda) \longrightarrow C_{*-n}(U_{CS})$$

so that the retraction R_{CS} is well-defined on the image of $\cap\tau_{CS}$. (See Appendix A, in particular equation (A.2), for more details.)

Definition 1.1. We define the loop product $\wedge_{Th}$ as the map

$$\wedge_{Th} \colon C_p(\Lambda) \otimes C_q(\Lambda) \longrightarrow C_{p+q-n}(\Lambda)$$

given by

$$A \wedge_{Th} B = \mathit{concat}\left(R_{CS}(\tau_{CS} \cap (A \times B)\right).$$

where $\mathit{concat} : \Lambda \times_M \Lambda \to \Lambda$ is the concatenation $\star$.

The map $(A, B) \to A \wedge_{Th} B$ is a composition of chain maps, and thus descends to a product on homology:

$$\wedge_{Th} \colon H_i(\Lambda) \otimes H_j(\Lambda) \to H_{i+j-n}(\Lambda)$$

Note that this is an integral singular chain level definition of a loop product, and that it does not involve any infinite dimensional tubular neighborhood, but only a simple retraction map and the pullback by the evaluation map of the Thom class of TM. Indeed the retraction map interacts only weakly with the loops in U_{CS}, adding "sticks" that depend only on the basepoints of γ and δ. We will show in Proposition 2.4 below that this definition of the Chas-Sullivan product coincides in homology with that of Cohen and Jones in [11].

We emphasized in the notation $\wedge_{Th}$ of the product that it is defined using a Thom-Pontrjagin like construction. To have nice algebraic properties, such as associativity and graded commutativity, one needs to correct the above product by a sign. (See Appendix B and Theorem 2.5 for more details.)

Definition 1.2. The *algebraic* loop product $\wedge$ is the map

$$\wedge\colon C_p(\Lambda)\otimes C_q(\Lambda) \longrightarrow C_{p+q-n}(\Lambda)$$

given by

$$A\wedge B := (-1)^{np+n} A\wedge_{Th} B.$$

We note that the idea of using "sticks" instead of deforming the loops can already be found in [34], where the same sign convention is also taken.

1.5. **The homology coproduct.** Let $F_{[0,1]}\subset \Lambda\times I$ be the subspace

$$F_{[0,1]} = \{(\gamma,s)\in \Lambda\times I \mid \gamma_0=\gamma(s)\},$$

and consider its neighborhood

$$U_{GH} = U_{GH,\varepsilon} := \{(\gamma,s)\in\Lambda\times I : |\gamma_0-\gamma(s)|<\varepsilon\} = (e_I)^{-1}(U_{M,\varepsilon}) \ \subset\ \Lambda\times I$$

where we again write γ_0 for $\gamma(0)$, and where

$$e_I\colon \Lambda\times I\longrightarrow M^2$$

is our notation for the map that takes a pair (γ,s) to the pair $(\gamma_0,\gamma(s))$. Define the retraction

$$R_{GH}: U_{GH}\longrightarrow F_{[0,1]}$$

by

$$R_{GH}(\gamma,s) = \Big(\big(\,\gamma[0,s]\ \star\ \overline{\gamma(s),\gamma_0}\,\big) *_s \big(\,\overline{\gamma_0,\gamma(s)}\star\ \gamma[s,1]\,\big)\ ,\ s\Big)$$

where $\gamma[a,b]$ denotes the restriction of γ to the interval $[a,b]\subset[0,1]$ precomposed by the scaling map $[0,1]\to[a,b]$, and where $\overline{xy}$ denotes again the shortest geodesic from x to y. (See Figure 1(b).) Note that the retraction is well-defined also when $s=0$ and $s=1$; in fact $R_{GH}(\gamma,s)=(\gamma,s)$ for any γ whenever $s=0$ or $s=1$.

Let

$$\tau_{GH} := (e_I)^*\tau_M\in C^n(\Lambda\times I, U^c_{GH,\varepsilon_0})$$

be the pulled-back Thom class. Taking $(X,U_0,U_1)=(\Lambda\times I, U_{GH,\varepsilon_0}, U_{GH,\varepsilon})$, the sequence of maps (A.2) composes to a map

$$\tau_{GH}\cap\, : C_*(\Lambda\times I)\longrightarrow C_{*-n}(U_{GH})$$

so that it is meaningful to apply the retraction map R_{GH} to the image of $\tau_{GH}\cap$.

Let

$$cut: F_{[0,1]}\to\Lambda\times\Lambda$$

be the cutting map, defined by $cut(\gamma,s)=(\gamma[0,s],\gamma[s,1])$.

Definition 1.3. We define the loop coproduct $\vee=\vee_{Th}$ as the map

$$C_*(\Lambda,M) = C_*(\Lambda)/C_*(M)\xrightarrow{\ \vee\ } C_*(\Lambda\times\Lambda)/\big(C_*(M\times\Lambda)+C_*(\Lambda\times M)\big)$$

given by

$$\vee A = cut(R_{GH}(\tau_{GH}\cap(A\times I));$$

that is, $\vee$ is the composition of the following maps of relative chains:

$$\begin{aligned}
C_*(\Lambda)/C_*(M) &\xrightarrow{\times I} C_{*+1}(\Lambda\times I)/\big(C_*(M\times I)+C_*(\Lambda\times\{0\})+C_*(\Lambda\times\{1\})\big)\\
&\xrightarrow{\tau_{GH}\cap} C_{*+1-n}(U_{GH})/\big(C_*(M\times I)+C_*(\Lambda\times\{0\})+C_*(\Lambda\times\{1\})\big)\\
&\xrightarrow{R_{GH}} C_{*+1-n}(F_{[0,1]})/\big(C_*(M\times I)+C_*(\Lambda\times\{0\})+C_*(\Lambda\times\{1\})\big)\\
&\xrightarrow{cut} C_{*+1-n}(\Lambda\times\Lambda)/\big(C_*(M\times\Lambda)+C_*(\Lambda\times M)\big)
\end{aligned}$$

Remark 1.4. We have chosen the quotient complex $C_{*+1-n}(\Lambda\times\Lambda)/\big(C_*(M\times\Lambda)+C_*(\Lambda\times M)\big)$ as our target for the coproduct $\vee$. The complex $C_*(M\times\Lambda)+C_*(\Lambda\times M)$ is a subcomplex of the complex $C_*(M\times\Lambda\cup\Lambda\times M)$, and the quotient map

$$C_{*+1-n}(\Lambda\times\Lambda)/\big(C_*(M\times\Lambda)+C_*(\Lambda\times M)\big)\twoheadrightarrow C_{*+1-n}(\Lambda\times\Lambda)/C_*(M\times\Lambda\cup\Lambda\times M)$$

is a quasi-isomorphism, so they both compute the relative homology group $H_*(\Lambda\times\Lambda, M\times\Lambda\cup\Lambda\times M)$. (This can for example be seen by replacing M by the homotopy equivalent subspace $\Lambda^{<\alpha}$ of small loops in Λ, with $\alpha<\rho$, and using "small simplices" associated to the covering $U_1=\Lambda\times\Lambda^{<\alpha}$ and $U_2=\Lambda^{\alpha}\times\Lambda$ of $\Lambda\times\Lambda^{<\alpha}\cup\Lambda^{<\alpha}\times\Lambda$.) As we will see, our chosen target complex is technically easier to work with in several situations.

As before this is a chain level definition, and involves only a simple retraction map and the pullback via the exponential map of the Thom class from TM. It is a composition of chain maps and descends to a map $H_k(\Lambda,M)\to H_{k-n+1}(\Lambda\times\Lambda, M\times\Lambda\cup\Lambda\times M)$. If we take coefficients in a field, or if the homology is torsion free, then we can compose with a (relative) Alexander-Whitney map to get a coproduct

$$\vee\colon H_k(\Lambda,M;k)\to\bigoplus_{k=p+q+n-1}H_p(\Lambda,M;k)\otimes H_q(\Lambda,M;k).$$

However there is in general no Kunneth isomorphism, and indeed no natural map $H_*(U\times V)\to H_*(U)\otimes H_*(V)$.

We will show in Proposition 2.12 below that this definition coincides on homology with that given by Goresky and Hingston in [18].

As there is no sign corrected version of the homology coproduct (as such a correction would require using the Kunneth isomorphism), we will suppress the "Th" from the notation in this case.

1.6. **The cohomology product.** The homology coproduct $\vee$ defined above induces a dual map

$$\vee^*:\Big(C_k(\Lambda\times\Lambda)/\big(C_k(M\times\Lambda)+C_k(\Lambda\times M)\big)\Big)^*\longrightarrow\quad C^{k+n-1}(\Lambda,M)$$

which we can precompose with a relative version of the cross-product

$$\times: C^i(\Lambda)\otimes C^j(\Lambda)\to C^{i+j}(\Lambda\times\Lambda):$$
$$a\times b=\pi_1^*a\cup\pi_2^*b.$$

Indeed, if the cochains a and b vanish on $C_*(M)$, then $a\times b$ will vanish on the image of $C_*(M\times\Lambda)$ and $C_*(\Lambda\times M)$ inside $C_*(\Lambda\times\Lambda)$.

Definition 1.5. The product

$$\circledast_{Th}\colon C^i(\Lambda,M)\otimes C^j(\Lambda,M)\to C^{i+j+n-1}(\Lambda,M)$$

is defined by

$$a\circledast_{Th}b=\vee^*(a\times b)$$

where $\vee$ is given in Definition 1.3.

Applying homology, this defines a cohomology product

$$\circledast_{Th}\colon H^p(\Lambda,M)\otimes H^q(\Lambda,M)\to H^{p+q+n-1}(\Lambda,M)$$

dual to the homology coproduct, but which is defined with integer coefficients as there is no Kunneth formula issue here. We will show in Section 2.3 that this product is equivalent on cohomology to the cohomology product defined in [18].

Just like for the loop product, the cohomology product needs to be corrected by a sign to be associative and graded commutative (see Appendix B and Theorem 2.13).

Definition 1.6. The *algebraic* product

$$\circledast\colon C^p(\Lambda,M)\otimes C^q(\Lambda,M)\to C^{p+q+n-1}(\Lambda,M)$$

is defined by

$$a\circledast b=(-1)^{(n-1)q}a\circledast_{Th}b=(-1)^{(n-1)q}\vee^*(a\times b).$$

2. Equivalence of the new and old definitions

Cohen and Jones' approach to defining the loop product, later also used in [18] for the coproduct, was via collapse maps associated to certain tubular neighborhoods in loop spaces. To show that our definition is equivalent with these earlier definitions, we will write down explicit such tubular neighborhoods, and show that the associated collapse maps can be modeled by sticks as described above. Chas and Sullivan originally defined their product in [8] by intersecting chains. This other approach has been made precise by several authors (see eg. [9, 28, 27]), and shown to be equivalent to the tubular neighborhood approach in eg. [28].

2.1. **Pushing points in M.** We define a map that allows us to "push points" continuously in M. This map will allow us in later sections to lift the tubular neighborhood of the diagonal to tubular neighborhoods of analogous subspaces of the loop space of M or related spaces.

Lemma 2.1. *Let M be a Riemannian manifold with injectivity radius ρ. Let $U_{M,\rho} \subset M \times M$ be the set of points (u,v) where $|u-v| < \rho$. There is a smooth map*

$$h : U_{M,\rho} \times M \subset M \times M \times M \quad \longrightarrow \quad M$$

with the following properties:

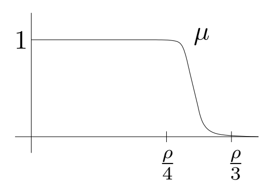
(i) For each $(u,v) \in U_{M,\frac{\rho}{14}}$, $h(u,v) := h(u,v,\_) : M \to M$ is a diffeomorphism.
(ii) $h(u,v)(u) = v$ if $|u-v| < \rho/14$.
(iii) $h(u,v)(w) = w$ if $|w-u| \geq \rho/2$.
(iv) $h(u,u)(w) = w$.

so $h(u,v)$ takes u to v if they are close enough and does not move w if w is far from u.

We encourage the reader to realize that it is quite clear that such a map should exist, but do give a proof for completeness.

Proof. Let $\mu : [0,\infty) \to \mathbb{R}$ be a smooth map with

$$\begin{aligned} \mu(r) &= 1 \text{ if } r \leq \rho/4 \\ \mu(r) &= 0 \text{ if } r \geq \rho/3 \\ \tfrac{-13}{\rho} &\leq \mu'(r) \leq 0 \end{aligned}$$



If $u,v \in M$ with $|u-v| < \rho$, we define $h(u,v) : M \to M$ by

$$h(u,v)(w) \quad = \quad \begin{cases} \exp_u\left(\exp_u^{-1}(w) + \mu(|u-w|)\exp_u^{-1}(v)\right) & \text{if } |u-w| \leq \rho/2 \\ w & \text{if } |u-w| \geq \rho/2 \end{cases}$$

First we will show that $h(u,v)(w)$ depends smoothly on u, v, and w. It is enough to show that $\exp_u^{-1}(w)$ depends smoothly on u and w if $|u-w| < \rho$. Recall that TM is the open disk bundle of radius ε, and $\varepsilon < \rho$. Recall from Section 1.3 the map $\nu_M : TM \to M \times M$ defined by

$$\nu_M(u,W) = (u, \exp_u W).$$

It is a smooth bijection onto $U_M = \{(u,w) \mid |u-w| < \varepsilon\}$, since there is a unique geodesic of length $< \varepsilon$ from u to w if $|u-w| < \varepsilon$. Thus it will be enough to show that the derivative $d\nu_M$ has maximal rank at each point in the domain TM of ν_M. But at a point (u,w) in the image, the image of $d\nu_M : TTM \to TM \times TM$ clearly contains all vectors of the form $(0,V)$, since $\exp_u : T_uM \to M$ is a diffeomorphism onto $\{w \mid |u-w| < \varepsilon\}$. And the projection of the image of $d\exp$ onto the first factor TM is clearly surjective, since "we can move u in TM". Thus $h(u,v)(w)$ is smooth in u, v, and w.

To see that $h(u,v)$ is a diffeomorphism when $|u-v| < \frac{\rho}{14}$, use $\exp_u$ to identify T_uM with the neighborhood $\{w \mid |u-w| < \varepsilon\}$ of u in M. In these coordinates, if $v = \exp_u V$ and $W \in T_uM$,

$$h(u,V)(W) =: \exp_u^{-1} h(u,v)(\exp_u W) = \begin{cases} W + \mu(|W|)V & \text{if } |W| \leq \rho/2 \\ W & \text{if } |W| \geq \rho/2. \end{cases}$$

Note that $h(u,V)$ preserves the "lines" $\{W_0+sV : s\in\mathbb{R}\}$. On a fixed line we have

$$\begin{aligned}\frac{d}{ds}h(u,V)(W_0+sV) &= \frac{d}{ds}\left((W_0+sV)+\mu(|W_0+sV|)V\right)\\ &= (1+\mu'(|W_0+sV|)\ \frac{d}{ds}(|W_0+sV|))V\end{aligned}$$

where the coefficient $(1+\mu'(|W_0+sV|)\ \frac{d}{ds}(|W_0+sV|))$ of V on the right is at least

$$(1-\frac{13}{\rho}\frac{\rho}{14})>0$$

because $\mu'(r)>-\frac{13}{\rho}$ for all r and $|V|=|u-v|<\frac{\rho}{14}$. Thus $h(u,V)$ is monotone on each line. It follows that, for each u,v, the map $h(u,V)$ is a local diffeomorphism and is bijective, and thus the same is true for $h(u,v):M\to M$. ☐

2.2. **Tubular neighborhood of the figure-eight space and equivalence to Cohen-Jones' definition of the Chas-Sullivan product.** There is a pull-back diagram

$$\begin{array}{ccc}\Lambda\times_M\Lambda & \hookrightarrow & \Lambda\times\Lambda\\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e\times e}\\ M & \xhookrightarrow{\Delta} & M\times M.\end{array}$$

Let $e^*(TM)\to\Lambda\times_M\Lambda$ be the pull-back of the tangent bundle of M along the evaluation at 0. A number of authors have already noticed that a tubular neighborhood of M sitting as the diagonal inside $M\times M$ can be lifted to a tubular neighborhood of $\Lambda\times_M\Lambda$ inside $\Lambda\times\Lambda$ (see eg. [2, 11, 28]). We give now an explicit such tubular neighborhood, compatible with the tubular neighborhood ν_M of the diagonal in M^2 constructed in Section 1.3. This construction is suggested in [2].

Proposition 2.2. *Let M be a Riemannian manifold with injectivity radius ρ and let $0<\varepsilon<\frac{\rho}{14}$. Identify as above TM with its ε–tangent bundle. The subspace $\Lambda\times_M\Lambda$ of $\Lambda\times\Lambda$ admits a tubular neighborhood*

$$\begin{array}{ccc}\nu_{CS}:\ e^*(TM) & \hookrightarrow & \Lambda\times\Lambda,\\ \downarrow & & \\ \Lambda\times_M\Lambda & & \end{array}$$

that is ν_{CS} restrict to the inclusion on the zero-section $\Lambda\times_M\Lambda$ and is a homeomorphism onto its image $U_{CS}=\{(\gamma,\lambda)\mid|\gamma(0)-\lambda(0)|<\varepsilon\}$. Moreover, one can choose this tubular neighborhood so that it is compatible with our chosen tubular neighborhood of the diagonal in M^2 in the sense that the following diagram commutes:

$$\begin{array}{ccc}e^*(TM) & \xhookrightarrow{\nu_{CS}} & \Lambda\times\Lambda\\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e\times e}\\ TM & \xhookrightarrow{\nu_M} & M\times M\end{array}$$

where ν_M is the tubular neighborhood of the diagonal (1.1).

Proof. Let $(\gamma,\delta,V)\in e^*(TM)$, so $(\gamma,\delta)\in\Lambda\times_M\Lambda$ and $V\in T_{\gamma_0}M=T_{\delta_0}M$, for $\gamma_0=\delta_0$ the startpoints of the loops. The map ν_{CS} is defined by $\nu_{CS}(\gamma,\delta,V)=(\gamma,\lambda(\delta,V))\in\Lambda\times\Lambda$ with

$$\lambda(\delta,V)(t)=h\big(\delta_0,\exp_{\delta_0}(V)\big)(\delta(t))$$

where h is the map of Lemma 2.1. This map is continuous with image $U_{CS}:=\{(\gamma,\lambda)\mid|\gamma_0-\lambda_0|<\varepsilon\}\subset\Lambda\times\Lambda$. Define now $\kappa_{CS}:U_{CS}\to e^*(TM)$ by $\kappa_{CS}(\gamma,\lambda)=(\gamma,\delta,V)$ where

$$\begin{aligned}\delta(t) &= (h(\gamma_0,\lambda_0))^{-1}(\lambda(t))\\ V &= \exp_{\gamma_0}^{-1}\lambda_0\end{aligned}$$

The map κ_{CS} is again continuous, and is an inverse for ν_{CS}. The compatibility of ν_{CS} and ν_M follows from the properties of h. □

Note that the inverse κ_{CS} of the tubular embedding ν_{CS} extends to to a "collapse map", for which we use the same notation,

$$\kappa_{CS}\colon \Lambda^2 \longrightarrow Th(ev_0^*TM) \tag{2.1}$$

by taking (γ,δ) to $\kappa(\gamma,\delta)=\nu_{CS}^{-1}(\gamma,\delta)$ if $(\gamma,\delta)\in U_{CS}$ and to the basepoint otherwise. Restricted to U_{CS}, this map has two components: $\kappa_{CS}=(k_{CS},V)$, where

$$k_{CS}\colon U_{CS}\longrightarrow \Lambda\times_M\Lambda$$

is a retraction map. We will now show that our retraction map R_{CS} of Section 1.4 that add sticks instead of deforming the loops, gives an approximation of the retraction component k_{CS} of the collapse map.

Lemma 2.3. *The maps $R_{CS}, k_{CS}\colon U_{CS}\longrightarrow \Lambda\times_M\Lambda$ are homotopic.*

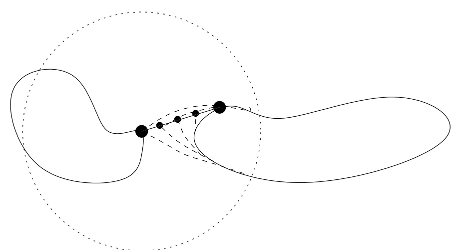


FIGURE 2. Homotopy between the retractions k_{CS} and R_{CS}.

Proof. Note that the only difference between the maps k_{CS} and R_{CS} is their value on the second factor of $\Lambda\times_M\Lambda$ when $|\gamma_0-\lambda_0|<\varepsilon$; it is given as

$$\delta(t)=(h(\gamma_0,\lambda_0))^{-1}\lambda(t)$$

in the first case and the loop $\overline{\gamma_0,\lambda_0}\star\lambda\star\overline{\lambda_0,\gamma_0}$ in the second case. To prove the Lemma, it is enough to define a homotopy $H: U_{CS}\times I\to\Lambda$ between these two maps, with the property that $H(\gamma,\lambda,s)(0)=\gamma_0$ for all s. Such a homotopy H can be given as follows: set $H(\gamma,\lambda,s)$ to be the loop

$$\overline{\gamma_0,\lambda_0}\,[0,s]\;\star\;\delta(\,\overline{\gamma_0,\lambda_0}(s),\lambda)\;\star\;\overline{\gamma_0,\lambda_0}\,[0,s]^{-1}$$

where $\overline{\gamma_0,\lambda_0}[0,s]$ is the geodesic from γ_0 to the point $\overline{\gamma_0,\lambda_0}(s)$ at time s on the geodesic, parameterized on $[0,1]$, from γ_0 to λ_0, and $\overline{\gamma_0,\lambda_0}\,[0,s]^{-1}$ is the same geodesic but in the reversed direction. (See Figure 2 for and illustration of this homotopy.) □

Cohen-Jones define the Chas-Sullivan product using a composition

$$C_*(\Lambda)\otimes C_*(\Lambda)\longrightarrow C_*(\Lambda\times\Lambda)\longrightarrow C_*(Th(e^*(TM))\longrightarrow \mathcal{C}_*(\Lambda)$$

where the first map is the Eilenberg-Zilber map, taking a pair of chains $A\otimes B$ to the chain $A\times B$ in the product, the second is the collapse map associated to a (not explicitly given) tubular neighborhood of $\Lambda\times_M\Lambda$ inside $\Lambda\times\Lambda$ of the form constructed in Proposition 2.2, the third one the concatenation and Thom isomorphism. (See [11, Thm 1] or [12, (1.7) and Prop 5] for a detailed description.)

Proposition 2.4. *The loop product $\wedge_{Th}$ of Definition 1.1 is chain homotopic to the explicit representative of the Cohen-Jones product of* [11] *obtained using Proposition 2.2.*

Proof. The map κ_{CS} of (2.1) is an explicit collapse map associated to a chosen tubular neighborhood of the form considered in [11, Sec 1]. Now $\kappa_{CS}=(k_{CS},V)$ may be replaced by $\widehat{R}_{CS}=(R_{CS},V)$ by Lemma 2.3. The result then follows from the naturality of the cap product (see Appendix A) and Thom isomorphism. □

Theorem 2.5. *The algebraic loop product* $\wedge\colon H_p(\Lambda)\otimes H_q(\Lambda)\to H_{p+q-n}(\Lambda)$ *is associative, unital with unit* $[M]\in H_n(\Lambda)$*, and commutative up to the following sign:*

$$[A]\wedge[B]=(-1)^{(n-p)(n-q)}[B]\wedge[A].$$

Proof. This result can be extracted from the litterature. However, as there are so many different definitions and sign conventions in the litterature, we give here instead a quick direct proof of the sign-commutativity using our definition. Consider the diagram

$$\begin{array}{ccccccccc} H_*(\Lambda)\otimes H_*(\Lambda) & \xrightarrow{\times} & H_*(\Lambda\times\Lambda) & \xrightarrow{t^*\tau_{CS}\cap} & H_*(U_{GH}) & \xrightarrow{R_{CS}} & H_*(\Lambda\times_M\Lambda) & \xrightarrow{concat} & H_*(\Lambda) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle id} \\ H_*(\Lambda)\otimes H_*(\Lambda) & \xrightarrow{\times} & H_*(\Lambda\times\Lambda) & \xrightarrow{\tau_{CS}\cap} & H_*(U_{GH}) & \xrightarrow{R_{CS}} & H_*(\Lambda\times_M\Lambda) & \xrightarrow{concat} & H_*(\Lambda) \end{array}$$

where t denotes each time the twist map that takes $[A]\otimes[B]$ to $(-1)^{|A||B|}[B]\otimes[A]$ in the case of the first map, and takes a pair (γ,δ) to (δ,γ) in the case of the second and third maps. The first square commutes up to a sign $(-1)^{pq}$ if we start with homology classes in $H_p(\Lambda)\otimes H_q(\Lambda)$. The second square commutes by the naturality of the cap product (Appendix A). The third square commutes up to homotopy on the space level by a homotopy that slides the basepoint along the geodesic stick added by the retraction. Finally the last square commutes up to the homotopy, using that the concatenation map is homotopic to the map that concatenates at time $s=\frac{1}{2}$, and that the identity map on Λ is homotopic to the map that reparametrizes the loops by precomposing with half a rotation of S^1. Now $t^*\tau_{CS}=(-1)^n\tau_{CS}$ because the corresponding fact holds for τ_M. Hence the top row is the $(1)^n$ times the product $\wedge_{Th}$, while the bottom row is $\wedge_{Th}$. Starting with $[A]\otimes[B]\in H_p(\Lambda)\otimes H_q(\Lambda)$, the diagram thus gives that $(-1)^n[A]\wedge_{Th}[B]=(-1)^{pq}[B]\wedge_{Th}[A]$. Hence

$$[A]\wedge[B]=(-1)^{n+np}[A]\wedge_{Th}[B]=(-1)^{np+pq}[B]\wedge_{Th}[A]=(-1)^{np+pq+n+nq}[B]\wedge[A],$$

which gives the result.

The associativity and unitality can likewise be proved by lifting the associativity and unitality of the algebraic intersection product to the loop space. ☐

2.3. **Tubular neighborhood for the coproduct and equivalence to Goresky-Hingston's definition of the coproduct.** We start by constructing a tubular neighborhood of Λ inside the path space PM.

Let

$$PM=\{\gamma:[0,1]\to M\mid \gamma\text{ is }H^1\}$$

be the space of H^1–paths in M, with evaluation maps e_0, $e_1: PM\to M$ at 0 and 1 so that

$$\Lambda=(e_0,e_1)^{-1}\Delta$$

for $(e_0,e_1):PM\to M^2$. Consider the vector bundle $e_1^*(TM)\to\Lambda$.

Lemma 2.6. *The embedding* $\Lambda\subset PM$ *admits a tubular neighborhood in the sense that there is an embedding*

$$\begin{array}{ccc} \nu_P:\ e_1^*(TM) & \hookrightarrow & PM \\ \downarrow & & \\ \Lambda & & \end{array}$$

restricting to the inclusion on the zero-section and with image $U_P=\{\gamma\in PM\mid \gamma(1)-\gamma(0)|<\varepsilon\}$*. Moreover, this tubular neighborhood is compatible with that of the diagonal in* M^2 *in the sense that the following diagram commutes:*

$$\begin{array}{ccc} e_1^*(TM) & \xhookrightarrow{\nu_P} & PM \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle (e_0,e_1)} \\ TM & \xhookrightarrow{\nu_M} & M\times M \end{array}$$

where ν_M is the map (1.1).

This lemma is parallel to Proposition 2.2, and so is its proof. In the current case though we cannot move the whole loop; we need to keep the starting point fixed while moving the endpoint.

Proof. Given a loop $\gamma \in \Lambda$ and a vector $V \in T_{\gamma_0}M$, we will define a path $\tau \in PM$. As before, we let $\gamma_0 = \gamma(0) = \gamma(1)$ denote the basepoint of γ, and we write $\tau_0 = \tau(0)$ and $\tau_1 = \tau(1)$ for the endpoints of a path τ.

The map ν_P is defined by $\nu_P(\gamma, V) = \tau(\gamma, V)$ where

$$\tau(\gamma, V)(t) = h(\gamma_0, \exp_{\gamma_0}(tV))(\gamma(t))$$

where h is the map of Lemma 2.1. Note that, from the properties of h we have $\nu_P(\gamma, V)(0) = \gamma_0$ and $\nu_P(\gamma, V)(1) = exp_{\gamma_0}(V)$ hence it is compatible with ν_M. Also it restricts to the identity on the zero section, i.e. when $V = 0$. The image of ν_P lies inside $U_P = \{\tau \in PM \mid |\tau_1 - \tau_0| < \varepsilon\}$, an open neighborhood of Λ inside P. We will show that ν_P is a homeomorphism onto U_P by defining an inverse.

Define $\kappa_P : U_P \to e_1^*(TM)$ by $\kappa_P(\tau) = \big(k_P(\tau), V_P(\tau)\big)$ where

$$k_p(\tau)(t) =: \gamma(\tau)(t) = (h(\tau_0, \overline{\tau_0\tau_1}(t)))^{-1}(\tau(t)) \tag{2.2}$$

$$V_P(\tau) = \exp_{\tau_0}^{-1}(\tau_1). \tag{2.3}$$

We have $\gamma(\tau)(1) = h(\tau_0, \tau_1)^{-1}\tau_1 = \tau(0) = \gamma(\tau)(0)$, and hence this does indeed define a loop. The maps ν_P and κ_P are continuous. Note that

$$\overline{\tau_0\tau_1}(t) = \exp_{\tau_0} tV_P(\tau).$$

It follows that ν_P and κ_P inverses of each other which finishes the proof. $\square$

Remark 2.7. Note that the tubular embedding ν_P takes constant paths to geodesics: if γ is constant, then $\nu_P(\gamma, V)$ is a geodesic.

Just as in our definitions of the loop product and coproduct, we can give a "sticky" version of the above retraction map: Let $U_P \subset PM$ be the ε-neighborhood of Λ inside PM as above and define

$$R_P : U_P \longrightarrow \Lambda$$

by

$$R_P(\tau) = \tau \star \overline{\tau_1, \tau_0}$$

That is, we add a "stick" $\overline{\tau_1, \tau_0}$ at the end of τ to get back to the starting point τ_0. Note that, by definition, $R_P(\tau) = \tau$ if $\tau \in \Lambda$.

Recall from (2.2) the map $k_P \colon U_P \longrightarrow \Lambda$. Note that k_P and R_P both keep the startpoint of the paths constant, i.e. they define maps over the evaluation at 0:

$$\begin{array}{ccc} PM & \xrightarrow{\;k_P, R_P\;} & \Lambda. \\ & {\scriptstyle e_0}\searrow \quad \swarrow{\scriptstyle e_0} & \\ & M & \end{array}$$

Lemma 2.8. *The maps $k_P, R_P \colon U_P \longrightarrow \Lambda$ are homotopic, through a homotopy leaving $\Lambda \subset U_P$ fixed at all time, and such that the homotopy is over the identity on M when evaluating at 0, i.e. the basepoints of the paths stay unchanged throughout the homotopy.*

Proof. Define a homotopy $H : U_P \times I \to \Lambda$ by

$$H(\tau, s) = k_P^s(\tau) \star \overline{\tau_0\tau_1}\,[0, s]^{-1} \tag{2.4}$$

where

$$k_P^s(\tau)(t) = h\big(\overline{\tau_0\tau_1}(st), \overline{\tau_0\tau_1}(t)\big)^{-1}(\tau(t)).$$

The right side of (2.4) is the $\star$-concatenation of two paths; the first, $k_P^s(\tau)$, begins ($t = 0$) at

$$h\big(\overline{\tau_0\tau_1}(0), \overline{\tau_0\tau_1}(0)\big)^{-1}(\tau_0) = h(\tau_0, \tau_0)^{-1}(\tau_0) = \tau_0$$

and ends ($t = 1$) at

$$h\big(\overline{\tau_0\tau_1}(s), \overline{\tau_0\tau_1}(1)\big)^{-1}(\tau_1) = h\big(\overline{\tau_0\tau_1}(s), \tau_1)\big)^{-1}(\tau_1) = \overline{\tau_0\tau_1}(s).$$

The second is a "stick" beginning at $\overline{\tau_0\tau_1}(s)$ and ending at τ_0. Thus $H(\tau, s) \in \Lambda$ for all $\tau \in U_P$ and $s \in [0, 1]$. When $s = 0$ the first path is

$$k_P^0(\tau)(t) = (h(\tau_0, \overline{\tau_0\tau_1}(t)))^{-1}\tau(t) = k_P(\tau)(t)$$

and the second is the constant path at τ_0. Thus

$$H(\tau, 0) = k_P(\tau)$$

We leave it to the reader to check that

$$H(\tau, 1) = R_P(\tau)$$

and that $H(-, s)$ fixes Λ for all s. Moreover, at all time s we have that $H(\tau, s)(0) = \tau_0$. □

In the case of the coproduct, the figure-eight space is replaced by the space

$$F = F_{[0,1]} := \{(\gamma, s) \in \Lambda \times I \mid \gamma(s) = \gamma(0)\} \subset \Lambda \times I.$$

This space can be defined as a pull-back in the following diagram:

$$\begin{array}{ccccc} & & F & \hookrightarrow & \Lambda \times I \\ & & \downarrow e & & \downarrow e_I \\ TM & \longrightarrow & M & \overset{\Delta}{\hookrightarrow} & M \times M \end{array}$$
$$TM \overset{\nu_M}{\dashrightarrow} M \times M$$

where e_I as before denotes the map taking (γ, s) to $(\gamma(0), \gamma(s))$. Again we can pull-back the ε–tangent bundle of M along the evaluation at 0 to get a bundle $e^*(TM) \to F$.

The original construction of the coproduct in [18] used that there exists a tubular neighborhood of $F_{(0,1)} := F \cap \Lambda \times (0, 1)$ inside $\Lambda \times (0, 1)$. It is argued in [18] that such a tubular neighborhood exists, but, just as for the Chas-Sullivan product, we need an explicit construction to be able to compare the resulting coproduct to the one defined in Section 1.5.

Consider the bundle $e^*(TM) \to F_{(0,1)}$.

Proposition 2.9. *The embedding $F_{(0,1)} \subset \Lambda \times (0, 1)$ admits a tubular neighborhood in the sense that there is an embedding*

$$\begin{array}{ccc} \nu_{GH}: e^*(TM) & \hookrightarrow & \Lambda \times (0,1) \\ \downarrow & & \\ F_{(0,1)} & & \end{array}$$

that restricts to the inclusion on the zero-section $F_{(0,1)}$ and is a homeomorphism onto its image $U_{GH} = \{(\gamma, s) \in \Lambda \times (0, 1) \mid |\gamma(0) - \gamma(s)| < \varepsilon\}$. Moreover this tubular neighborhood restricts to a tubular embedding over each $s \in (0, 1)$ and is compatible with that of the diagonal in M^2 in the sense that the following diagram commutes:

$$\begin{array}{ccc} e^*(TM) & \overset{\nu_{GH}}{\hookrightarrow} & \Lambda \times (0,1) \\ \downarrow e & & \downarrow e_I \\ TM & \overset{\nu_M}{\hookrightarrow} & M \times M \end{array}$$

where ν_M is the map (1.1) and $e_I : \Lambda \times (0, 1) \to M$ takes (γ, s) to $(\gamma(0), \gamma(s))$.

Proof. Let $(\gamma, s, V) \in (e^*(TM) \to F_{(0,1)})$, so γ satisfies $\gamma(s) = \gamma(0)$ and $V \in T_{\gamma(0)}M = T_{\gamma(s)}M$ is a vector with $|V| < \varepsilon$ and $0 < s < 1$.

Recall from Lemma 2.6 the map $\nu_P : (e_1^*(TM) \to \Lambda) \longrightarrow PM$. To define ν_{GH}, we apply ν_P to $\gamma[0, s]$, the restriction of γ to the interval $[0, s]$ (rescaled to be parametrized again by $[0, 1]$), and to $\gamma[s, 1]^{-1}$, the path $\gamma[s, 1]$ taken in the reverse direction: Define $\nu_{GH}(\gamma, s, V) = (\tau, s)$ with

$$\tau = \big(\nu_P(\gamma[0, s], V)\big) *_s \big(\nu_P(\gamma[s, 1]^{-1}, V)\big)^{-1}$$

and where $*_s$ indicates that the concatenation takes place at time s.

Note first that this is a loop, as $\nu_P(\gamma[0, s], V)$ is a path starting at $\gamma(0) = \gamma(1)$ and ending at $\exp_{\gamma(s)} V$, just like $\nu_P(\gamma[s, 1]^{-1}, V)$. As the latter is reversed one more time, the two paths can be glued together. The glued path τ has $\tau(0) = \gamma(0)$ and $\tau(s) = \exp_{\gamma(s)} V$. In particular, $|\tau(s) - \tau(0)| < \varepsilon$.

Let $U_{GH}^{(0,1)} = \{(\tau, s) \in \Lambda \times (0, 1) \mid |\tau(s) - \tau(0)| < \varepsilon\}$. We define an inverse map

$$\kappa_{GH} : U_{GH}^{(0,1)} \longrightarrow \big(e^*(TM) \to F_{(0,1)}\big)$$

taking (τ, s) to (γ, s, V) with $V = \exp^{-1}_{\tau(0)}(\tau(s))$, and

$$\gamma =: k_{GH}(\tau, s) = \big(k_P(\tau[0, s])\big) *_s \big(k_P(\tau[s, 1]^{-1})\big)^{-1}$$

where k_P is the map of Lemma 2.6. We check that γ is well-defined: $\tau[0, s]$ is a path with endpoints at distance at most ε, so it is in the source of the generalized map k_P. Likewise for $\tau[s, 1]^{-1}$. Moreover, the image of those paths under k_P are loops based at $\gamma(0) = \gamma(1)$. Hence it makes sense to concatenate them, yielding a loop based at that point, now parametrized by $[0, 1]$ and with a self-intersection at time s.

The two maps ν_{GH} and κ_{GH} are continuous and are inverse for each other because ν_P and κ_P are inverse of each other. □

Remark 2.10. The constructions on $F_{(0,1)}$ could also be done on

$$F_{\frac{1}{2}} = \{\gamma \in \Lambda \mid \gamma(\tfrac{1}{2}) = \gamma(0)\}$$

and then expanded to the remainder of $F_{(0,1)}$ using the natural map $F_s \overset{\approx}{\to} F_{\frac{1}{2}}$ induced by precomposing with the piecewise linear bijection $\theta_{\frac{1}{2}\to s} \colon [0, 1] \to [0, 1]$ that takes $0 \to 0$, $\frac{1}{2} \to s$, $1 \to 1$. These maps appear in the definition of the cohomology loop product in [18, Sec 9.1]. Note that when $s = 0$ we have a map but not a bijection $F_0 \to F_{\frac{1}{2}}$; the map ν_{GH} also does not extend to $s = 0, 1$.

The map

$$k_{GH} : U_{GH}^{(0,1)} \longrightarrow F_{(0,1)}$$

occurring in the definition of the collapse map in the proof above is a retraction that preserves $s \in (0, 1)$. We will now show that it is homotopic to our stick map R_{GH} from Section 1.5 restricted to $U_{GH}^{(0,1)} = U_{GH} \cap \Lambda \times (0, 1)$.

Lemma 2.11. *The maps $k_{GH}, R_{GH} : U_{GH}^{(0,1)} \longrightarrow F_{(0,1)}$ are homotopic.*

Proof. This follows from applying Lemma 2.8 on both sides of the concatenation at s. □

Proposition 2.12. *The tubular neighborhood ν_{GH} allows us to define an explicit description of the Goresky-Hingston coproduct* [18, 8.4]*, and the coproduct $\vee$ of Definition 1.3 is a chain model for it.*

Proof. The definition of the Goresky-Hingston coproduct is made most precise in the proof of Lemma 8.3 of the paper [18], and we recall it here. The coproduct is defined via a sequence of maps

$$\begin{aligned} H_*(\Lambda, M) &\longrightarrow H_{*+1}(\Lambda \times I, M \times I \cup \Lambda \times \{0, 1\}) \\ &\longrightarrow H_{*+1-n}(F_{[0,1]}, F_{[0,1]} \cap (M \times I \cup \Lambda \times \{0, 1\})) \\ &\longrightarrow H_{*+n-1}(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M) \end{aligned}$$

where the first map crosses with I and the last map is the cut-map, just as in our definition of the coproduct $\vee$ (Definition 1.3). The middle map needs to be constructed, and the paper [18] produces it as follows.

For a small $\alpha > 0$, let

$$V_\alpha = \Lambda^{<\alpha} \times I \ \cup \ \Lambda \times \big([0,\alpha) \cup (1-\alpha, 1]\big),$$

where $\Lambda^{<\alpha}$ is the space of loops of energy smaller than α^2. Let $V_\alpha^{(0,1)} = V_\alpha \cap \big(\Lambda \times (0,1)\big)$. A tubular neighborhood of $F_{(0,1)}$ inside $\Lambda \times (0,1)$ together with excision allows to define maps

$$\begin{array}{ccc} H_*(\Lambda \times (0,1), V_\alpha^{(0,1)}) & \longrightarrow & H_{*-n}(F_{(0,1)}, F_{(0,1)} \cap V_\alpha^{(0,1)}) \\ \cong \downarrow & & \downarrow \cong \\ H_*(\Lambda \times I, V_\alpha) & & H_{*-n}(F_{[0,1]}, F_{[0,1]} \cap V_\alpha) \end{array}$$

where the horizontal map is defined by capping with the Thom class and retracting (or equivalently applying the Thom collapse map followed by the Thom isomorphism), as we have done above. Then it is argued in [18] that taking a limit for $\alpha \to 0$ gives a well-defined map

$$H_*(\Lambda \times I, \Lambda \times \partial I \cup M \times I) \longrightarrow H_{*-n}(F_{[0,1]}, \Lambda \times \partial I \cup M \times I),$$

which finishes the definition of the coproduct.

We have here given an explicit tubular neighborhood of $F_{(0,1)}$ inside $\Lambda \times (0,1)$ and Lemma 2.11 shows that the retraction map k_{GH} associated to this tubular neighborhood is homotopic to the map R_{GH} used in our definition of the coproduct. We thus have chain homotopic maps

$$\begin{array}{ccc} C_*(\Lambda \times (0,1), V_\alpha^0) & \xrightarrow{\tau_{GH}\cap} & C_{*-n}(U_{GH} \cap \Lambda \times (0,1), U_{GH} \cap V_\alpha^0) \\ & & k_{GH} \downarrow \quad \downarrow R_{GH} \\ & & C_{*-n}(F_{(0,1)}, F_{(0,1)} \cap V_\alpha^0). \end{array}$$

Now the definition of R_{GH} does not need a tubular neighborhood and is well-defined also when $\alpha = 0$. This provides a different proof that the limit as α tends to 0 exists, already on the chain level using the map R_{GH}. At the same time it shows that our definition of the coproduct agrees in homology with that of [18]. □

The cohomology product is defined in [18] using a slightly different sequence of maps than the dual of the above maps, to avoid complications with taking limits. As we do not need to take limits with our new definition, it is now easier to show that this other construction is indeed the dual of the coproduct $\vee$.

Theorem 2.13. *The algebraic cohomology product*

$$\circledast \colon H^*(\Lambda, M) \otimes H^*(\Lambda, M) \to H^*(\Lambda, M)$$

induced in homology by the map $\circledast$ of Definition 1.6 agrees with the cohomology product defined in [18, Sec 9]. *In particular,*

(a) the cohomology product $\circledast$ is associative and satisfies the following graded commutativity relation

$$x \circledast y = (-1)^{(p+n-1)(q+n-1)} y \circledast x$$

for any $x \in H^p(\Lambda, M)$, $y \in H^q(\Lambda, M)$;

(b) it is dual to the homology coproduct in the sense that

$$\langle x \circledast y, Z\rangle = (-1)^{q(n-1)} \langle x \times y, \vee Z\rangle$$

for any $x \in H^p(\Lambda, M)$, $y \in H^q(\Lambda, M)$, and $Z \in H_{p+q+n-1}(\Lambda, M)$.

Proof. Recall from Definition 1.6 that

$$x \circledast y = (-1)^{q(n-1)} x \circledast_{Th} y = (-1)^{q(n-1)} \vee^* (x \times y).$$

In [18], the cohomology product is instead defined as $(-1)^{q(n-1)}\widetilde{\vee}^*(x \times y)$, for a map $\widetilde{\vee}$ which we recall below. We will show that $\vee$ and $\widetilde{\vee}$ induce the same map on cohomology.

Recall from Remark 2.10 the reparametrization map $\theta_{\frac{1}{2}\to s}$, which is defined for all $s \in [0,1]$. This can be used to define a map $J : \Lambda \times I \to \Lambda$ by $J(\gamma, s) = \gamma \circ \theta_{\frac{1}{2}\to s}$. Note that J restricts to a map $J\colon F_{[0,1]} \to F_{\frac{1}{2}}$ and also commutes with the evaluation at 0 and defines a map of pairs:

$$\begin{array}{ccc} (\Lambda \times I, \Lambda \times \partial I \cup M \times I) & \xrightarrow{\quad J \quad} & (\Lambda, F^{\bullet,0} \cup F^{0,\bullet}) \\ & {\scriptstyle e}\searrow \quad M \quad \swarrow{\scriptstyle e} & \end{array}$$

where $F^{\bullet,0} \cup F^{0,\bullet}$ is the subspace of Λ of loops that are constant on their first or second half. Now consider the diagram

$$\begin{array}{ccc}
H^*(\Lambda, M) & = & H^*(\Lambda, M) \\
\uparrow {\scriptstyle (\times I)^*} & & \uparrow {\scriptstyle (\times I)^*} \\
H^{*+1}(\Lambda \times I, \Lambda \times \partial I \cup M \times I) & = & H^{*+1}(\Lambda \times I, \Lambda \times \partial I \cup M \times I) \\
\uparrow {\scriptstyle J^*} & & \\
H^{*+1}(\Lambda, F^{\bullet,0} \cup F^{0,\bullet}) & & \uparrow {\scriptstyle \tau_{GH}\cup} \\
\uparrow {\scriptstyle e^*\tau_M \cup} & & \\
H^{*+1-n}(U_{\frac{1}{2}}, F^{\bullet,0} \cup F^{0,\bullet}) & \xrightarrow[J^*]{} & H^{*+1-n}(U_{GH}, F_0 \cup F_1 \cup M \times I) \\
\uparrow {\scriptstyle k^*_{GH}} & & \uparrow {\scriptstyle R^*_{GH}} \\
H^{*+1-n}(F_{\frac{1}{2}}, F^{\bullet,0} \cup F^{0,\bullet}) & \xrightarrow[J^*]{} & H^{*+1-n}(F_{[0,1]}, F_0 \cup F_1 \cup M \times I) \\
\uparrow {\scriptstyle cut^*} & & \uparrow {\scriptstyle cut^*} \\
H^{*+1-n}(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M) & = & H^{*+1-n}(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M)
\end{array}$$

where $U_{\frac{1}{2}} := U_{GH} \cap \Lambda \times \{\frac{1}{2}\}$, and k_{GH} is the restriction to $U_{\frac{1}{2}}$ of the retraction map of the same name defined above. The right vertical composition is our definition of $\vee^*$, while the left composition is an explicit description of the map $\widetilde{\vee}^*$ defining the cohomology product in [18] (see Figure 3 in Section 9 of that paper, where the sign $(-1)^{q(n-1)}$ is encoded in the letter ω). So we are left to show that this diagram commutes. The first square trivially commutes. The second square commutes by naturality of the cap product (see Appendix A) using that $\tau_{GH} = J^*e^*\tau_M$ by the commutativity of the triangle above. The third square commutes up to homotopy on the space level by Lemma 2.11. Finally the last square commutes on the space level.

This shows that our definition is equivalent to that of [18]. Now statement (a) follows from [18, Prop 9.2] and statement (b) from our definition 1.6. □

3. The extended cohomology product and lifted homology coproduct

In this section we will define a lift of the coproduct $\vee$ to the non-relative chains $C_*(\Lambda)$, and an extension of the product $\circledast$ to the non-relative cochains $C^*(\Lambda)$. We start by the extension of the product.

As above, we let $e\colon \Lambda \longrightarrow M$ denote the evaluation map at 0, and we write $i : M \to \Lambda$ for the inclusion as constant loops. We also write 1 for the map induced by the identity on chains, cochains, homology or cohomology.

The maps

$$\Lambda \underset{e}{\overset{i}{\rightleftarrows}} M$$

satisfy that $e \circ i$ is the identity map on M. Hence they induce a decomposition

$$C^*(\Lambda) \cong C^*(\Lambda, M) \oplus C^*(M) \quad \text{and} \quad H^*(\Lambda) \cong H^*(\Lambda, M) \oplus H^*(M)$$

(see eg. [20, p147]). Likewise we get

$$C^*(\Lambda) \otimes C^*(\Lambda) = \big(C^*(\Lambda, M) \otimes C^*(\Lambda, M)\big) \oplus \big(C^*(M) \otimes C^*(\Lambda) + C^*(\Lambda) \otimes C^*(M)\big)$$

and the same for cohomology. This allows to define an "*extension by* 0" $\widehat{\circledast}$ of the product $\circledast$ by setting

- $\widehat{\circledast} = \circledast$ on $C^*(\Lambda, M) \otimes C^*(\Lambda, M)$;
- $\widehat{\circledast} = 0$ on $C^*(M) \otimes C^*(\Lambda)$ and $C^*(\Lambda) \otimes C^*(M)$.

The extension will in addition have the property that its image lies in $C^*(\Lambda, M) \subset C^*(\Lambda)$. Note also that the splitting map $C^*(\Lambda) \to C^*(\Lambda, M)$ is simply the map

$$(1 - e^*i^*) \colon C^*(\Lambda) \longrightarrow C^*(\Lambda, M). \tag{3.1}$$

This is the standard formula for the splitting, and one can check that for any cochain $x \in C^*(\Lambda)$, the cochain $(1 - e^*i^*)x$ vanishes on any chain $A \in i_*C_*(M)$, so the map $(1 - e^*i^*)$ has image in $C^*(\Lambda, M)$. Moreover, the map takes $e^*C^*(M)$ to 0 and $C^*(\Lambda, M)$ to itself via the identity.

Theorem 3.1. *The maps $e : \Lambda \leftrightarrows M : i$ induce a natural extension*

$$\widehat{\circledast} : C^p(\Lambda) \otimes C^q(\Lambda) \longrightarrow C^{p+q+n-1}(\Lambda)$$

of the product $\circledast$ of Definition 1.5. It is given by the following formula: If $x, y \in C^*(\Lambda)$, and $i : M \to \Lambda$ is the inclusion as before, then

$$\begin{aligned} x \widehat{\circledast} y &= (1 - e^*i^*)x \circledast (1 - e^*i^*)y \\ &= (-1)^{q(n-1)}(1 - e^*i^*)x \circledast_{Th} (1 - e^*i^*)y \end{aligned}$$

considering $(1-e^*i^*)$ as a map from $C^*(\Lambda)$ to $C^*(\Lambda, M) \subset C^*(\Lambda)$, and has the following properties:

(a) The product $\widehat{\circledast}$ is the unique "extension by 0" on the trivial loops, that is satisfying that

$$\langle x \widehat{\circledast} y, Z\rangle = 0 \ \text{if } x \in e^*C^*(M), y \in e^*C^*(M), \ \text{or } Z \in i_*C_*(M).$$

(b) The associated cohomology product is associative and satisfies the following graded commutativity relation:

$$[x]\widehat{\circledast}[y] = (-1)^{(p+n-1)(q+n-1)}[y]\widehat{\circledast}[x]$$

for any $[x] \in H^p(\Lambda)$ and $[y] \in H^q(\Lambda)$.

(c) Morse theoretic inequality: Let $[x], [y] \in H^*(\Lambda)$, then

$$Cr([x]\widehat{\circledast}[y]) \geq Cr([x]) + Cr([y])$$

where $Cr([x]) = \sup\{l \in \mathbb{R} \mid [x]$ is supported on $\Lambda^{\geq \ell}\}$ for $\Lambda^{\geq \ell}$ the subspace of loops of energy at least ℓ^2.

Recall from Section 1.1 that the loops of energy at most ℓ^2 also have length at most ℓ as $\mathcal{L}(\gamma)^2 \leq E(\gamma)$, for $\mathcal{L}$ the length function and E the energy.

Proof. We define the lifted product $\widehat{\circledast}$ by the following commutative diagram:

$$\begin{array}{ccc} C^j(\Lambda) \otimes C^k(\Lambda) & \xrightarrow{\ \widehat{\circledast}\ } & C^{j+k+n-1}(\Lambda) \\ \Big\downarrow{\scriptstyle p^* \otimes p^*} & & \Big\uparrow{\scriptstyle q^*} \\ C^j(\Lambda, M) \otimes C^k(\Lambda, M) & \xrightarrow{\ \circledast\ } & C^{j+k+n-1}(\Lambda, M) \end{array}$$

where $p_* = (1 - i^*e^*)$ is the splitting map (3.1) and q^* is the canonical inclusion. We see that this gives the formula in the statement, where we have though suppressed the inclusion q^*. As the maps e and i are natural, this extension is natural in M.

We check (a): We have that $\langle x\widehat{\circledast}y, Z\rangle = \langle p^*x \circledast p^*y, q_*Z\rangle$ for $q_*\colon C_*(\Lambda) \to C_*(\Lambda, M)$ the dual of the map q^*. Now the latter bracket is 0 if $x \in e^*C^*(M)$ as $p^*(x) = 0$ in that case, if $y \in e^*C^*(M)$ as $p^*y = 0$ in that case, or if $Z \in i_*C_*(M)$ as $q_*(Z) = 0$ in that case. And because $C_*(\Lambda) \cong C_*(\Lambda, M) \oplus C_*(M)$ and $C_*(\Lambda\times\Lambda) \cong \big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big) \oplus C_*(\Lambda\times\Lambda)/\big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big)$ (see Lemma 3.2 below), this determines the extension.

The last properties will be deduced from the corresponding results for $\circledast$ in [18], which is justified by our Theorem 2.13. Graded commutativity follows directly from the graded commutativity of $\circledast$, as proved in [18, Prop 9.2]. For associativity, if $[x]$, $[y]$, $[z] \in H^*(\Lambda)$,

$$\begin{aligned}
([x]\widehat{\circledast}[y])\widehat{\circledast}[z] &= (1-e^*i^*)\Big((1-e^*i^*)[x] \circledast (1-e^*i^*)[y]\Big) \circledast (1-e^*i^*)[z] \\
&= \Big((1-e^*i^*)[x] \circledast (1-e^*i^*)[y]\Big) \circledast (1-e^*i^*)[z] \\
&= (1-e^*i^*)[x] \circledast \Big((1-e^*i^*)[y] \circledast (1-e^*i^*)[z]\Big) \\
&= (1-e^*i^*)[x] \circledast (1-e^*i^*)\Big((1-e^*i^*)[y] \circledast (1-e^*i^*)[z]\Big) \\
&= [x]\widehat{\circledast}([y]\widehat{\circledast}[z])
\end{aligned}$$

where the first and last equality hold by definition of $\widehat{\circledast}$, the second and fourth using that any product $[a] \circledast [b] \in H^*(\Lambda, M) \cong \ker i^*$ so $(1 - e^*i^*) = 1$ on such products. Finally the middle equality is the associativity of $\circledast$ [18, Prop 9.2]. This proves statement (b).

Finally, given $[x] \in H^*(\Lambda)$, write $[x] = [x_0] + [x_1]$ with $x_0 \in e^*C^*(M)$ and $x_1 \in q^*C^*(\Lambda, M)$. Then $Cr([x]) = Cr([x_1])$ as $[x_0]$ is represented by loops of length 0. Now $(1 - i^*e^*)[x] = [x_1]$, which gives that $Cr\big((1 - i^*e^*)[x]\big) = Cr([x_1]) = Cr([x])$. So

$$\begin{aligned}
Cr([x]\widehat{\circledast}[y]) &= Cr((1-i^*e^*)[x] \circledast (1-i^*e^*)[y]) \\
&\geq Cr\big((1-i^*e^*)[x]\big) + Cr\big((1-i^*e^*)[y]\big) = Cr([x]) + Cr([y]),
\end{aligned}$$

where the middle inequality is given by [18, (1.7.3)]. Statement (c) follows, which finishes the proof of the theorem. □

The evaluation at 0 also induces the following splitting of the chain complex $C_*(\Lambda \times \Lambda)$.

Lemma 3.2. *There is a splitting of chain complexes*

$$C_*(\Lambda\times\Lambda) \cong \big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big) \oplus C_*(\Lambda\times\Lambda)/\big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big)$$

with splitting map

$$(1-i_*e_*)\times(1-i_*e_*)\colon C_*(\Lambda\times\Lambda)/\big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big) \longrightarrow C_*(\Lambda\times\Lambda).$$

Proof. The map $(\mathrm{id}, e)_* + (e, \mathrm{id})_* - (e, e)_*\colon C_*(\Lambda\times\Lambda) \longrightarrow C_*(\Lambda\times M) + C_*(M\times\Lambda)$ defines a splitting of the short exact sequence

$$C_*(\Lambda\times M) + C_*(M\times\Lambda) \longrightarrow C_*(\Lambda\times\Lambda) \longrightarrow C_*(\Lambda\times\Lambda)/\big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big)$$

and the associated splitting $C_*(\Lambda\times\Lambda)/\big(C_*(\Lambda\times M) + C_*(M\times\Lambda)\big) \to C_*(\Lambda\times\Lambda)$ is the map $1 - j((\mathrm{id}, e)_* + (e, \mathrm{id})_* - (e, e)_*)$, for j the inclusion, and this is the stated map. □

Recall from Remark 1.4 that the inclusion $C_*(\Lambda\times M) + C_*(M\times\Lambda) \longrightarrow C_*(\Lambda\times M \cup M\times\Lambda)$ is a quasi-isomorphism; the above splitting in homology becomes the splitting

$$H_*(\Lambda\times\Lambda) \cong H_*(\Lambda\times M \cup M\times\Lambda) \oplus H_*(\Lambda\times\Lambda, \Lambda\times M\cup M\times\Lambda).$$

We are now ready to define the extension of the coproduct $\vee$, which is the dual of the extension $\widehat{\circledast}$ of the product $\circledast$:

Theorem 3.3. *The maps $e : \Lambda \leftrightarrows M : i$ induce a natural extension*

$$\widehat{\vee} : C_*(\Lambda) \longrightarrow C_{*+1-n}(\Lambda \times \Lambda)$$

of the coproduct $\vee$ on relative loop homology. It is given by the following formula: If $Z \in C_*(\Lambda)$, then

$$\widehat{\vee} Z = (1 - i_*e_*) \times (1 - i_*e_*) \vee (q_*Z)$$

with $(1 - i_*e_*) \times (1 - i_*e_*) \colon C_*(\Lambda \times \Lambda)/\big(C_*(\Lambda \times M) + C_*(M \times \Lambda)\big) \longrightarrow C_*(\Lambda \times \Lambda)$ the map of Lemma 3.2, $\vee$ as in Definition 1.3, and $q_* : H_*(\Lambda) \to H_*(\Lambda, M)$ the projection. It has the following properties :

(A) The coproduct $\widehat{\vee}$ is the unique "extension by 0" on the trivial loops, satisfying

$$\langle x, \widehat{\vee} Z\rangle = 0 \ \text{ if } \ x \in (e^* \times 1)C^*(M \times \Lambda) \text{ or } (1 \times e^*)C^*(\Lambda \times M), \text{ or } Z \in i_*C_*(M).$$

(B) Duality: The extended product, the lifted coproduct, and the Kronecker product satisfy

$$\langle x \widehat{\circledast} y, Z\rangle = (-1)^{q(n-1)} \langle x \times y, \widehat{\vee} Z\rangle$$

for any $x \in C^p(\Lambda)$, $y \in C^q(\Lambda)$ and $Z \in C_{p+q+n-1}(\Lambda)$.

Note again that, whenever the Kunneth formula holds for Λ, for example with field coefficients, we can post-compose the coproduct $\widehat{\vee}$ with the AW map and get an actual coproduct in homology

$$H_*(\Lambda) \xrightarrow{\widehat{\vee}} H_*(\Lambda \times \Lambda) \xrightarrow{AW} H_*(\Lambda) \otimes H_*(\Lambda),$$

which is of degree $1 - n$.

Proof. We define the extended coproduct $\widehat{\vee}$ to be the map making the following diagram commutative

$$\begin{array}{ccc} C_*(\Lambda) & \xrightarrow{\quad\widehat{\vee}\quad} & C_*(\Lambda \times \Lambda) \\ {\scriptstyle q_*}\downarrow & & \uparrow{\scriptstyle P_*} \\ C_*(\Lambda, M) & \xrightarrow{\ \vee\ } & C_*(\Lambda \times \Lambda)/\big(C_*(M \times \Lambda) + C_*(\Lambda \times M)\big) \end{array}$$

where the inclusion $P_* = (1 - i_*e_*) \times (1 - i_*e_*)$ comes from the splitting in Lemma 3.2. This gives precisely the formula in the statement. Note that in order to lift the coproduct $\vee$, we need to "complete" $\vee$ by subtracting something coming from $C_*(\Lambda \times M) + C_*(M \times \Lambda)$ that has the same boundary in that complex, so that if $\vee A$ was a relative cycle, $\widehat{\vee} A$ becomes an actual cycle. This is precisely what the map $(1 - i_*e_*) \times (1 - i_*e_*)$ does.

Statement (A) follows from the fact that $q_*(Z) = 0$ if $Z \in i_*H_*(M)$ and the fact that P^* vanishes on the image of $C^*(M \times \Lambda)$ and $C^*(\Lambda \times M)$ as $(1 - e^*i^*) = 0$ on $e^*C^*(M)$. And the unicity follows again from the splittings, as in statement (a) in Theorem 3.1.

Statement (B) follows from the fact that

$$\begin{aligned} \langle x \widehat{\circledast} y, Z\rangle &= (-1)^{q(n-1)} \langle (1 - e^*i^*)x \circledast_{Th} (1 - e^*i^*)y, Z\rangle \\ &= (-1)^{q(n-1)} \langle \vee^*\big((1 - e^*i^*)x \times (1 - e^*i^*)y\big), Z\rangle \\ &= (-1)^{q(n-1)} \langle x \times y, \big((1 - i_*e_*) \times (1 - i_*e_*)\big)\vee Z\rangle \quad = (-1)^{q(n-1)} \langle x \times y, \widehat{\vee} Z\rangle \end{aligned}$$

when $x \in C^p(\Lambda)$, $y \in C^q(\Lambda)$ and $Z \in C_{p+q+n-1}(\Lambda)$. □

3.1. **Iterates of the lifted coproduct.** For our applications in Section 4, we will need an "iterated" coproduct, which we define now. Let

$$\Delta^k = \{(s_1, \dots, s_k) \in [0, 1]^k \mid 0 \le s_1 \le \cdots \le s_k \le 1\}$$

and let

$$F_{\Delta^k} := \{(\gamma, s_1, \dots, s_k) \in \Lambda \times \Delta^k \mid \gamma(0) = \gamma(s_1) = \cdots = \gamma(s_k)\} \subset \Lambda \times \Delta^k.$$

Consider also

$$U^k_{GH} := \{(\gamma, s_1, \dots, s_k) \in \Lambda \times \Delta^k \mid |\gamma(s_i) - \gamma(0)| < \varepsilon \text{ for all } 1 \le i \le k\}$$

the ε–neighborhood of F_{Δ^k} in $\Lambda \times \Delta^k$. As in the case $k=1$, one can define a retraction map

$$R_k \colon U^k_{GH} \longrightarrow F_{\Delta^k}$$

by adding sticks to the loops, leaving the time coordinates constant.

The diagonal embedding $M \longrightarrow M^{k+1}$ has normal bundle isomorphic to TM^k and the class

$$\tau_k := e^*_{I_k}\tau_M \cup \cdots \cup e^*_{I_1}\tau_M \in C^*(U^k_{GH})/\sum_i C_*((U^{k,i}_{GH,\varepsilon,\varepsilon_0})^c) \tag{3.2}$$

with $U^{k,i}_{GH,\varepsilon,\varepsilon_0} = \{(\gamma, s_1, \ldots, s_k) \in U^k_{GH} \mid |\gamma(s_i) - \gamma(0)| < \varepsilon_0\}$, and $e_{I_j} \colon \Lambda \times \Delta^k \to M^2$ defined by $e_{I_j}(\gamma, s_1, \ldots, s_k) = (\gamma(0), \gamma(s_j))$. This class can also be interpreted as the pull-back to $U^k_{GH} \subset \Lambda^k$ of a Thom class for TM^k.

Definition 3.4. The *iterated coproduct* $\widehat{\vee}^k : C_*(\Lambda) \to C_{*-k(n-1)}(\Lambda^k)$ is the composition

$$\widehat{\vee}^k = (1 - i_*e_*)^{\times k} \circ \vee^k \circ p_*$$

where $p_* \colon C_*(\Lambda) \longrightarrow C_*(\Lambda, M)$ is the projection as above and $\vee^k$ is defined as the composition

$$\begin{aligned}
\vee^k : C_*(\Lambda, M) = C_*(\Lambda)/C_*(M) \ &\xrightarrow{\times \Delta^k}\ C_{*+k}(\Lambda \times \Delta^k)/\big(C_*(M \times \Delta^k) + \textstyle\sum_i C_*(\Lambda \times \partial_i \Delta^k)\big)\\
&\xrightarrow{\cap \tau_k}\ C_{*+k-nk}(U^k_{GH})/\big(C_*(M \times \Delta^k) + \textstyle\sum_i C_*(\Lambda \times \partial_i \Delta^k)\big)\\
&\xrightarrow{R_k}\ C_{*+k-nk}(F_{\Delta^k})/\big(C_*(M \times \Delta^k) + \textstyle\sum_i C_*(\Lambda \times \partial_i \Delta^k)\big)\\
&\xrightarrow{cut_k}\ C_{*+k-nk}(\Lambda^k)/\textstyle\sum_i C_*(\Lambda^{i-1} \times M \times \Lambda^{k-i})
\end{aligned}$$

with cut_k is the k-fold cutting map, and where we have subdivided Δ^k so that for any $A \in C_*(\Lambda)$, the chain $A \times \partial\Delta^k$ lies in $\sum_i C_{*-1}(\Lambda \times \partial_i \Delta^k)$.

As it is a composition of chain maps, it descends to a map

$$\widehat{\vee}^k : H_*(\Lambda) \to H_{*-k(n-1)}(\Lambda^k).$$

It is an iterated coproduct in the following sense.

Proposition 3.5. *The map $\widehat{\vee}^k$ is dual to the iterated lifted cohomology product $\widehat{\circledast}^k$ in the sense that*

$$\big\langle x_1 \widehat{\circledast}(x_2 \widehat{\circledast}(\ldots(x_{k-1} \widehat{\circledast} x_k)\ldots), Z\big\rangle = (-1)^\sigma \big\langle x_1 \times \ldots \times x_k, \widehat{\vee}^k Z\big\rangle$$

for any $x_i \in C^*(\Lambda)$ and $Z \in C_*(\Lambda)$, where σ is a computable function of n and the degrees of the x_i's. Moreover,

$$\widehat{\vee}^k = (1 \times \ldots \times 1 \times \widehat{\vee}) \circ \cdots \circ (1 \times \widehat{\vee}) \circ \widehat{\vee}.$$

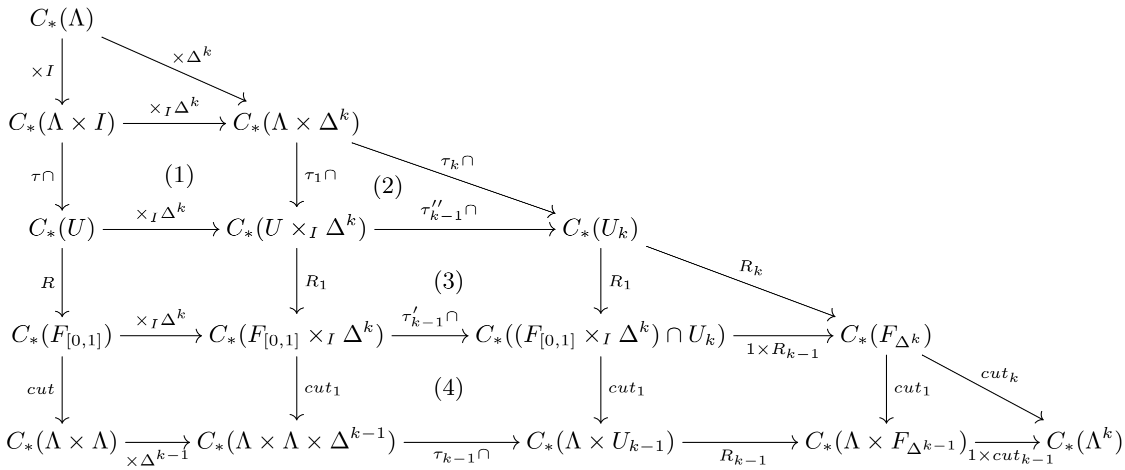
Note that if the Kunneth formula holds for Λ, e.g. with field coefficients, then in homology

$$AW^k \circ \widehat{\vee}^k = (1 \otimes \ldots \otimes 1 \otimes (AW \circ \widehat{\vee})) \circ \cdots \circ (AW \circ \widehat{\vee}) \colon H_*(\Lambda; k) \to H_*(\Lambda; k)^{\otimes k}$$

is the actual iterated composition of the coproduct, each time applied in the last position.

Remark 3.6. The coproduct $AW \circ \widehat{\vee}$ is only coassociative up to sign (as it is defined with the Thom sign, which is the "non-algebraic sign"). This explains why we are careful in the statement about which iteration of the coproduct $\widehat{\vee}$ the map $\widehat{\vee}^k$ corresponds to. The product $\widehat{\circledast}$ on the other hand has been defined with the algebraic sign, and it is associative in homology, but we still have to be a little careful on chains.

Proof of Proposition 3.5. Consider the diagram



where we have suppressed which relative chains one needs to work with and the degree shifts for readability. The diagonal composition is the map $\vee^k$, while going along the two other sides of the diagram gives the composition $(1 \times \vee^{k-1}) \circ \vee$. The spaces in the middle of the diagram are defined as follows:

$$U \times_I \Delta^k = \{(\gamma, s_1, \dots, s_k) \mid |\gamma(s_1) - \gamma_0| < \varepsilon\}$$

$$F_{[0,1]} \times_I \Delta^k = \{(\gamma, s_1, \dots, s_k) \mid \gamma(s_1) = \gamma_0\}$$

and the classes τ_j are each time the cup product of the pull-backs along the appropriate evaluation maps of the class τ_M, adapting appropriately the formula (3.2). We claim that the diagram commutes. The less obvious commuting subdiagrams are those labeled (1) through (4). Diagram (1) commutes by naturality of the cap product because both Thom classes are the pull-back of τ_M along the compatible evaluation at 0 in the diagram. Diagram (2) commutes by the compatibility between the cup and cap product because $\tau_k = \tau_{k-1} \cup \tau_1$ by definition. Diagram (3) commutes by naturality of the cap product because the retraction R_1 commutes with the relevant evaluation maps. Finally diagram (4) commutes for the same reason, with the cutting map replacing the retraction.

By induction, we thus get that

$$\vee^k = (1 \times \ldots \times 1 \times \vee) \circ \cdots \circ (1 \times \vee) \circ \vee \colon C_*(\Lambda, M) \longrightarrow C_{*-k(n-1)}(\Lambda^k) / \sum_i C_*(\Lambda^{i-1} \times M \times \Lambda^{k-i}),$$

and the corresponding equation with $\vee$ replaced by $\widehat{\vee}$ using that

$$(1 - i_* e_*)^{\times k} \colon C_*(\Lambda^k) / \sum_i C_*(\Lambda^{i-1} \times M \times \Lambda^{k-i}) \to C_*(\Lambda)$$

is a right inverse to the natural projection, which gives the second part of the statement.

To prove the first part of the statement, we have that

$$\begin{aligned}
&\big\langle x_1 \widehat{\circledast} (x_2 \widehat{\circledast} (\ldots \widehat{\circledast} x_k) \ldots), Z \big\rangle \\
&\quad = \pm \big\langle (1 - e^* i^*) x_1 \circledast_{Th} (1 - e^* i^*)((1 - e^* i^*) x_2 \circledast_{Th} \cdots \circledast_{Th} (1 - e^* i^*) x_k) \ldots), Z \big\rangle \\
&\quad = \pm \big\langle x_1 \times ((1 - e^* i^*) x_2 \circledast_{Th} \cdots \circledast_{Th} (1 - e^* i^*) x_k) \ldots), \widehat{\vee} Z \big\rangle \\
&\quad = \pm \big\langle x_1 \times x_2 \times ((1 - e^* i^*) x_3 \circledast_{Th} \cdots \circledast_{Th} (1 - e^* i^*) x_k) \ldots), (1 \times \widehat{\vee}) \widehat{\vee} Z \big\rangle \\
&\quad \vdots \\
&\quad = \pm \big\langle (x_1 \times \ldots \times x_k), (1 \times \ldots 1 \times \widehat{\vee}) \circ \cdots \circ \widehat{\vee} Z \big\rangle \\
&\quad = \pm \big\langle x_1 \times \ldots \times x_k, \widehat{\vee}^k Z \big\rangle
\end{aligned}$$

where the sign depends on the sign coming from the duality between $\widehat{\circledast}$ and $\widehat{\vee}$ as given in Theorem 3.3(C), and the sign this induces on the duality between $1 \times \widehat{\circledast}$ and $1 \times \widehat{\vee}$, which involves a sign coming from the cross product. ∎

3.2. **Triviality of product following the coproduct.** The Chas-Sullivan product $\wedge_{Th}$ is defined as a composition

$$H_*(\Lambda)\otimes H_*(\Lambda) \xrightarrow{\times} H_*(\Lambda\times\Lambda) \xrightarrow{\wedge^{\times}_{Th}} H_*(\Lambda)$$

for $\wedge^{\times}_{Th}$ is the *short* Chas-Sullivan product.

Theorem 3.7. *The composition*

$$H_*(\Lambda) \xrightarrow{\widehat{\vee}} H_{*+1-n}(\Lambda\times\Lambda) \xrightarrow{\wedge^{\times}_{Th}} H_{*+1-2n}(\Lambda)$$

of the lifted coproduct followed by the (short) Chas-Sullivan product is the zero map. In particular, with field coefficients, we have the formula

$$\wedge_{Th}\circ AW\circ\widehat{\vee}=0$$

with $\widehat{\vee}\equiv AW\circ\widehat{\vee}: H_*(\Lambda)\to H_*(\Lambda;k)\otimes H_*(\Lambda;k)$.

Remark 3.8. The above result implies that the composition $\wedge\circ AW\circ\widehat{\vee}$ is trivial mod 2. As we will see in Remark 3.16, reducing mod 2 is necessary for this second vanishing.

We will give a geometric proof of the above theorem. We will see in the proof that this vanishing is closely related to the vanishing of the so-called trivial coproduct. An alternative algebraic proof using signs, at least with field coefficients, is sketched in Remark 3.12.

We recall first the *trivial coproduct*, whose triviality was pointed out by Tamanoi [34]. One can define a coproduct using the sequence of maps analogous to that of the coproduct studied here, but fixing the parameter $s=\frac{1}{2}$. More precisely, let

$$F_{\frac12}=\{\gamma\in\Lambda\mid\gamma(\frac{1}{2})=\gamma(0)\}\ \subset\Lambda$$

and consider its neighborhood

$$U_T=U_{T,\varepsilon}=\{\gamma\in\Lambda\mid|\gamma(\frac{1}{2})-\gamma(0)|<\varepsilon\}\ \subset\Lambda.$$

Then restricting the tubular neighborhood ν_{GH} to $s=\frac{1}{2}$ defines a tubular neighborhood

$$\begin{array}{c}\nu_T:\ e^*(TM)\longrightarrow\Lambda\\ \downarrow\\ F_{\frac12}\end{array}$$

of $F_{\frac12}$ inside Λ, with image U_T. Denoting by $\kappa_T=(k_T,V_T)$ the associated collapse map, $R_T\simeq k_T$ the restriction of R_{GH} to time $s=\frac{1}{2}$, and

$$\tau_T=e^*(\tau_M)\in C^n(\Lambda\times\Lambda,U^c_{T,\varepsilon_0})$$

the associated Thom class, this yields a degree $-n$ coproduct which we can model via the sequence of maps

$$\vee_{\frac12}: C_*(\Lambda)\xrightarrow{\tau_T\cap} C_{*-n}(U_T)\xrightarrow{R_T} C_{*-n}(F_{\frac12})\xrightarrow{cut} C_{*-n}(\Lambda\times\Lambda).$$

Tamanoi, who also suggested the description of the coproduct in terms of the retraction map R_T, showed in [34] that this operation is almost identically zero! More precisely, he shows that

$$\vee_{\frac12}([A])=\chi(M)([A]\wedge[*])\times[*]=\chi(M)[*]\times([A]\wedge[*])\in H_*(\Lambda\times\Lambda),$$

for $[*]\in H_0(\Lambda)$ the basepoint in the component of the constant loops. It follows, as pointed out by Tamanoi, that $\vee_{\frac12}([A])=0$ is unless $\chi(M)\neq 0$ and A is of degree n with $[A]\wedge[*]$ a non-zero multiple of $[*]\in H_0(\Lambda)$. (See Theorem B and the more detailed Theorem 3.1 in that paper.)

Let

$$\vee^F_{\frac12}: C_*(\Lambda)\xrightarrow{\tau_T\cap} C_{*-n}(U_T)\xrightarrow{R_T} C_{*-n}(F_{\frac12})$$

be the trivial coproduct prior to the cutting map, so that $\vee_{\frac12}=cut\circ\vee^F_{\frac12}$. We start by giving a refinement of Tamanoi's result to $\vee^F_{\frac12}$:

Lemma 3.9. *For $[A] \in H_*(\Lambda)$, the pre-coproduct $\vee^F_{\frac{1}{2}}([A]) = 0$ unless $\chi(M) \neq 0$ and $[A]$ is of degree n intersecting the component of the constant loops non-trivially. In that case,*

$$\vee^F_{\frac{1}{2}}([A]) = k\,\chi(M)[*] \in H_0(F_{\frac{1}{2}})$$

for some integer k.

Proof. We follow the argument for the proof of Theorem 3.1 in [34] which gives the analogous result for $\vee_{\frac{1}{2}}$. Consider the diagram

$$\begin{array}{ccc}
H_*(\Lambda)\otimes H_n(\Lambda) & \xrightarrow{1\otimes\vee^F_{\frac{1}{2}}} & H_*(\Lambda)\otimes H_0(F_{\frac{1}{2}}) \\
{\scriptstyle\times}\downarrow & & \downarrow{\scriptstyle\times} \\
H_{*+n}(\Lambda\times\Lambda) & \xrightarrow{1\times\vee^F_{\frac{1}{2}}} & H_*(\Lambda\times F_{\frac{1}{2}}) \\
{\scriptstyle\wedge^\times_{Th}}\downarrow & & \downarrow{\scriptstyle\wedge^\times_{Th}} \\
H_*(\Lambda) & \xrightarrow{\vee^F_{\frac{1}{2}}} & H_{*-n}(F_{\frac{1}{2}})
\end{array}$$

where $\wedge^\times_{Th}$ is the Chas-Sullivan product (with Thom sign) modulo the cross-product, so that the vertical compositions are the Chas-Sullivan product $\wedge_{Th}$, and the map $1 \times \vee^F_{\frac{1}{2}}$ is defined just as $\vee^F_{\frac{1}{2}}$ but carrying an extra factor Λ along. The top square in the diagram commutes by naturality of the cross product, while the bottom square is, up to a sign $(-1)^n$ in the bottom map, the left half of the third diagram in the proof of Theorem 2.2 in [34], where it is shown to commute up to the sign $(-1)^n$ as part of the Frobenius identity for $\vee_{\frac{1}{2}}$ and $\wedge$. Hence with the given signs it commutes. Starting with $[A] \otimes [M]$ in the top left corner and following the left and bottom of the diagram, we get that it is mapped to $\vee^F_{\frac{1}{2}}([A] \wedge_{Th} [M]) = \vee^F_{\frac{1}{2}}([A])$ as $[M]$ is the unit for the Chas-Sullivan product. Going along the top and right side instead, we get that $[A] \otimes [M]$ is mapped to $[A] \wedge_{Th} \vee^F_{\frac{1}{2}}([M])$. Now $\vee^F_{\frac{1}{2}}([M]) = [M] \cap e = \chi(M)[*]$ for e the Euler class of M as $[M]$ comes from the zero section of TM. The commutativity of the diagram then gives that $\vee^F_{\frac{1}{2}}([A]) = \chi(M)([A] \wedge_{Th} [*])$. Now [34, Thm C] shows that $\chi(M)([A] \wedge_{Th} [*]) = 0$ unless $[A]$ has degree n and intersect non-trivially the constant loops. And for degree and component reasons, $[A] \wedge_{Th} [*] = k[*]$ in that case, which finishes the proof. (Note that in this proof, the signs do not actually matter and are absorbed by the unknown integer k.) □

We now show that the short Chas-Sullivan product $\wedge^\times_{Th}$, when restricted to pairs of loops that already intersect at zero, is just as trivial as the trivial coproduct. Let $j \colon \Lambda \times_M \Lambda \to \Lambda \times \Lambda$ denote the inclusion.

Lemma 3.10. *The restriction of the (short) Chas-Sullivan product $\wedge^\times_{Th}$ to $j_*(H_*(\Lambda \times_M \Lambda)) \subset H_*(\Lambda \times \Lambda)$ vanishes except for classes of degree n intersecting the constant loops non-trivially, for which we have*

$$\wedge_{Th}(j_*([A])) = k\,\chi(M)[*] \in H_*(\Lambda)$$

for k some integer.

Proof. Note first that concatenation gives an identification

$$\Lambda \times_M \Lambda \xrightarrow{\cong} F_{\frac{1}{2}} \subset \Lambda$$

for $F_{\frac{1}{2}}$ as defined above. Let $\tilde{j}\colon F_{\frac{1}{2}} \to \Lambda$ denote the canonical inclusion and consider the diagram

$$\begin{array}{ccccccc}
H_*(\Lambda\times\Lambda) & \xrightarrow{\tau_{CS}\cap} & H_{*-n}(U_{CS}) & \xrightarrow{R_{CS}} & H_{*-n}(\Lambda\times_M\Lambda) & \xrightarrow{concat} & H_{*-n}(\Lambda)\\
\uparrow{\scriptstyle j} & & {\scriptstyle j}\uparrow & & \Vert & & \\
H_*(\Lambda\times_M\Lambda) & \xrightarrow{j^*\tau_{CS}\cap} & H_{*-n}(\Lambda\times_M\Lambda) & \xrightarrow{id} & H_{*-n}(\Lambda\times_M\Lambda) & & \\
{\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \\
H_*(F_{\frac{1}{2}}) & \xrightarrow{\tilde{j}^*\tau_T\cap} & H_{*-n}(F_{\frac{1}{2}}) & \xrightarrow{id} & H_{*-n}(F_{\frac{1}{2}}) & & \\
\downarrow{\scriptstyle\tilde{j}} & & \downarrow{\scriptstyle\tilde{j}} & & \Vert & & \\
H_*(\Lambda) & \xrightarrow{\tau_T\cap} & H_{*-n}(U_T) & \xrightarrow{R_T} & H_{*-n}(F_{\frac{1}{2}}). & &
\end{array}$$

The top row of the diagram is the Chas-Sullivan product $\wedge^{\times}_{Th}$, while the bottom row is the trivial pre-coproduct $\vee^F_{\frac{1}{2}}$ defined above. The subspaces $\Lambda\times_M\Lambda \cong F_{\frac{1}{2}}$ are both fixed by the retractions, which gives the commutativity of the right column. The top and bottom squares in the left column commute by naturality of the cap product, and the middel square because the classes $j^*\tau_{CS}$ and $\tilde{j}^*\tau_T$ can both be more directly defined as $e^*\tau_M$, compatibly under the isomorphism $\Lambda\times_M\Lambda \xrightarrow{\cong} F_{\frac{1}{2}} \subset \Lambda$.

Now by Lemma 3.9 the bottom row is the zero-map except on classes of degree n that intersect the constant loops non-trivially, in which case it takes the value $k\,\chi(M)[*]$ for some k. Let $[A] \in H_*(\Lambda\times_M\Lambda) \cong H_*(F_{\frac{1}{2}})$. Going along the top of the diagram computes $\wedge^{\times}_{Th}(j_*([A]))$. On the other hand, going along the bottom of the diagram gives the zero map unless $[A]$ has degree n and $\tilde{j}_*([A]) \in H_*(\Lambda)$ intersects the component of the trivial loops. But this is only possible if $[A]$ itself is of that degree and intersects the component of the trivial loops. The result then follows from Lemma 3.9, where we write $[*]$ both for the component of the constant loops in $\Lambda\times_M\Lambda$ and in Λ after concatenation. □

The following, probably well-known fact, follows immediately:

Corollary 3.11. *The Chas-Sullivan product is trivial on the based loop space, except on classes of degree n intersecting non-trivially the constant loops.*

Proof of Theorem 3.7. By definition, the coproduct $\widehat{\vee}$ has image inside $H_*(\Lambda\times_M\Lambda)\subset H_*(\Lambda\times\Lambda)$. Moreover, by Theorem 3.3 (B) its image does not intersect the component of the constant loops $H_*(M\times M)\subset H_*(\Lambda\times\Lambda)$, hence also not its diagonal subspace $H_*(M)\subset H_*(\Lambda\times_M\Lambda)\subset H_*(\Lambda\times\Lambda)$. The result is then a consequence of Lemma 3.10. □

Remark 3.12 (Signs forcing the vanishing). Suppose for simplicity that we work over a field. One can check that the graded commutativity of the "non-algebraic" Chas-Sullivan product and cohomology product is as follows: for $[A]\in H_p(\Lambda;k)$, $[B]\in H_q(\Lambda;k)$, $[a]\in H^p(\Lambda;k)$ and $[b]\in H^q(\Lambda;k)$,

$$[A]\wedge_{Th}[B] = (-1)^{qp+n}[B]\wedge_{Th}[A] \quad\text{ and }\quad [a]\circledast_{Th}[b] = (-1)^{pq+n-1}[b]\circledast_{Th}[a].$$

This difference by precisely (-1) in the sign forces the composition $\wedge_{Th}\circ\widehat{\vee}$ to be zero as each term $[X]\otimes[Y]\in H_p(\Lambda;k)\otimes H_q(\Lambda;k)$ in $\widehat{\vee}([A])$ will have a companion term $(-1)^{pq+n-1}[Y]\otimes[X]$. But their product $[X]\wedge_{Th}[Y]$ and $(-1)^{pq+n-1}[Y]\wedge_{Th}[X]$ differ by a sign $(-1)^{pq+n-1+pq+n} = (-1)$! Signs are though subtle, so we prefer to avoid relying on sign computations.

3.3. **Computations in the case of odd dimensional spheres.** We will now compute the coproduct when M is an odd dimensional sphere, and investigate formulas involving the product and coproduct in that case. Before moving to the particular case of spheres, we will prove the following result, stated in terms of the unlifted coproduct, which says that the coproduct is given by the expected intersection under an appropriate transversality assumption, where transversality happens in the manifold M. The corresponding statement for the Chas-Sullivan product is Proposition 5.5 in [18].

Proposition 3.13. *Let $Z : \Sigma \to \Lambda$ be a $k-$cycle with the property that the restriction of*

$$E(Z) =: e_I \circ (Z \times id)\colon (\Sigma \times [0,1], \Sigma \times \partial[0,1]) \longrightarrow (M \times M, \Delta M)$$
$$(\sigma, t) \mapsto \big(Z(\sigma)(0), Z(\sigma)(t)\big)$$

to the preimage of $U_{M,\varepsilon}$ is a smooth map of manifolds with boundary, whose restriction to the space $E(Z)^{-1}(U_{M,\varepsilon}) \cap \Sigma \times (0,1)$ is transverse to the diagonal ΔM. Let $\Sigma_\Delta := E(Z)^{-1}(\Delta M) \cap \Sigma \times (0,1)$ be oriented so that at each $(\sigma,t) \in \Sigma_\Delta$, the isomorphism

$$T_{(\sigma,t)}\Sigma \cong N_{E(Z)(\sigma,t)}\Delta(M) \oplus T_{(\sigma,t)}\Sigma_\Delta$$

induced by $E(Z)$ respects the orientations, and suppose that the closure

$$(\overline{\Sigma_\Delta}, \partial\overline{\Sigma_\Delta}) \subset (\Sigma \times [0,1], \Sigma \times \{0,1\})$$

is a codimension n submanifold with (possibly empty) boundary. Then

$$[\vee Z] = [cut \circ (Z \times id)|_{\overline{\Sigma_\Delta}}] \in H_{k+n-1}(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M).$$

Note that under the identification $TM \cong N(\Delta M)$ of Section 1.3, the fibers of the normal bundle are oriented as $(-1)^n$ times the canonical orientation of those in the tangent bunle; this is forced by the relation $u_M \cap [M \times M] = [M]$. A simple application of the result can be found in Example 4.4.

Proof. By definition, $[\vee Z] = cut \circ R_{GH}(e_I^*(\tau_M) \cap (Z \times I))$. We will first compute $e_I^*(\tau_M) \cap (Z \times I) \in H_*(U_{GH}, M \times I \cup \Lambda \times \partial I)$. By the proof of Proposition 2.12, we can compute $[\vee Z]$ replacing the pair $(U_{GH}, M \times I \cup \Lambda \times \partial I)$ with $((\Lambda \times (0,1)) \cap U_{GH}, V_\alpha^{(0,1)} \cap U_{GH})$ for some small α. Restricting to $\Lambda \times (0,1)$, we can do a non-relative computation. By naturality of the cap product, we have that $e_I(e_I^*(\tau_M) \cap (Z \times (0,1))) = \tau_M \cap e_I(Z \times (0,1)) \in H_*(U_{M,\varepsilon})$. As the restriction of $e_I \circ (Z \times \mathrm{id}) = E(Z)$ to $E(Z)^{-1}(U_{M,\varepsilon}) \cap \Sigma \times (0,1)$ is transverse to the diagonal, by our choice of orientation we have $\tau_M \cap e_I(Z \times (0,1)) = E(Z)|_{\Sigma_\Delta}$. By transversality, it follows that $e_I^*(\tau_M) \cap (Z \times (0,1)) = Z|_{\Sigma_\Delta} \in H_*((\Lambda \times (0,1)) \cap U_{GH}, V_\alpha^{(0,1)} \cap U_{GH}) \cong H_*(U_{GH}, V_\alpha)$, where we now in addition passed back to relative homology. Under the last isomorphism, $Z|_{\Sigma_\Delta}$ is mapped to $Z|_{\overline{\Sigma_\Delta}}$. As $Z|_{\Sigma_\Delta}$ already has image in $F_{[0,1]}$, the retraction map fixes $Z|_{\Sigma_\Delta}$ and the result follows. □

We now consider the case when M is an odd sphere. The Chas-Sullivan ring for S^n, $n \geq 3$ odd was computed in the paper of Cohen, Jones and Yan [12, Thm 2] to be

$$H_*(\Lambda S^n) = \bigwedge(A) \otimes \mathbb{Z}[U]$$

where $A \in H_0(\Lambda S^n)$, and $U \in H_{2n-1}(\Lambda S^n)$. This ring has a unit $1 = [M] \in H_n(\Lambda S^n)$, represented by the constant loops.

Proposition 3.14. *The coproduct on $H^*(\Lambda S^n)$, for $n \geq 3$ odd, is given on generators by the formula*

$$\widehat{\vee}(A \wedge U^{\wedge k}) = \Sigma_{j=1}^{k-2}(A \wedge U^{\wedge j}) \times (A \wedge U^{\wedge(k-1-j)})$$
$$\widehat{\vee} U^{\wedge k} = \Sigma_{j=1}^{k-2}(A \wedge U^{\wedge j} \times U^{\wedge(k-1-j)}) + (U^{\wedge j} \times A \wedge U^{\wedge(k-1-j)})$$

Before proving the proposition, we use it to confirm our result of the previous section in the case of odd spheres:

Corollary 3.15. *In $H_*(\Lambda S^n)$ for S^n an odd sphere with $n \geq 3$, we have that*

$$\wedge_{Th}\widehat{\vee}(A \wedge U^{\wedge k}) = 0 = \wedge_{Th}\widehat{\vee}(U^{\wedge k}),$$

which confirms Theorem 3.7 in the case of odd spheres.

Proof of Corollary 3.15. The fact that $\wedge_{Th}\widehat{\vee}(A \wedge U^{\wedge k}) = 0$ follows from the formula obtained for $\widehat{\vee}(A \wedge U^{\wedge k})$ in Proposition 3.14 together with the fact that $A \wedge A = 0 = A \wedge_{Th} A$. For the other computations, as $\deg(A \wedge U^{\wedge j}) = j(n-1)$ and $\deg(U^{\wedge(k-1-j)}) = (k-j)n - k + 1 + j$, we have that

$$(A \wedge U^{\wedge j}) \wedge_{Th} U^{\wedge(k-1-j)} = (-1)^{n((k-j)n-k+1+j)+n} A \wedge U^{\wedge k-1} = A \wedge U^{\wedge k-1}$$
$$U^{\wedge(k-1-j)} \wedge_{Th} (A \wedge U^{\wedge j}) = (-1)^{n(j(n-1))+n} A \wedge U^{\wedge k-1} = -A \wedge U^{\wedge k-1}$$

where we each time used that n is odd. Hence when we use the Thom sign for the product, the left terms in the formula for $\widehat{\vee} U^{\wedge k}$ obtained in Proposition 3.14 cancel with the right terms, which proves the result. □

Remark 3.16. Note that $\wedge\widehat{\vee}(U^{\wedge k}) = (2k-4)A \wedge U^{\wedge k-1} \neq 0$ so we cannot replace $\wedge_{Th}$ by $\wedge$ in the formula $\wedge_{Th} \circ \widehat{\vee} = 0$.

Recall from [18, Sec 15] that the relative cohomology ring for an odd sphere with the product $\circledast$ is generated by four classes: ω, X, Y and Z of degrees $n-1, 2n-2, 2n-1$ and $3n-2$ respectively, and

$$H^*(\Lambda S^n, S^n) \;=\; \bigwedge(Y,Z) \otimes \mathbb{Z}[\omega, X]/\sim$$

for $\sim$ generated by the two relations $X \circledast X = \omega^{\circledast 3}$ and $X \circledast Y = \omega \circledast Z$. As explained in [18], this ring can also be written as $H^*(\Lambda S^n, S^n) = \bigwedge(u) \otimes \mathbb{Z}[t]_{\geq 2}$ where $\omega = t^{\circledast 2}$, $X = t^{\circledast 3}$, $Y = u \circledast t^{\circledast 2}$ and $Z = u \circledast t^{\circledast 3}$. Here $\mathbb{Z}[t]_{\geq 2}$ denotes the ideal generated by $t^{\circledast 2}$ and $t^{\circledast 3}$ in the polynomial ring.

To deduce from this a computation of the homology coproduct, we need the following compatibility between the homology and cohomology generators.

Lemma 3.17. *The generators $A, U \in H_*(\Lambda S^n)$ and ω, X, Y and Z in $H^*(\Lambda S^n, S^n)$ can be chosen so that for every $k \geq 1$,*

$$\begin{aligned}
\left\langle \omega^{\circledast k}, A \wedge U^{\wedge 2k-1}\right\rangle &= 1\\
\left\langle X \circledast \omega^{\circledast k-1}, A \wedge U^{\wedge 2k}\right\rangle &= 1\\
\left\langle Y \circledast \omega^{\circledast k-1}, U^{\wedge(2k-1)}\right\rangle &= 1\\
\left\langle Z \circledast \omega^{\circledast k-1}, U^{\wedge(2k)}\right\rangle &= 1
\end{aligned}$$

The case $k = 1$ in the lemma can be taken as our definition of the cohomology generators, once the homology generators are fixed. For $k \geq 2$ this is a signed version of a general principle established in [18, Sec 13,14] saying that "to go up one level in cohomology on a manifold with all geodesics closed, you multiply by ω", and "to go up one level in homology, you multiply by $\boldsymbol{\Theta}$", $\boldsymbol{\Theta}$ being $U^{\wedge 2}$ in the case of spheres. See Remark 3.18 for a graphic representation of this phenomenon.

Proof of Lemma 3.17. We fix an orientation for S^n. From [12], we have that $H_*(\Lambda S^n) = \bigwedge(A) \otimes \mathbb{Z}[U]$ with generators $A \in H_0(\Lambda S^n)$, $1 = [S^n] \in H_n(\Lambda S^n)$ and $U \in H_{2n-1}(\Lambda S^n)$. We fix the orientation of the class $A \wedge U \in H_{n-1}(\Lambda S^n)$ by requiring that, for $Z\colon \Sigma_{n-1} \to \Omega S^n$ a cycle representing it, the map $\Sigma_{n-1} \times S^1 \to S^n$ taking (a,t) to $Z(a)(t)$ in homology is $(-1)^n$ times the fundamental class $[S^n]$. (The sign $(-1)^n = -1$ is chosen for compatibility with the normal orientation of ΔS^n inside $S^n \times S^n$, as above.) This fixes the orientation of U, and hence determines the orientation of each class $U^{\wedge k}$ and $A \wedge U^{\wedge k}$.

We choose the orientation of X and ω in such a way that

$$\langle \omega, A \wedge U\rangle = 1 \quad \text{and} \quad \langle X, A \wedge U^{\wedge 2}\rangle = 1,$$

which one can check is compatible with the equation $X \circledast X = \omega^{\circledast 3}$ [18, (15.5.1)] using the proof of that equation in [18] together with Proposition 3.13 for the signs, showing that both products are dual to $A \wedge U^{\wedge 5}$. (This comes down to computing $\vee(A \wedge U^{\wedge 5})$ and $\vee^2(A \wedge U^{\wedge 5})$ and uses that $A \wedge U^{\wedge 5} = (A \wedge U) \cdot (A \wedge U^{\wedge 2}) \cdot (A \wedge U^{\wedge 2})$ in the Pontrjagin ring $H_*(\Omega S^n)$, a fact that follows from the "Gysin formulas" [18, p151]. Here $\cdot$ denotes the Pontrjagin product.) We then define $Y = \iota^! \iota^* \omega$ and $Z = \iota^! \iota^* X$ [18, (15.6.2)] for $\iota\colon \Omega S^n \to \Lambda S^n$ the inclusion. The equation $X \circledast Y = \omega \circledast Z$ [18, (15.6.4)] then follows. The computation of $\vee^k(A \wedge U^{\wedge 2k-1})$ and $\vee^k(A \wedge U^{\wedge 2k})$ is similar, using Proposition 3.13 again, and the fact that $A \wedge U^{\wedge 2k-1} = (A \wedge U) \cdot (A \wedge U^{\wedge 2})^{\cdot k-1}$ while $A \wedge U^{\wedge 2k} = (A \wedge U^{\wedge 2}) \cdot (A \wedge U^{\wedge 2})^{\cdot k-1}$. This leads to

$$\langle \omega^{\circledast k}, A \wedge U^{\wedge 2k-1}\rangle = 1 \quad \text{and} \quad \langle X \circledast \omega^{\circledast k-1}, A \wedge U^{\wedge 2k}\rangle = 1.$$

Finally from this it follows that

$$\begin{aligned}\left\langle Y \circledast \omega^{\circledast i}, U^{\wedge(2i+1)}\right\rangle &= \left\langle \iota^! \iota^* \omega \circledast \omega^{\circledast i}, U^{\wedge(2i+1)}\right\rangle \\ &= \left\langle \iota^!(\iota^*\omega \circledast_\Omega i^*\omega^{\circledast i}), U^{\wedge(2i+1)}\right\rangle \\ &= \left\langle \omega^{\circledast i+1}, \iota_* \iota_! U^{\wedge(2i+1)}\right\rangle \\ &= \left\langle \omega^{\circledast i+1}, A \wedge U^{\wedge(2i+1)}\right\rangle = 1\end{aligned}$$

and

$$\begin{aligned}\left\langle Z \circledast \omega^{\circledast i}, U^{\wedge(2i+2)}\right\rangle &= \left\langle \iota^! \iota^* X \circledast \omega^{\circledast i}, U^{\wedge(2i+2)}\right\rangle \\ &= \left\langle \iota^!(\iota^* X \circledast_\Omega i^*\omega^{\circledast i}), U^{\wedge(2i+2)}\right\rangle \\ &= \left\langle X \circledast \omega^{\circledast i}, i_* i_! U^{\wedge(2i+2)}\right\rangle \\ &= \left\langle X \circledast \omega^{\circledast i}, A \wedge U^{\wedge(2i+2)}\right\rangle = 1\end{aligned}$$

where we used the Gysin formulas given in [18, p151]. □

Remark 3.18. For the convenience of the reader, we recall from [18] that the homology and cohomology fit together as follows:

degree	0	$n-1$	n	$2n-2$	$2n-1$	$3n-3$	$3n-2$	$4n-4$	$4n-3$	$5n-5$	$5n-4$
$H_*(M)$	A		U^0								
$H_*(\Lambda^{\le 2\pi}, \Lambda^{<2\pi})$		AU		AU^2	U		U^2				
$H_*(\Lambda^{\le 4\pi}, \Lambda^{<4\pi})$						AU^3		AU^4	U^3		U^4
$H_*(\Lambda^{\le 6\pi}, \Lambda^{<6\pi})$										AU^5	
$H^*(\Lambda^{\le 2\pi}, \Lambda^{<2\pi})$		$\omega = t^2$		$X = t^3$	$Y = ut^2$		$Z = ut^3$				
$H^*(\Lambda^{\le 4\pi}, \Lambda^{<4\pi})$						ω^2		ωX	ωY		ωZ
$H^*(\Lambda^{\le 6\pi}, \Lambda^{<6\pi})$										ω^3	

In each degree k we have either that both $H_k(\Lambda S^n)$ and $H^k(\Lambda S^n)$ are 0, or both are equal to $\mathbb{Z}$. In the latter case, we wrote a generator. To save space we have left out the products $\wedge$ and $\widehat{\circledast} = \circledast$ from the notation in the table; so for example $AU^2 =: A \wedge U^{\wedge 2}$ in the table. Note that in the case of odd spheres, with the algebraic signs as here, both products are commutative. The boxes in the table correspond to the homology of the unit tangent bundle, shifted up by successive multiplication with U^2 in homology and ω in cohomology.

Proof of Proposition 3.14. By Lemma 3.17 and the identification $\omega^{\circledast k} = t^{\circledast 2k}$ and $X \circledast \omega^{\circledast k-1} = t^{\circledast 2k+1}$, we have that the class $A \wedge U^{\wedge k}$ is dual to $t^{\circledast k+1}$. As this product can be decomposed as $t^{\circledast j+1} \circledast t^{\circledast k-j}$ for any $1 \le j \le k-2$, it follows from the duality between the $\vee$ and $\circledast$ (Theorem 2.13) and the fact that $\widehat{\vee}$ is an extension by 0 (Theorem 3.3) that

$$\widehat{\vee}(A \wedge U^{\wedge k}) = \Sigma_{j=1}^{k-2} \pm (A \wedge U^{\wedge j}) \times (A \wedge U^{\wedge(k-1-j)}).$$

Likewise, $Y \circledast \omega^{\circledast k-1} = u \circledast t^{\circledast 2k}$ and $Z \circledast \omega^{\circledast k-1} = u \circledast t^{\circledast 2k+1}$, so Lemma 3.17 shows that $U^{\wedge k}$ is dual to $u \circledast t^{\circledast k+1}$. This last product can be decomposed as

$$(u \circledast t^{\circledast j+1}) \circledast t^{\circledast k-j} = (-1)^{(jn+n-j-(n-1))((k-j-1)(n-1)-(n-1))} t^{\circledast k-j} \circledast (u \circledast t^{\circledast j+1}) = t^{\circledast k-j} \circledast (u \circledast t^{\circledast j+1})$$

for any $1 \le j \le k-2$ when n is odd, where $jn+n-j$ is the degree of $u \circledast t^{\circledast j+1}$ and $(k-j-1)(n-1)$ is the degree of $t^{\circledast k-j}$. Hence

$$\widehat{\vee} U^{\wedge k} = \Sigma_{j=1}^{k-2} \pm (A \wedge U^{\wedge j} \times U^{\wedge(k-1-j)}) + \pm(U^{\wedge j} \times A \wedge U^{\wedge(k-1-j)}).$$

Left is to work out the signs for each term. We have already seen that there are no signs coming from the decomposition as products. There two potential additional two signs coming from the duality:

(i) $\langle c\times d, \vee E\rangle = (-1)^{\deg(d)(n-1)}\langle c \circledast d, E\rangle$,
(ii) $\langle c\times d, C\times D\rangle = (-1)^{\deg(D)\deg(c)}\langle c, C\rangle\langle d, D\rangle$.

Now in the case at hand, we have assumed that n is odd, so the first equation does not give a sign. Also, in all cases either C or D will be a term of the form $A\wedge U^{\wedge j}$, which is of degree $j(n-1)$, that is always even dimensional when n is odd. So the second equation will likewise not give any sign. □

Remark 3.19 (Higher genus non-trivial operations). If we represent the product and coproduct by pairs of pants, the composition of the coproduct followed by the product is represented by a genus one surface with two boundary components. We have shown above that this genus one operation is trivial if we use the Thom sign for the product. If one inserts the operation $(1\times\Delta)$ in between the product and the coproduct, for Δ the degree 1 operation coming from the circle action, the resulting operation is non-trivial whether we use $\wedge$ or $\wedge_{Th}$: for $n\geq 3$ odd, the operation

$$t := \wedge\circ(1\times\Delta)\circ\widehat{\vee}\colon H_*(\Lambda S^n)\longrightarrow H_{*-2n+2}(\Lambda S^n)$$

is non-trivial on the classes $A\wedge U^{\wedge k}$ with $k\geq 3$, and likewise with $\wedge_{Th}$ replacing $\wedge$. Indeed, from Proposition 3.14, we have that

$$\widehat{\vee}(A\wedge U^{\wedge k}) = \Sigma_{j=1}^{k-2}(A\wedge U^{\wedge j})\times(A\wedge U^{\wedge(k-1-j)}).$$

From [22, Lem 6.2], we know moreover that

$$\Delta(A\wedge U^{\wedge k}) = (-1)^k k\, U^{\wedge(k-1)}$$

(taking into account our sign convention). Putting these two computations together we get that

$$\begin{aligned}\wedge\circ(1\times\Delta)\circ\widehat{\vee}(A\wedge U^{\wedge k}) &= \Sigma_{j=1}^{k-2}(-1)^{k-1-j}(k-1-j)A\wedge U^{\wedge(k-2)}\\ &= (-1)^k\lfloor\frac{k-1}{2}\rfloor A\wedge U^{\wedge(k-2)}\end{aligned}$$

which in particular is non-zero. As $A\wedge U^{\wedge j}$ has even degree when $n\geq 3$ is odd, one gets a global sign $(-1)^n=-1$ if we use $\wedge_{Th}$ instead of $\wedge$.

The operation t would still be associated to a genus 1 surface, and composing t with itself g times can be associated to a genus g surface. Note also that the operation t^g is likewise non-trivial on $A\wedge U^{\wedge k}$ whenever $k\geq 2g+1$. Similar computations in algebraic models of the loop space can be found in [35, Prop 4.1] (see also [5]), and in fact, these papers indicate that there should be many such non-trivial operations combining the product, coproduct and Δ operations, associated to classes in the homology of the Harmonic compactification of Riemann surfaces. These more general operations will be studied in our following paper [23].

Remark 3.20 (Failure of the Frobenius formula). It has been suggested that there should be a formula of the form

$$\vee(A\wedge B) = (\vee A)\wedge B + A\wedge(\vee B) \tag{3.3}$$

i.e.

$$\vee\circ\wedge \overset{?}{=} (1\times\wedge)\circ\vee\times 1 + (\wedge\times 1)\circ 1\times\vee$$

[33, p 349] relating the coproduct and the product. It is very difficult to make sense of this formula in relative homology. It is natural to ask if the product and *lifted* coproduct satisfy

$$\widehat{\vee}\circ\wedge = (1\times\widehat{\wedge})\circ\vee\times 1 + (\widehat{\wedge}\times 1)\circ 1\times\vee \tag{3.4}$$

Our choice of lift $\widehat{\vee}$ does not satisfy the formula (3.4): one can check by a computation similar to that in Remark 3.19 for odd spheres. If the formula (3.4) were true we would have for example

$$\widehat{\vee}(\mathbf{\Theta}\wedge\mathbf{\Theta}) = \widehat{\vee}(\mathbf{\Theta})\wedge\mathbf{\Theta} + \mathbf{\Theta}\wedge\widehat{\vee}(\mathbf{\Theta}) \tag{3.5}$$

where $\mathbf{\Theta} =: U^{\wedge 2} \in H_{3n-2}(\Lambda S^n)$. Now the right hand side is 0 since $\widehat{\vee}(\mathbf{\Theta}) = 0$ by Theorem 3.7; indeed $\mathbf{\Theta}$ is represented by the space of circles and has support in the simple and constant loops. But the left hand side of (3.5) must *not* be 0 since $\mathbf{\Theta} \wedge \mathbf{\Theta}$ is not in the image of $\Lambda^{\leq \ell}$ and thus $\mathbf{\Theta} \wedge \mathbf{\Theta}$ is dual to a decomposable cohomology class. (The coproduct $\widehat{\vee}(\mathbf{\Theta} \wedge \mathbf{\Theta}$ has one nonzero term for each way of writing the dual $u \circledast t^{\circledast 5}$ of $\mathbf{\Theta} \wedge \mathbf{\Theta}$ as a product.) The conclusion is that our lifted coproduct fails to satisfy (3.5) for fundamental reasons.

More generally, using direct computation and testing $A = B = U^{\wedge 2k} = \mathbf{\Theta}^{\wedge k}$ in (3.3), we did not find equality, but rather that the two sides differ by

$$\begin{aligned} & \widehat{\vee}(U^{\wedge 2k} \wedge U^{\wedge 2k}) - \big((\widehat{\vee} U^{\wedge 2k}) \wedge U^{\wedge 2k} + U^{\wedge 2k} \wedge (\widehat{\vee} U^{\wedge 2k})\big) \\ = \;& A \wedge U^{\wedge(2k-1)} \times U^{\wedge 2k} + A \wedge U^{\wedge 2k} \times U^{\wedge(2k-1)} - U^{\wedge(2k-1)} \times A \wedge U^{\wedge 2k} - U^{\wedge 2k} \times A \wedge U^{\wedge(2k-1)} \end{aligned}$$

yielding four nonzero terms which do not add to 0. It is also notable that the four terms are "in the middle". From a naive geometric perspective, (3.3) is not to be expected to be true: it says that, "the self-intersections in $A \wedge B$, (i.e. $\vee(A \wedge B)$) come from the self-intersections in A (i.e. $(\vee A) \wedge B$) and from the self-intersections in B (i.e. $A \wedge (\vee B)$)". But what about the self-intersections in $A \wedge B$ of the form $\alpha * \beta$ where $\alpha \in \operatorname{Im} A$ and $\beta \in \operatorname{Im} B$ are both simple? The evidence from the finite dimensional approximation is that the difference between the left- and right-hand side in equation (3.3) picks up the second-order intersections of A and B.

4. Support of the cohomology product and homology coproduct

We will show in this section that the coproduct $\widehat{\vee}$ vanishes on the simple loops, and more generally that the iterated coproduct $\widehat{\vee}^k$ vanishes on loops having at most k-fold self-intersections. As a consequence, we are guaranteed to find loops with multiple self-intersections when a power of the coproduct is not 0. Examples will follow. We start by defining what we mean by multiple self-intersections.

Definition 4.1. We will say that a (non-constant) loop $\gamma \in \Lambda$ has a *k-fold intersection at $p \in M$* if $\gamma^{-1}\{p\}$ consists of k points.

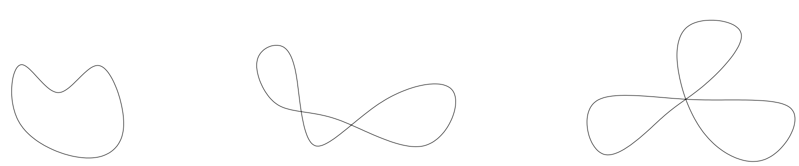


Figure 3. Simple loop, loop with two 2-fold intersections and loop with a 3-fold intersection

Note that if γ is parameterized proportional to arc length (see Section 1.1) then $\gamma \in \Lambda$ has a k-fold intersection at the basepoint if and only if it is the concatenation of k nontrivial loops (and no more), so roughly speaking

$$k\text{-fold intersection} \quad \leftrightsquigarrow \quad k \text{ loops}.$$

Recall the operator

$$\Delta\colon H_*(\Lambda) \longrightarrow H_{*+1}(\Lambda)$$

coming from the S^1–action on Λ that rotates the loops.

Theorem 4.2. *The coproduct $\widehat{\vee}$ and its iterate $\widehat{\vee}^k$ have the following support property:*

(i) If $Z \in C_*(\Lambda)$ has the property that every nonconstant loop in the image of Z has at most k-fold intersections, then

$$\widehat{\vee}^k[Z] = \widehat{\vee}^k(\Delta[Z]) = 0 \in H_*(\Lambda^k).$$

(ii) If $Z \in C_*(\Lambda)$ is a cycle with the property that every nonconstant loop in the image of Z has at most a k-fold intersection at the basepoint, then

$$\widehat{\vee}^k[Z] = 0 \in H_*(\Lambda^k).$$

Remark 4.3. As an immediate consequence of (ii) we have the following (stronger) statement:

(ii') Suppose $Z \in C_*(\Lambda)$ is a cycle with the property that for *some* $s \in S^1$, every nonconstant loop γ in the image of Z has at most a k-fold intersection $\gamma(s)$. Then

$$\widehat{\vee}^k[Z] = 0 \in H_*(\Lambda^k).$$

On the chain level the coproduct $\widehat{\vee}$ sees only self-intersections at the basepoint; $\widehat{\vee} \circ \Delta$ picks up on other self-intersections. See Example 4.4 below.

The equations in Theorem 4.2 are valid in $H_*(\Lambda \times \Lambda)$ and therefore, when applicable, after the Kunneth map in $H_*(\Lambda) \otimes H_*(\Lambda)$. They are valid for any choice of coefficients.

Example 4.4. Here is an example that illustrates the consequences of Theorem 4.2. Let M be a surface of genus 3 and let $\gamma(t)$ be the loop pictured in Figure 4, starting at the intersection point p, and let $X = \{\gamma\} \in C_0(\Lambda)$. Then γ has a 2-fold intersection at the basepoint. Despite this, $\vee[X] = \widehat{\vee}[X] = 0$ for (at least) two reasons: (i) $\widehat{\vee}[X]$ has degree -1 and (ii) X is homologous to $Y = \{\delta\} \in C_0(\Lambda)$ where δ traces out the same path as γ, but starts at the point q. Since δ has no self-intersection at the basepoint, $\widehat{\vee}[X] = \widehat{\vee}[Y] = 0$ by Theorem 4.2.

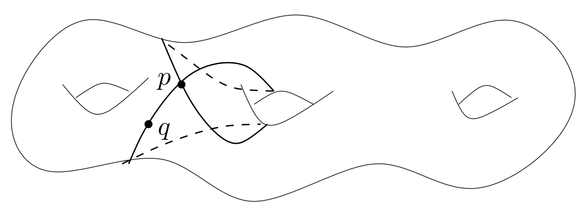


FIGURE 4. Loop in a genus 3 surface

Now consider the cycle $\Delta Y \in C_1(\Lambda)$. Suppose $p = \delta(t_0) = \delta(t_1)$ with $0 < t_0 < t_1 < 1$. To compute $\widehat{\vee}\Delta Y$, we use Proposition 3.13. Consider

$$\begin{array}{ccccc} I \times I & \xrightarrow{\Delta Y \times I} & \Lambda \times I & \xrightarrow{e_I} & M \times M \\ (t,s) & \mapsto & (\delta_t, s) & \mapsto & (\delta_t(0), \delta_t(s)) = (\delta(t), \delta(t+s)) \end{array}$$

where δ_t is the loop defined by $\delta_t(s) = \delta(s+t)$. On the *interior* of $I \times I$, the image of this map intersects the diagonal $\Delta \subset M \times M$ exactly twice, at the image of the points $(t,s) = (t_0, t_1 - t_0)$ and $(t,s) = (t_1, t_0 - t_1 + 1)$. We invite the reader to check from the definitions that the intersection will be transverse if δ self-intersects transversely at p, which we may assume. Let

$$\begin{array}{rcl} \zeta & = & \delta[t_0, t_1] \\ \xi & = & \delta[t_1, t_0 + 1] \end{array}$$

be the two loops that make up δ. Then Proposition 3.13 gives that (up to sign), the unlifted coproduct is

$$\vee \Delta Y = \{(\zeta, \xi), (\xi, \zeta)\} \in C_0(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M),$$

which happens to be a cycle also in non-relative chains. (The two points will have with opposite signs, but the actual sign depends on choices of orientation.) The homology class $[\vee \Delta Y]$ is nontrivial because (ζ, ξ) and (ξ, ζ) belong to different components of $\Lambda \times \Lambda$, and neither of which contains a pair in $M \times \Lambda \cup \Lambda \times M$. The lifted coproduct is (up to sign)

$$[\widehat{\vee}\Delta Y] = [\widehat{\vee}\Delta X] = [\{(\zeta, \xi)\} + (\xi, \zeta) + \{(p, \xi)\} + \{(\zeta, p)\} + \{(p, \zeta)\} + \{(\xi, p)\} + 2\{(p, p)\}]$$

in $H_0(\Lambda \times \Lambda)$, which is also clearly nonzero. (Here p denotes the constant loop at p.)

To sum up: we can perturb away the self intersection of γ at the basepoint, from which it follows that $\widehat{\vee}[X] = 0$. But $\widehat{\vee}[\Delta X] = \widehat{\vee}[\Delta Y] \neq 0$, and from this we can conclude that every representative Z of $[\Delta X]$ contains a nonconstant loop with nontrivial self-intersection.

Note that in order to compute $[\vee\Delta Y]$ and $[\widehat{\vee}\Delta Y]$, we only looked at intersections in the *interior* of $I \times I$, despite the fact that the edges $s = 0$ and $s = 1$ map to the diagonal in $M \times M$. This was justified by Proposition 3.13 for the coproduct $[\vee\Delta Y]$, which originates in the fact that the coproduct has image in $H_*(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M)$, and by the fact that the lifted coproduct $\widehat{\vee}$ is a lift by 0: any chain with support on the edges $s = 0$ and $s = 1$ will be killed by $(1-i_*e_*)\times(1-i_*e_*)$.

For $k = 1$, Theorem 4.2 says that $\widehat{\vee}[Z] = \widehat{\vee} \circ \Delta[Z] = 0$ for any homology class $[Z]$ that can be represented using simple and trivial loops only, and that $\widehat{\vee}[Z] = 0$ for any $[Z]$ that has a representative consisting entirely of trivial loops and loops that have no self-intersection at the basepoint (that is, no self-intersections of order $k > 1$ at the basepoint). This support statement confirms our intuition that the coproduct

"*looks for self-intersections, and cuts them apart*".

From the point of view of geometry, if there are no self-intersections, the coproduct should vanish. Moreover the lifted coproduct is not foolish enough to mistake the tautology $\gamma(0) = \gamma(0)$ (the equation you get when you set $\gamma(0) = \gamma(s)$ and let $s \to 0$) for a self-intersection! For example, the union of "all circles, great and small" on a sphere, where a *circle* on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is by definition the non-empty intersection of S^n with a 2-plane in $\mathbb{R}^{n+1}$, defines a non-trivial homology class in $H_{3n-2}(\Lambda S^n)$. We can use our theorem to conclude that the coproduct vanishes on this homology class, even though it includes the constant loops. (Note that this agrees with our computation in Proposition 3.14.)

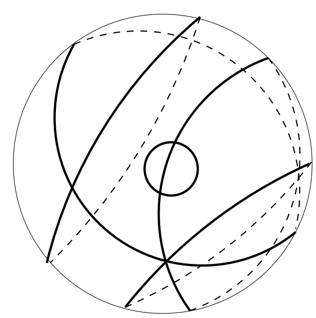


FIGURE 5. Connected union of 5 circles on S^2

More generally let $G \subset M = S^n$ be a connected union of k circles on M (e.g. as in Figure 5). Let $\Lambda_G \subset \Lambda$ be the set of loops $\gamma : S^1 \to M$ with image G that are injective except at the "vertices": We allow $\gamma \in \Lambda_G$ to have a k-fold self-intersections with $k > 1$ only at a point where two distinct circles in G intersect, or if G consists of a single point. (In other words, γ is an "Eulerian paths", as in the Königsberg's bridges problem.) Note that Λ_G is nonempty by Euler's theorem. If $\gamma \in \Lambda_G$, it is easy to see that γ has at most k-fold self-intersections, since there are at most $2k$ edges at each vertex. Let $\mathcal{G}_k \subset \Lambda$ be the union of all the Λ_G, where G is the union of k circles:

$$\mathcal{G}_k = \bigcup_{\substack{G = \{c_1, ..., c_k\} \\ \gamma \in \Lambda_G}} \gamma \quad \subset \Lambda$$

Then it follows from Theorem 4.2 that

$$\widehat{\vee}^k = 0 \text{ on } \mathcal{G}_k.$$

The space $\underset{k}{\cup}\mathcal{G}_k$ (or some subset thereof; such as the subset of loops parameterized proportional to arclength, or with an orientation condition) has been suggested as a "small model", a subspace of

Λ that is invariant under the natural $O(2)$-action, and that reflects in some sense the equivariant topology of Λ.

Remark 4.5. If $\Sigma \subset \Lambda$ is a cycle with Σ a smooth submanifold, then the map

$$\begin{array}{rcl}\Sigma \times \Delta^k & \to & M^k \\ (\gamma, s_1, ..., s_k) & \to & (\gamma(s_1), ..., \gamma(s_k))\end{array}$$

will generically intersect the diagonal $\Delta_k M = \{(p, ..., p) \in M^k\} \subset M^k$ (signalling the existence of a k-fold intersection at some point of Σ that can not be avoided) only if

$$\dim(\Sigma \times \Delta^k) + \dim \Delta_k M \geq \dim M^k$$

which implies

$$\dim \Sigma \geq (k-1)(n-1) - 1.$$

Similarly, the map

$$\begin{array}{rcl}\Sigma \times \Delta^{k-1} & \to & M^k \\ (\gamma, s_1, ..., s_{k-1}) & \to & (\gamma(0), \gamma(s_1), ..., \gamma(s_{k-1}))\end{array}$$

will generically intersect the diagonal $\Delta_k M \subset M^k$ (signalling the existence of a k-fold intersection at the basepoint at some point of Σ that can not be avoided) only if

$$\dim(\Sigma \times \Delta^{k-1}) + \dim \Delta_k \geq \dim M^k$$

which implies

$$\dim \Sigma \geq (k-1)(n-1).$$

So for example in dimension $n = 2$, with $k = 2$ and $\dim \Sigma = 0$: if a loop intersects itself transversely at the basepoint we cannot by perturbation remove the self-intersection entirely, given that $0 \geq (k-1)(n-1)-1$, but we can move the self-intersection away from the basepoint, as $0 \not\geq (k-1)(n-1)$. Note though that in general cycles are not represented by smooth submanifolds.

To prove Theorem 4.2, we will start with the case $k = 1$ for simplicity. We have that

$$\hat{\vee} A = (1 - i_* e_*)^{\times 2} \circ \mathit{cut} \circ R_{GH}\,(\tau_{GH} \cap (A \times I)) \tag{4.1}$$
$$= \mathit{cut} \circ (1 - e_{L*}) \circ (1 - e_{R*}) \circ R_{GH}\,(\tau_{GH} \cap (A \times I)) \tag{4.2}$$

where

$$e_L, e_R \colon F_{[0,1]} = \{(\gamma, s) \in \Lambda \times I \,|\, \gamma(0) = \gamma(s)\} \longrightarrow F_{[0,1]}$$

are the left and right "deflation maps", defined by $e_L(\gamma, s) = \gamma_0 *_s \gamma[s, 1]$ and $e_R(\gamma, s) = \gamma[0, s] *_s \gamma_0$. Note that $e_L \circ e_R = e_R \circ e_L$ is also the restriction of $(i \circ e) \times \mathrm{id}$ to $F_{[0,1]} \subset \Lambda \times I$.

Let $\overline{\Lambda} \subset \Lambda$ be the space of loops parameterized proportional to arc length. There is a (homotopy) retraction of Λ onto $\overline{\Lambda}$ (see Section 1.1). Given $\ell > 0$ and $0 < \alpha < \rho/2$, let

$$V_{\alpha,\ell} = \{(\gamma, s) \in \overline{\Lambda} \times I \mid \mathcal{L}(\gamma) < \alpha, \text{ or } \mathcal{L}(\gamma) < \ell \text{ and } s \in [0, \alpha/\ell) \cup (1 - \alpha/\ell, 1]\}.$$

Lemma 4.6. *Let $B \in C_*(\Lambda \times I)$ be a (compact, continuous) singular chain with image in the subspace $\mathcal{S}^{\leq \ell} \times I \cap \overline{\Lambda} \times I$, for $\mathcal{S}^{\leq \ell} \subset \Lambda$ the subspace of loops of length $\leq \ell$ that are either trivial, or have a simple (i.e. 1-fold) intersection at the basepoint. Then for any $0 < \alpha < \rho/2$, there exists $\tilde{\varepsilon} > 0$ so that $\mathrm{Im}\, B \cap U_{GH,\tilde{\varepsilon}} \subset V_{\alpha,\ell}$.*

Proof. Otherwise $\exists \alpha > 0$ and $(\gamma_i,s_i) \in \mathrm{Im}\, B : |\gamma_i(0) - \gamma_i(s_i)| \to 0$, $\alpha \leq \mathcal{L}(\gamma_i)$, and $s_i \in [\alpha/\ell, 1-\alpha/\ell]$. The sequence (γ_i,s_i) has a subsequence converging to $(\gamma,s) \in \mathrm{Im}\, B$ with $\gamma(0) = \gamma(s)$ but $\mathcal{L}(\gamma) \geq \alpha$ and $s \in [\alpha/\ell, 1 - \alpha/\ell]$, which contradicts the fact that $(\gamma,s) \in \mathrm{Im}\, B$, because γ is nonconstant, parametrized proportional to arc length, and γ has a base-point self-intersection at $s \in [\alpha/\ell, 1 - \alpha/\ell]$. □

Lemma 4.7. *Let ℓ and α be as above, and let $\varepsilon < \rho$. The retraction*

$$R_{GH} : U_{GH,\varepsilon} \to F_{[0,1]}$$

is homotopic to a map

$$\widetilde{R_{GH}} : U_{GH,\varepsilon} \to F_{[0,1]}$$

that takes $U_{GH,\varepsilon} \cap V_{\alpha,\ell} \cap \overline{\Lambda} \times I$ to

$$F^{triv} = \{(\zeta *_s \xi, s) \in F_{[0,1]} : \mathcal{L}(\zeta) = 0, \text{ or } \mathcal{L}(\xi) = 0\}.$$

as follows: Suppose $(\gamma, s) \in U_{GH,\varepsilon} \cap V_{\alpha,\ell} \cap \overline{\Lambda} \times I$. Then $\widetilde{R_{GH}}(\gamma, s) = (\zeta *_s \xi, s)$ with $\mathcal{L}(\zeta) = 0$ if $s \in [0, \alpha/\ell)$, with $\mathcal{L}(\xi) = 0$ if $s \in (1 - \alpha/\ell, 1]$, and with $\mathcal{L}(\zeta) = \mathcal{L}(\xi) = 0$ if $\mathcal{L}(\gamma) < \alpha$.

Proof. If $(\gamma, s) \in U_{GH,\varepsilon} \cap \overline{\Lambda} \times I$, its image in $F_{[0,1]}$ is of the form $R_{GH}(\gamma, s) = (\zeta *_s \xi, s)$ where ζ and ξ are the closed loops

$$\begin{aligned} \zeta &= \gamma[(0, s] \star \overline{\gamma_s \gamma_0} \\ \xi &= \overline{\gamma_0 \gamma_s} \star \gamma[(s, 1]. \end{aligned}$$

Assume $(\gamma, s) \in U_{GH,\varepsilon} \cap V_{\alpha,\ell} \cap \overline{\Lambda \times I}$. If $\mathcal{L}(\gamma) < \alpha$, then $\mathcal{R}_{GH}(\gamma, s)$ has length $< 2\alpha = \rho$, and the two loops ζ and ξ can be contracted to their common basepoint along geodesics, yielding two constant loops. If $s \in [0, \alpha/\ell)$ then $\mathcal{L}(\gamma|_{[0,s]}) < \alpha$ (since (γ, s) is optimally parameterized), so γ_1 has length $< 2\alpha = \rho$ and can be contracted to its basepoint along geodesics. Similarly, if $s \in (1-\alpha/\ell, 1]$, then $\mathcal{L}(\gamma|_{[s,1]}) < \alpha$, so γ_2 has length $< 2\alpha = \rho$ and can be contracted to its basepoint. Clearly a continuous retraction can do all these things at once. ☐

Proof of Theorem 4.2. Let $Z \in C_*(\Lambda)$ be a cycle with image in the constant and simple loops, that is with image in $\mathcal{S}$, and fix ℓ large enough so that $Z \in C_*(\mathcal{S}^{\leq \ell})$. Let $\overline{Z}$ be its image under the Anosov map. Pick $0 < \alpha < \rho/2$ with $\alpha/\ell < \frac{1}{3}$. By Lemma 4.6 we can find $\tilde{\varepsilon} > 0$ so that $\mathrm{Im}(\overline{Z} \times I) \cap U_{GH,\tilde{\varepsilon}} \subset V_{\alpha,\ell}$. Let $\varepsilon < \min\{\tilde{\varepsilon}, \frac{\rho}{14}\}$. If we use sufficiently fine simplices on $\overline{Z}$ and I (which will not affect the homology class or the set $\mathrm{Im}(\overline{Z} \times I) \subset \overline{\Lambda} \times I$), on every simplex σ of $\overline{Z} \times I$ we will have

$$s < 1 - \alpha/\ell \quad \text{or} \quad s > \alpha/\ell. \tag{4.3}$$

Recall from (A.2) that we have chosen τ_{GH} so that capping with this class kills any simplex of $\overline{Z} \times I$ that does not lie in $U_{GH,\varepsilon}$. Thus by Lemma 4.6, $\tau_{GH} \cap (\overline{Z} \times I)$ is supported on $U_{GH,\varepsilon} \cap V_{\alpha,\ell}$. It follows from Lemma 4.7 that $\widetilde{R_{GH}}(\tau_{GH} \cap (\overline{Z} \times I))$ lies in F^{triv}. Moreover every simplex in $\widetilde{R_{GH}}(\tau_{GH} \cap (\overline{Z} \times I))$ lies in one of the subsets $M \times \Lambda$ or $\Lambda \times M$ of F^{triv} by (4.3).

To the expression (4.2) for the coproduct, we now apply the identities

$$\begin{aligned} \mathrm{id} &= e_L \quad \text{on } M \times \Lambda \\ \mathrm{id} &= e_R \quad \text{on } \Lambda \times M \end{aligned}$$

Note that, as e_L and e_R commute,

$$(1 - e_{R*})(1 - e_{L*}) = (1 - e_{L*})(1 - e_{R*}).$$

So as a consequence of our constructions, every simplex in $\widetilde{R_{GH}}_*((\overline{Z} \times I) \cap \tau)$ lies in a subset where $1 - e_{L*} = 0$ or $1 - e_{R*} = 0$. Thus

$$(1 - e_{R*})(1 - e_{L*})\widetilde{R_{GH}}((\overline{Z} \times I) \cap \tau) = 0$$

which gives statement (ii) of the theorem when $k = 1$.

Statement (i) in the case $k = 1$ follows from statement (ii) as, if Z consists only of simple and trivial loops, then both Z and ΔZ consists only of trivial loops and those with no self-intersection at the basepoint. This completes the proof in the case $k = 1$.

Finally we consider the case $k > 1$. We need to show that if the image of $Z \in C_*(\Lambda)$ has no non-trivial loop that has an intersection of order $> k$ at the basepoint, then

$$\widehat{\vee}^k[Z] = 0.$$

This follows from a direct generalization of the argument in the case $k = 1$. We have that

$$\begin{aligned}\widehat{\vee}^k Z &= (1 - i_* e_*)^{\times k} \circ cut_{(k)} \circ R_k \left(\tau_k \cap (A \times \Delta^k)\right) \\ &= cut_{(k)} \circ \big((1 - e_k) \circ \cdots \circ (1 - e_0)\big) \circ R_k \left((A \times \Delta^k) \cap \tau_k\right)\end{aligned}$$

where $e_i \colon F_{\Delta^k} \to F_{\Delta^k}$ is the ith "deflating map", which takes a tuple $(\gamma, s_1, \ldots, s_k)$ to the same loop though with $\gamma[s_i, s_{i+1}]$ replaced by the trivial loop at $\gamma(s_i) = \gamma(0)$, where we set $s_0 = 0$ and $s_{k+1} = 1$. Recall here that $F_{\Delta^k} = \{(\gamma, s_1, \ldots, s_k) \in \Lambda \times \Delta^k \mid \gamma(0) = \gamma(s_1) = \cdots = \gamma(s_k)\}$ and the retraction

$$R_k \colon U^k_{GH} \subset \Lambda \times \Delta^k \longrightarrow F_{\Delta^k}$$

is defined just like R_{GH} by adding sticks. Let

$$F^{triv}_{\Delta^k} = \{(\gamma, s_1, \ldots, s_k) \in F_{\Delta^k} \mid \exists 0 \le i \le k \text{ with } \gamma[s_i, s_{i+1}] \text{ constant}\} \subset F_{\Delta^k}.$$

Given $\ell > 0$ and $0 < \alpha < \rho/2$, define

$$V^k_{\alpha,\ell} = \{(\gamma, s_1, \ldots, s_k) \in \overline{\Lambda} \times \Delta^k \mid \mathcal{L}(\gamma) < \alpha, \text{ or } \mathcal{L}(\gamma) < \ell \text{ and } |s_{i+1} - s_i| < \alpha/\ell \text{ for some } 0 \le i \le k\}.$$

Consider $\mathcal{S}_k \subset \Lambda$ the subspace of loops that are either trivial, or have a j-fold intersection at the basepoint with $j \le k$. In particular, $\mathcal{S} = \mathcal{S}_1$. Just as in the case $k = 1$, one can show that if $B \in C_*(\Lambda \times \Delta^k)$ is a (compact, continuous) singular chain with image in $\mathcal{S}^{\le \ell}_k \times \Delta^k \cap \overline{\Lambda} \times \Delta^k$, then there exists $\tilde{\varepsilon} > 0$ so that $\operatorname{Im} B \cap U^k_{GH,\tilde{\varepsilon}} \subset V^k_{\alpha,\ell}$. Also there is a retraction

$$\widetilde{R_k} : U^k_{GH,\varepsilon} \to F_{\Delta^k}$$

that is homotopic to R_k and takes $U^k_{GH,\varepsilon} \cap V^k_{\alpha,\ell} \cap \overline{\Lambda} \times \Delta^k$ to $F^{triv}_{\Delta^k}$.

Now these facts, combined with the fact that the map f_k defined above vanishes on $F^{triv}_{\Delta^k}$ finishes the proof of (ii) in the general case. Statement (i) again follows from statement (ii). □

Finally we give the dual statement to Theorem 4.2.

Theorem 4.8. *The product $\widehat{\circledast}$ and its iterate $\widehat{\circledast}^k$ have the following support property:*

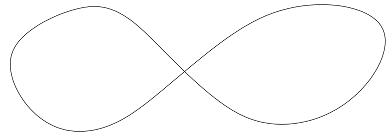
(i) If $[Z] \in H_*(\Lambda)$ has a representative with the property that every nonconstant loop has at most k-fold intersections, and if $[x_1], \ldots, [x_k] \in H^*(\Lambda)$, then

$$\langle [x_1] \widehat{\circledast} [x_2] \widehat{\circledast} \ldots \widehat{\circledast} [x_k], Z \rangle = 0 \text{ and } \langle [x_1] \widehat{\circledast} [x_2] \widehat{\circledast} \ldots \widehat{\circledast} [x_k], \Delta[Z] \rangle = 0.$$

(ii) If $[Z] \in H_*(\Lambda)$ has a representative with the property that every nonconstant loop has at most a k-fold intersection at the basepoint, and if $[x_1], \ldots, [x_k] \in H^*(\Lambda)$, then

$$\langle [x_1] \widehat{\circledast} [x_2] \widehat{\circledast} \ldots \widehat{\circledast} [x_k], [Z] \rangle = 0.$$

Remark 4.9. There is another, well-known "dual" statement to Theorem 4.2, which is concerned with the Chas-Sullivan product instead: If $[X]$ and $[Y]$ are homology classes that have representatives whose evaluations at 0 are disjoint, then their product $[X] \wedge [Y] = 0$. To see the "duality", consider the picture one must see to have a nontrivial product or coproduct: if $\widehat{\vee}[Z] \neq 0$ or $[X] \wedge [Y] \neq 0$, one must see a picture that looks like this:



Proof. The statement is a direct consequence of Theorem 4.2 and Theorem 3.3 (D) together with Proposition 3.5. □

4.1. **Application to spheres and projective spaces.** The iterated coproduct $\widehat{\vee}^k$, via Theorem 4.2, gives us a way to study the following geometric invariant of homology classes in the loop space:

Definition 4.10. Given $[X] \in H_*(\Lambda)$, we define the *intersection multiplicity* of $[X]$ as follows:

$$\operatorname{int}([X]) =: \inf_{\substack{A \in C_*(\Lambda) \\ [A]=[X]}} \sup_{\substack{\gamma \in \operatorname{Im} A \\ \mathcal{L}(\gamma)>0 \\ p \in M}} \#(\gamma^{-1}\{p\}).$$

and the *basepoint intersection multiplicity* of $[X]$ by

$$\operatorname{int}_0([X]) =: \inf_{\substack{A \in C_*(\Lambda) \\ [A]=[X]}} \sup_{\substack{\gamma \in \operatorname{Im} A \\ \mathcal{L}(\gamma)>0}} \#(\gamma^{-1}\{\gamma(0)\}).$$

Note that we are not counting the number of pairs (s,t) with $\gamma(s) = \gamma(t)$, but rather for a fixed point p how many times the loop γ goes through p. A representative A for $[X]$ consisting entirely of piecewise geodesic loops, each with at most N pieces and each piece of length $\leq \rho/2$, and parameterized proportional to arc length, will have finite intersection multiplicities, since a given point p intersects each piece at most once. As such representatives always exist, the intersection multiplicity is necessarily a finite number. Also, by definition and by Theorem 4.2

$$\operatorname{int}([X]) \leq k \implies \operatorname{int}_0([X]) \leq k \implies \widehat{\vee}^k[X] = 0. \tag{4.4}$$

The following result shows that the reverse implications hold in the case of spheres and projective spaces.

Theorem 4.11. *If $M = S^n$, $\mathbb{R}P^n$, $\mathbb{C}P^n$ or $\mathbb{H}P^n$, then for any $[X] \in H_*(\Lambda)$, for any coefficients, and $k \geq 1$,*

$$\operatorname{int}([X]) \leq k \iff \operatorname{int}_0([X]) \leq k \iff \widehat{\vee}^k[X] = 0. \tag{4.5}$$

Thus for these manifolds the vanishing of the iterated coproduct is a perfect predictor of the intersection multiplicity of each homology class $[X]$, except that it cannot be used to distinguish between intersection multiplicity 0 and 1: we have that $\widehat{\vee}^1 = 0$ on both $H_*(M)$ (where $\operatorname{int} = \operatorname{int}_0 = 0$ by definition) and on $H_*(\Lambda^{\leq \ell}) - H_*(M)$, (where $\operatorname{int} = \operatorname{int}_0 = 1$).

We will show below that (4.5) follows from the following two hypotheses:

(A) M is a compact Riemannian manifold all of whose geodesics are closed and of the same minimal period ℓ, and
(B) $H_*(\Lambda^{\leq \ell})$ is supported on the union of the simple and the constant loops.

Ultimately the proposition is a topological statement, so (4.5) will also follow if M is a compact smooth manifold that carries a metric satisfying (A) and (B).

Lemma 4.12. *Properties (A) and (B) hold for spheres and projective spaces with their standard metric.*

Proof. The standard metric on a sphere or projective space has the property (A) [4, 3.31] We will show that (B) is also satisfied for the standard metric on these spaces. Recall that a circle on the sphere $S^n \subset \mathbb{R}^{n+1}$ is a nonempty intersection of S^n with a 2-plane, considered here as parameterized injectively and proportional to arc length on S^1. A circle on a projective space $\mathbb{K}P^n$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, is a circle on a standard real, (round) sphere S^q in $\mathbb{K}P^n$, where $q = \dim_{\mathbb{R}} \mathbb{K}$. Thus

$$\Theta =: \{\text{circles}\} \subset \Lambda^{\leq \ell}$$

for ℓ the minimal period of geodesics. A circle is *great* if the plane goes through the origin in $\mathbb{R}^{n+1}$ (resp. $\mathbb{R}^{q+1}$). The simple closed geodesics on M are precisely the (optimally parameterized) great circles (see [21] for the case of projective spaces); they constitute a Morse-Bott nondegenerate submanifold for the energy function on Λ [36, 29]. Each circle is simple or trivial and has length $\leq \ell$, so it is enough to show that $H_*(\Lambda^{\leq \ell})$ is supported on Θ. Because Θ is an embedded submanifold, lying below level ℓ except along the critical submanifold of great circles, and of dimension equal to the sum of the index ($n-1$, resp. $q-1$) and the dimension of the space of great circles ($2n-1$, resp. $2q-1$), we have an isomorphism, for any coefficients,

$$H_*(\Theta, \Theta^{<\ell}) \xrightarrow{\cong} H_*(\Lambda, \Lambda^{<\ell})$$

by [18, Thm D2]. We also have

$$H_*(\Theta^{=0}) \xrightarrow{\cong} H_*(M)$$

Putting these together, using the fact there are no critical points with length in $(0, \ell)$, we see that the inclusion of Θ in Λ induces an isomorphism

$$H_*(\Theta) \xrightarrow{\cong} H_*(\Lambda^{\leq \ell})$$

which gives (B). □

Proof of Theorem 4.11. Let M be a sphere or projective space. By the lemma we may assume that M satisfies properties (A) and (B). We will derive (4.5) from these two properties. By Theorem 4.2, we only need to check the implications

$$\widehat{\vee}^k[X] = 0 \implies \operatorname{int}_0([X]) \leq k \text{ and } \operatorname{int}([X]) \leq k.$$

So assume $k \geq 1$ and $\widehat{\vee}^k[X] = 0$. We claim that $[X]$ is supported on $\Lambda^{\leq k\ell}$. Assume otherwise; then the image of $[X]$ in $H_*(\Lambda, \Lambda^{\leq k\ell})$ is nonzero. Let k be a field so that the image of $[X]$ in $H_*(\Lambda, \Lambda^{\leq k\ell}; k)$ is nonzero, and let $[x] \in H^*(\Lambda, \Lambda^{\leq k\ell}; k)$ be so that the Kronecker product satisfies

$$\langle [x], [X] \rangle \neq 0.$$

By [18, Thm 14.2, Cor 14.8], the class $[x]$ is a finite sum of terms of the form $[x_1] \circledast \cdots \circledast [x_j] = [x_1] \widehat{\circledast} \cdots \widehat{\circledast} [x_j]$, where $[x_i] \in H^*(\Lambda^{\leq \ell}, M; F)$ and where $j > k$. But then for some term we have

$$0 \neq \langle [x_1] \widehat{\circledast} \cdots \widehat{\circledast} [x_j], [X] \rangle = \langle [x_1] \otimes \cdots \otimes [x_j], \widehat{\vee}^{j-1}[X] \rangle$$

which is a contradiction since $j > k$ and $\widehat{\vee}^k[X] = 0$. So $[X]$ is supported on $\Lambda^{\leq k\ell}$.

By [18, Prop 5.3, Thm 13.4], the Chas Sullivan ring is generated by $H_*(\Lambda^{\leq \ell})$. Moreover, $H_*(\Lambda^{\leq k\ell})$, $k > 1$, is supported on

$$\mathbf{\Theta}^{\wedge k} = \{\gamma_1 \star \gamma_2 \star \cdots \star \gamma_k \mid \gamma_i \in \Theta \text{ and } \gamma_1(0) = \cdots = \gamma_k(0)\}.$$

It follows that X has a representative supported in $\mathbf{\Theta^{\wedge k}}$, and thus

$$\operatorname{int}(X) \leq \operatorname{int}(\mathbf{\Theta}^{\wedge k}) =: \sup_{\substack{\gamma \in \mathbf{\Theta}^{\wedge k} \\ \ell(\gamma) > 0 \\ p \in M}} \#(\gamma^{-1}\{p\}).$$

The reader can check that if $\gamma \in \mathbf{\Theta}^{\wedge k}$, and γ is not constant, then γ is the $\star$-concatenation of j nontrivial circles, for some j with $1 \leq j \leq k$, and thus that $\gamma^{-1}\{p\}$ consists of at most k points, so $\operatorname{int}(\mathbf{\Theta}^{\wedge k}) = k$. As the same holds for int_0, this proves the proposition. □

Remark 4.13. If M is a sphere or projective space, the space of circles

$$\cup_{k \geq 1} \mathbf{\Theta}^{\wedge k} \subset \Lambda M$$

is a small simple space carrying the topology of ΛM in the sense described above. Unfortunately it is not invariant under the natural S^1 action on ΛM. For an equivariant model, see [7].

Example 4.14 (Theorem 4.11 in the case of odd dimensional spheres)**.** Let $M = S^n$ with n odd and greater than 2. We use again the notation from Section 3.3 so that

$$H_*(\Lambda S^n) = \wedge(A) \otimes \mathbb{Z}[U];$$

thus every homology class is a multiple of $A \wedge U^{\wedge k}$, $k \geq 0$ (in dimension $k(n-1)$), or a multiple of $U^{\wedge k}$, $k \geq 0$ (in dimension $(k+1)n-k$). In the metric in which all geodesics are closed with minimal length ℓ, the homology classes $A \wedge U^{\wedge(2m-1)}$, $A \wedge U^{\wedge 2m}$, $U^{\wedge(2m-1)}$, and $U^{\wedge 2m}$ are "at level m", that is, they are supported on $\Lambda^{\leq m\ell}$ (but not on $\Lambda^{(m-1)\ell}$); see [18, 15.2 and Fig 9]. Thus, by the proof of Theorem 4.11, they are also supported on the cycle $\mathbf{\Theta}^{\wedge m}$, the space of all concatenations of

m circles with a common basepoint. It follows that they have representatives with at most m-fold intersections, and $\vee^m$ should annihilate each of these classes. Using Proposition 3.14 we compute: We have

$$\vee^{m-1}(A \wedge U^{\wedge 2m-1}) = (A \wedge U)^{\times m} \neq 0$$

but

$$\vee^m(A \wedge U^{\wedge 2m-1}) = 0.$$

Also

$$\vee^{m-1}(A \wedge U^{\wedge 2m}) = \Sigma_{j=0}^{m-1}(A \wedge U)^{\times(m-(j+1))} \times (A \wedge U^{\wedge 2}) \times (A \wedge U)^{\times j} \neq 0$$

but

$$\vee^m(A \wedge U^{\wedge k}) = 0.$$

The reader can likewise check that $\vee^{m-1}(U^{\wedge(2m-1)}) \neq 0$ and $\vee^{m-1}(U^{\wedge 2m}) \neq 0$ but $\vee^m(U^{\wedge 2m-1}) = \vee^m(U^{\wedge 2m}) = 0$. Thus for each of the four non-trivial homology classes Z at level m, we have

$$\vee^m Z = 0$$

which confirms (4.4), and also

$$\vee^{m-1} Z \neq 0.$$

It follows that

$$\vee^m Z = 0 \iff Z \text{ is supported on } \Lambda^{\leq m\ell} \iff \operatorname{int}([Z]) \leq m.$$

This confirms Theorem 4.11 in the case of an odd sphere S^n, $n > 2$. Note that the level of a homology class is not monotone in the degree; the class of lowest degree at level $m+1$ is in a lower degree than the class in higest degree in level m: $A \wedge U^{\wedge(2m+1)}$ is in dimension $(2m+1)(n-1)$ and level $m+1$ but $U^{\wedge 2m}$ is in dimension $(2m+1)(n-1)+1$ and level m.

5. Degree 1 maps and homotopy invariance

We consider here invariance of the product and coproduct under degree one maps $f : M_1 \to M_2$ between two closed oriented manifolds M_1 and M_2, that is continuous maps such that $f([M_1]) = [M_2]$. Note that a homotopy equivalence necessarily maps $[M_1]$ to $\pm[M_2]$, and so defines a degree 1 map if choose the orientations of M_1 and M_2 to be compatible under f. The map f induces a map $\Lambda f : \Lambda M_1 \to \Lambda M_2$, which is again a homotopy equivalence if f was one. We will show here that Λf respects both the product and the coproduct if f is a degree one map. This generalizes and extends an earlier result of Cohen-Klein-Sullivan, and later Gruher-Salvatore and Crabb, who show that Λf gives an isomorphism of algebras under the Chas-Sullivan product when f is a homotopy equivalence (see [13, Thm 1],[19, Prop 23],[14, Thm 3.7]). Also, in [21], the first author proved the invariance under Λf^* of the cohomology product restricted to the based loop space, under the assumption that M_2 is simply-connected.

Theorem 5.1. *Let $f : M_1 \to M_2$ be a degree 1 map between closed oriented manifolds. Then $\Lambda f : \Lambda M_1 \to \Lambda M_2$ respects both the product and the coproduct. More precisely, the following diagrams commute:*

$$\begin{array}{ccc} H_*(\Lambda M_1 \times \Lambda M_1) & \xrightarrow{\wedge^\times} & H_{*-n}(\Lambda M_1) \\ \downarrow{\scriptstyle \Lambda f \times \Lambda f} & & \downarrow{\scriptstyle \Lambda f} \\ H_*(\Lambda M_2 \times \Lambda M_2) & \xrightarrow{\wedge^\times} & H_{*-n}(\Lambda M_2) \end{array} \qquad \begin{array}{ccc} H_*(\Lambda M_1) & \xrightarrow{\hat{\vee}} & H_{*+1-n}(\Lambda M_1 \times \Lambda M_1) \\ \downarrow{\scriptstyle \Lambda f} & & \downarrow{\scriptstyle \Lambda f \times \Lambda f} \\ H_*(\Lambda M_2) & \xrightarrow{\hat{\vee}} & H_{*+1-n}(\Lambda M_2 \times \Lambda M_2) \end{array}$$

In particular, if f is a homotopy equivalence, then the map Λf induces an isomorphism $H_*(\Lambda M_1) \xrightarrow{\cong} H_*(\Lambda M_2)$ of algebra and coalgebra over the Chas-Sullivan product and extended coproduct.

Remark 5.2. We have stated the result using the pre-Alexander-Whitney/Kunneth isomorphism for the product and coproducts, using in particular the "algebraic short" Chas-Sullivan product $\wedge^\times = (-1)^{np+n} \wedge^\times_{Th}$, for $\wedge^\times_{Th}$ as in section 3.2. As both the Alexander-Whitney map and the Kunneth isomorphisms are natural, the invariance immediately follows for composition with such maps, yielding the invariance of the product as a map $H_*(\Lambda) \otimes H_*(\Lambda) \to H_*(\Lambda)$ and of the coproduct as a map $H_*(\Lambda) \to H_*(\Lambda) \otimes H_*(\Lambda)$ under the suitable assumptions for the map to be defined in the latter case.

For the rest of the section, we will write $\Lambda_i := \Lambda M_i$ for the free loop space of M_i, for $i = 1, 2$. We first show that the theorem holds for the lifted homology coproduct if and only if it holds for the unlifted one.

Lemma 5.3. *Let $f : M_1 \to M_2$ be a continuous map. Then the map $\Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2)$ respects the lifted coproduct $\widehat{\vee}$, in the sense of Theorem 5.1, if and only if the map $\Lambda f_* : H_*(\Lambda_1, M_1) \to H_*(\Lambda_2, M_2)$ respects the unlifted coproduct*

$$H_*(\Lambda_i, M_i) \xrightarrow{\vee} H_*(\Lambda_i \times \Lambda_i, M_i \times \Lambda_i \cup \Lambda_i \times M_i).$$

Proof. Recall from Section 3 that the maps $M_i \xrightarrow{i} \Lambda_i \xrightarrow{e} M_i$ including M_i as the constant loops and evaluating at 0 define a canonical splitting $H_*(\Lambda_i) \cong H_*(M_i) \oplus H_*(\Lambda_i, M_i)$, and the lifted coproduct is the extension by 0 on $H_*(M)$ of the unlifted coproduct. (See Theorem 3.3.) The result follows from the fact that a continuous map $f : M_1 \to M_2$ induces a map $\Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2)$ that respects the splitting as it commutes with both the maps i and e. □

We will for the proof of Theorem 5.1 use the unlifted coproduct.

Any continuous map $f : M_1 \to M_2$ between compact manifolds is homotopy equivalent to a Lipschitz map (because it is homotopy equivalent to a smooth map and the manifolds are compact). So we may assume that f is Lipschitz for some constant K. As before, let

$$U_{M_i} = U_{M_i,\varepsilon} := \{(m_1, m_2) \mid |m_1 - m_2| < \varepsilon\} \subset M_i \times M_i$$

denote the ε–neighborhood of the diagonal in $M_i \times M_i$. Note that such a Lipschitz map f takes $U_{M_1,\varepsilon}$ to $U_{M_2,K\varepsilon}$. We choose ε so that $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = K\varepsilon$ are both smaller than $\min(\frac{\rho_1}{14}, \frac{\rho_2}{14})$, for ρ_i the injectivity radius of M_i.

As f is of degree 1, the manifolds M_1 and M_2 are necessarily of the same dimension n. For $i = 1, 2$, let $\tau_{M_i} \in C^n(U_{M_i}, U^c_{\varepsilon_0})$ be the pull-back of the Thom class of TM_i as in Section 1.3. Here $\varepsilon_0 < \varepsilon_i$ depends on i just like $\varepsilon = \varepsilon_i$ though we will suppress this from the notation: we choose $\varepsilon_{1,0} < \varepsilon_1$ and set $\varepsilon_{2,0} = K\varepsilon_{1,0}$.

Recall that the coproduct is defined as the composition

$$C_*(\Lambda_i) \xrightarrow{\times I} C_{*+1}(\Lambda_i \times I) \xrightarrow{\tau_{GH,i}\cap} C_{*+1-n}(U_{GH,i}) \xrightarrow{R_i} C_{*+1-n}(\mathcal{F}_i) \xrightarrow{cut} C_{*+1-n}(\Lambda_i \times \Lambda_i)$$

in appropriate relative chains, where

$$U_{GH,i} := e_I^{-1}(U_{M_i}) = \{(\gamma, s) \mid |\gamma(0) - \gamma(s)| < \varepsilon_i\} \subset \Lambda_i \times I$$

is a neighborhood of $\mathcal{F}_i = \{(\gamma, t) \in \Lambda_i \times I \mid \gamma(0) = \gamma(t)\}$, and

$$\tau_{GH,i} = ev_I^* \tau_{M_i} \in C^n(U_{GH,i}, U^c_{\varepsilon_0})$$

the Thom class τ_{GH} for M_i. By our assumption on f, we have that $(\Lambda f \times I)$ takes the triple $(\Lambda_1 \times I, U_{GH,1}, U_{\varepsilon_0})$ to $(\Lambda_2 \times I, U_{GH,2}, U_{\varepsilon_0})$. Applying the map f we thus get a diagram

$$\begin{array}{ccccccccc} C_*(\Lambda_1) & \xrightarrow{\times I} & C_{*+1}(\Lambda_1 \times I) & \xrightarrow{\tau_{GH,1}\cap} & C_{*+1-n}(U_{GH,1}) & \xrightarrow{R_1} & C_{*+1-n}(F_1) & \xrightarrow{cut} & C_*(\Lambda_1 \times \Lambda_1) \\ \downarrow{\scriptstyle \Lambda f} & (1) & \downarrow{\scriptstyle \Lambda f \times I} & (2) & \downarrow{\scriptstyle \Lambda f \times I} & (3) & \downarrow{\scriptstyle \Lambda f \times I} & (4) & \downarrow{\scriptstyle \Lambda f \times \Lambda f} \\ C_*(\Lambda_2) & \xrightarrow{\times I} & C_{*+1}(\Lambda_2 \times I) & \xrightarrow{\tau_{GH,2}\cap} & C_{*+1-n}(U_{GH,2}) & \xrightarrow{R_2} & C_{*+1-n}(F_2) & \xrightarrow{cut} & C_*(\Lambda_2 \times \Lambda_2) \end{array} \tag{5.1}$$

and a corresponding diagram in relative chains. The first and last squares will easily be seen to commute, and we will show that the second and third commute in homology, square (2) using that f is a degree 1 map and square (3) using that f is Lipschitz. We start by showing the latter commutativity.

Lemma 5.4. *Suppose that $f : M_1 \to M_2$ is a Lipschitz map. Then the square*

$$\begin{array}{ccc} U_{GH,1} & \xrightarrow{R_1} & F_1 \\ {\scriptstyle \Lambda f \times I}\downarrow & & \downarrow{\scriptstyle \Lambda f \times I} \\ U_{GH,2} & \xrightarrow{R_2} & F_2 \end{array}$$

commutes up to homotopy, where $R_i : U_{GH,i} \to F_i$ is the map R_{GH} of Section 1.5 for the manifold M_i. Moreover, the homotopy is fiberwise over s and leaves the constant loops fixed. In particular, it will give a chain homotopy commutative square for the chains relative to constant loops and to the subspace $\Lambda_i \times \{0,1\}$, corresponding to $s = 0, 1$, i.e. relative to the subspaces

$$M_1 \times I \cup \Lambda_1 \times \{0,1\} \ \subset \ U_{GH,1} \quad \text{and} \quad M_2 \times I \cup \Lambda_2 \times \{0,1\} \ \subset \ F_2.$$

Proof. As before, we let $\gamma_0 = \gamma(0)$ denote the basepoint of a loop γ. Going via the top of the diagram, the composition takes an element (γ, s) to (γ', s) where

$$\gamma' = f(\gamma[0,s]) * f\big(\overline{\gamma(s), \gamma_0}\big) *_s f\big(\overline{\gamma_0, \gamma(s)}\big) * f(\gamma[s,1])$$

with $\overline{\gamma(s), \gamma_0}$ the geodesic from $\gamma(s)$ to γ_0 in M_1. Going along the bottom of the diagram we get instead (γ'', s) with

$$\gamma'' = f(\gamma[0,s]) * \overline{f(\gamma(s)), f(\gamma_0)} *_s \overline{f(\gamma_0), f(\gamma(s))} * f(\gamma[s,1]),$$

where $\overline{f(\gamma(s)), f(\gamma_0)}$ is now the geodesic in M_2 from $f(\gamma(s))$ to $f(\gamma_0)$. Now these loops only differ on the paths $f\big(\overline{\gamma(t), \gamma_0}\big)$ and $\overline{f(\gamma(t)), f(\gamma_0)}$ from $f(\gamma(t))$ to $f(\gamma_0)$. Because f is Lipschitz, both of these paths lie in the ball of radius ε_2 around $f(\gamma_0)$ in M_2, and thus are homotopic, fixing their endpoints, through linear interpolation in $T_{f(\gamma_0)} M_2$. So γ' and γ'' are homotopic paths inside F_2. As this linear interpolation is continuous, this gives the required homotopy. And note that the homotopy leaves the constant loops invariant, and is fiberwise over s, as required. ☐

Next we will reduce the commutativity of square (2) in diagram (5.1) to the commutativity of an analogous square in manifolds. We will apply the naturality of the cap product (see (A.1) in several ways to triples (X, X_0, X_1) mostly of the form $(U_\varepsilon, U^c_{\varepsilon_0}, U^0)$ for $U_\varepsilon = U_{GH,i}$ or U_{M_i} as above, with U_{ε_0} the corresponding subspace, and $U^0 = U^0_i = M_i \times I \cup \Lambda_i \times \{0,1\} \subset U_i$ or $M_i \subset U_{M_i}$. Note that evaluating at 0 and s defines a map of triples

$$e_I : (U_{GH,i}, U^c_{\varepsilon_0}, U^0_i) \longrightarrow (U_{M_i}, U^c_{M_i,\varepsilon_0}, \Delta M_i) = (U_{M_i}, U^c_{M_i,\varepsilon_0}, M_i)$$

while f defines a map of triples

$$\Lambda f \times I : (U_{GH,1}, U^c_{\varepsilon_0}, U^0_1) \longrightarrow (U_{GH,2}, U^c_{\varepsilon_0}, U^0_2).$$

We put all these spaces and maps together to decompose the second square in diagram (5.1) and show that it commutes in homology:

Proposition 5.5. *Let $f\colon M_1 \to M_2$ be a degree 1 map. Then the following diagram commutes*

$$\begin{array}{ccccccc}
H_*(U_{GH,1}, U^c_{\varepsilon_0} \cup U^0_1) & & & \xrightarrow{\tau_{GH,1}\cap} & & & H_{*-n}(U_{GH,1}, U^0_1) \\
& \searrow^{e_I} & & & & \swarrow^{e_I} & \\
& & H_*(U_{M_1}, U^c_{M_1,\varepsilon_0} \cup M_1) & \xrightarrow{\tau_{M_1}\cap} & H_{*-n}(U_{M_1}, M_1) & & \\
{\scriptstyle \Lambda f \times I}\downarrow & & {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle \Lambda f \times I} \\
& & H_*(U_{M_2}, U^c_{M_2,\varepsilon_0} \cup M_2) & \xrightarrow{\tau_{M_2}\cap} & H_{*-n}(U_{M_2}, M_2) & & \\
& \nearrow^{e_I} & & & & \nwarrow^{e_I} & \\
H_*(U_{GH,2}, U^c_{\varepsilon_0} \cup U^0_2) & & & \xrightarrow{\tau_{GH,2}\cap} & & & H_{*-n}(U_{GH,2}, U^0_2).
\end{array}$$

The main ingredient for proving the proposition is the following lemma, which will allow to prove that the middle square in the above diagram commutes:

Lemma 5.6. *Let $f : M_1 \to M_2$ be a degree d map. Then the classes $d\,\tau_{M_1}$ and $(f \times f)^*\tau_{M_2}$ are equal in $H^n(U_{M_1}, U^c_{M_1,\varepsilon_0})$.*

Proof. This will follow from analysing the following diagram:

$$\begin{array}{ccccccc}
d[M_1] \in & H_d(M_1) & \xrightarrow{i_*} & H_d(DTM_1) & \xleftarrow{\cap[DTM_1]} & H^d(DTM_1, D_{\varepsilon_{1,0}}TM_1^c) & \ni d\,u_{M_1} \\
& \| & & \uparrow{\scriptstyle \kappa_{1*}} & & \downarrow{\scriptstyle \kappa_1^*} & \\
& H_d(M_1) & \xrightarrow{\Delta} & H_d(U_{M_1}) & \xleftarrow{\cap[U_{M_1}]} & H^d(U_{M_1}, U^c_{M_1,\varepsilon_0}) & \ni d\,\tau_{M_1} \\
& \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times f} & & \uparrow{\scriptstyle (f\times f)^*} & \\
& H_d(M_2) & \xrightarrow{\Delta} & H_d(U_{M_2}) & \xleftarrow{\cap\, d^2[U_{M_2}]} & H^d(U_{M_2}, U^c_{M_2,\varepsilon_0}) & \ni \tau_{M_2} \\
& \| & & \downarrow{\scriptstyle \kappa_{2*}} & & \uparrow{\scriptstyle \kappa_2^*} & \\
d^2[M_2] \in & H_d(M_2) & \xrightarrow{i_*} & H_d(DTM_2) & \xleftarrow{\cap\, d^2[DTM_2]} & H^d(DTM_2, D_{\varepsilon_{2,0}}TM_2^c) & \ni u_{M_2}
\end{array}$$

The squares in the left column commute because they come from commutative squares in spaces: the map i is the inclusion of the zero section, and the maps κ_1 and κ_2 take that zero section to the diagonal in $U_{M_i} \subset M_i \times M_i$. For the right column, we first note that

$$(\kappa_i)_*([U_{M_i}]) = [DTM_i] \in H_{2d}(DTM_i, STM_i)$$

as κ_i is the "inverse" of an embedding. Also,

$$(f_* \times f_*)[M_1 \times M_1] = d^2[M_2 \times M_2] \in H_{2d}(M_2 \times M_2, U^c_{M_2,\varepsilon_0})$$

as f is a degree d map, and hence, after excision we get

$$(f_* \times f_*)[U_{M_1}] = d^2[U_{M_2}] \in H_{2d}(U_{M_2}, U^c_{M_2,\varepsilon_0}).$$

This gives the commutativity of the squares in the right column, using the naturality of the cap product (A.1).

Now d times the fundamental class of M_1 goes to d times the Thom class u_{M_1} in the top right corner of the diagram (see eg. [6, Def VI.11.1]), which maps to $d\,\tau_{M_1}$ under κ_1^* by definition of the latter. On the other hand it goes to $d^2\,[M_2]$ in the bottom left corner as f has degree d, and $d^2[M_2]$ goes to u_{M_2} in the bottom right corner which likewise maps to τ_{M_2}. The result then follows from the commutativity of the diagram. □

Proof of Proposition 5.5. We first check that the four squares around the middle square commute in homology. For the left and right square, this is because they come from a commutative square in spaces, while for the top and bottom, this is by the naturality of the cap product (A.1). Finally the middle square commutes by the naturality of the cap product (A.1) using Lemma 5.6. □

Proof of Theorem 5.1 part 1. We are now ready to prove that the map Λf induced by a degree one map f preserves the coproduct. We need to show that diagram (5.1) commutes in relative homology. We write each of the four squares, one at a time, and prove that they commute.

The first the square

$$\begin{array}{ccc}
H_*(\Lambda_1, M_1) & \xrightarrow{-\times I} & H_{*+1}(\Lambda_1 \times I, U_1^0) \\
\downarrow{\scriptstyle \Lambda f} & & \downarrow{\scriptstyle \Lambda f \times I} \\
H_*(\Lambda_2, M_2) & \xrightarrow{-\times I} & H_{*+1}(\Lambda_2 \times I, U_2^0)
\end{array}$$

commutes as the diagram commutes on the level of chains. The second square expands to the diagram

$$\begin{array}{ccccc} H_*(\Lambda_1 \times I, U_1^0) & \longrightarrow & H_*(U_{GH,1}, U_1^0 \cup U_{\varepsilon_0}^c) & \xrightarrow{\tau_{GH,1}\cap} & H_{*-n}(U_{GH,1}, U_1^0) \\ \downarrow{\scriptstyle \Lambda f \times I} & & \downarrow{\scriptstyle \Lambda f \times I} & & \downarrow{\scriptstyle \Lambda f \times I} \\ H_*(\Lambda_2 \times I, U_2^0) & \longrightarrow & H_*(U_{GH,2}, U_2^0 \cup U_{\varepsilon_0}^c) & \xrightarrow{\tau_{GH,2}\cap} & H_{*-n}(U_{GH,2}, U_2^0) \end{array}$$

where the first horizontal maps are projections followed by excision (as in equation (A.1)) and the commutativity of the first square comes from the naturality of these maps, while the second square commutes by Proposition 5.5. The third square in diagram (5.1) in relative homology is the square

$$\begin{array}{ccc} H_*(U_{GH,1}, U_1^0) & \xrightarrow{R_1} & H_*(F_1, F_1^0) \\ \downarrow{\scriptstyle \Lambda f \times I} & & \downarrow{\scriptstyle \Lambda f \times I} \\ H_*(U_{GH,2}, U_2^0) & \xrightarrow{R_2} & H_*(F_2, F_2^0) \end{array}$$

which is commutative by Lemma 5.4. Finally the last square

$$\begin{array}{ccc} H_*(F_1, U_1^0) & \xrightarrow{cut} & H_*(\Lambda_1 \times \Lambda_1, M_1 \times \Lambda_1 \cup \Lambda_1 \times M_1) \\ \downarrow{\scriptstyle \Lambda f \times I} & & \downarrow{\scriptstyle \Lambda f \times \Lambda f} \\ H_*(F_2, U_2^0) & \xrightarrow{cut} & H_*(\Lambda_2 \times \Lambda_2, M_2 \times \Lambda_2 \cup \Lambda_2 \times M_2) \end{array}$$

commutes because it comes from a commutative diagram in spaces. □

To finish the proof of Theorem 5.1, we are left to show that a degree one map also preserves the Chas-Sullivan product on the loop spaces. We write

$$U_{CS,i} = \{(\gamma, \delta) \in \Lambda_i \times \Lambda_i \mid |\gamma_0 - \delta_0| < \varepsilon_i\}$$

with $U_{\varepsilon_0}^c = U_{CS,i} \backslash U_{i,\varepsilon_{0,i}}$ defined analogously. For the product, one needs to consider the diagram

$$\begin{array}{ccccccc} C_*(\Lambda_1 \times \Lambda_1) & \xrightarrow{\tau_{CS,1}\cap} & C_{*+1-n}(U_{CS,1}) & \xrightarrow{R_1} & C_{*+1-n}(\Lambda_1 \times_{M_1} \Lambda_1) & \xrightarrow{concat} & C_*(\Lambda_1) \\ \downarrow{\scriptstyle f} & (1) & \downarrow{\scriptstyle f} & (2) & \downarrow{\scriptstyle f} & (3) & \downarrow{\scriptstyle f} \\ C_*(\Lambda_2 \times \Lambda_2) & \xrightarrow{\tau_{CS,2}\cap} & C_{*+1-n}(U_{CS,2}) & \xrightarrow{R_2} & C_{*+1-n}(\Lambda_2 \times_{M_2} \Lambda_2) & \xrightarrow{concat} & C_*(\Lambda_2) \end{array} \tag{5.2}$$

where $\tau_{CS,i} = (e \times e)^* \tau_{M_i} \in C^n(U_{CS,i}, U_{i,\varepsilon_0}^c)$ and R_1, R_2 now denote the retractions R_{CS} of section 1.4 for M_1 and M_2. To show the commutativity of diagrams (1) and (2), we prove the analogues of Lemma 5.4 and Proposition 5.5 for the product.

Lemma 5.7. *Let $f : M_1 \to M_2$ be a Lipschitz map. Then the diagram*

$$\begin{array}{ccc} U_{CS,1} & \xrightarrow{R_1} & \Lambda_1 \times_{M_1} \Lambda_1 \\ \downarrow{\scriptstyle \Lambda f \times \Lambda f} & & \downarrow{\scriptstyle \Lambda f \times \Lambda f} \\ U_{CS,2} & \xrightarrow{R_1} & \Lambda_2 \times_{M_2} \Lambda_2 \end{array}$$

commutes up to homotopy.

Proof. Going along the top of the diagram, a pair of loops (γ, δ) is taken to the loops

$$(f(\gamma), f(\overline{\gamma_0, \delta_0}) \star f(\delta) \star f(\overline{\delta_0, \gamma_0}))$$

while going along the bottom of the diagram takes it to

$$(f(\gamma), \overline{f(\gamma_0), f(\delta_0)} \star f(\delta) \star \overline{f(\delta_0), f(\gamma_0)})).$$

Now both path $f(\overline{\gamma_0,\delta_0})$ and $\overline{f(\gamma_0),f(\delta_0)}$ live in a ball of radius ε_2 around $f(\gamma_0)$ in M_2, and have the same endpoints. Linear interpolation in $T_{f(\gamma_0)}M_2$ thus defines a homotopy between the two paths. As this homotopy depends continuously on γ and δ, this proves the lemma. $\square$

Lemma 5.8. *Let $f: M_1 \to M_2$ be a degree one map. Then the diagram*

$$\begin{array}{ccc} H_*(U_{CS,1},U^c_{\varepsilon_0}) & \xrightarrow{\tau_{CS,1}\cap} & H_{*-n}(U_{CS,1}) \\ {\scriptstyle \Lambda f\times\Lambda f}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda f\times\Lambda f} \\ H_*(U_{CS,2},U^c_{\varepsilon_0}) & \xrightarrow{\tau_{CS,2}\cap} & H_{*-n}(U_{CS,2}) \end{array}$$

commutes.

Proof. Consider the diagram

$$\begin{array}{ccccc} H_*(U_{CS,1},U^c_{\varepsilon_0}) & & \xrightarrow{\tau_{CS,1}\cap} & & H_{*-n}(U_{CS,1}) \\ \big\downarrow & \searrow^{e\times e} & & \swarrow^{e\times e} & \big\downarrow \\ {\scriptstyle \Lambda f\times\Lambda f} & H_*(U_{M_1},U^c_{M_1,\varepsilon_0}) & \xrightarrow{\tau_{M_1}\cap} & H_{*-n}(U_{M_1}) & {\scriptstyle \Lambda f\times\Lambda f} \\ \big\downarrow & {\scriptstyle f\times f}\big\downarrow & & \big\downarrow{\scriptstyle f\times f} & \big\downarrow \\ \big\downarrow & H_*(U_{M_2},U^c_{M_2,\varepsilon_0}) & \xrightarrow{\tau_{M_2}\cap} & H_{*-n}(U_{M_2}) & \big\downarrow \\ \big\downarrow & \nearrow_{e\times e} & & \nwarrow_{e\times e} & \big\downarrow \\ H_*(U_{CS,2},U^c_{\varepsilon_0}) & & \xrightarrow{\tau_{CS,2}\cap} & & H_{*-n}(U_{CS,2}). \end{array}$$

The middle square is the same as the middel square in Proposition 5.5 except for the fact that it is not relative to the diagonal M_i's. It commutes by the naturality of the cap product (A.1) using again Lemma 5.6. The top and bottom squares commute likewise by the naturality of the cap product, while the left and right square come from commuting squares in spaces. $\square$

Proof of Theorem 5.1 part 2. We are left to show that Λf respects the product when f has degree one, where we assume again without loss of generality that f is Lipschitz. We need to show that diagram (5.2) commutes in homology, which we verify one square at a time. The first square commutes by Lemma 5.8, the second square commutes by Lemma 5.7, and the last square commutes because it comes from a commuting square in spaces. $\square$

Remark 5.9. Applying Lemma 5.6 in the case of a degree d map, one see that the classes $\tau_{GH,1}$ and $\tau_{CS,1}$ need to be replaced by d times themselves for the commutativity fo the diagrams in the proof of Theorem 5.1, which shows that a degree d map more generally takes d times time the product/coproduct in ΛM_1 to the product/coproduct in ΛM_1.

Appendix A. Cap product

Let $X_0, X_1 \subset X$ and $Y_0, Y_1 \subset Y$ be spaces and (possibly empty) subspaces satisfying that $C_*(X_0)+C_*(X_1) = C_*(X_0\cup X_1)$ and the same for Y_0,Y_1. Suppose $f:(X,X_0,X_1)\to(Y,Y_0,Y_1)$ is a continuous map from X to Y taking X_i to Y_i for $i=0,1$. Then the cap product and the map f define maps

$$\begin{array}{ccc} C^q(X,X_0)\otimes C_p(X,X_0\cup X_1) & \xrightarrow{\cap} & C_{p-q}(X,X_1) \\ \big\uparrow{\scriptstyle f^*} \quad\quad {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ C^q(Y,Y_0)\otimes C_p(Y,Y_0\cup Y_1) & \xrightarrow{\cap} & C_{p-q}(Y,Y_1). \end{array}$$

Naturality of the cap product then says that for any $\alpha \in C^q(Y, Y_0)$ and $B \in C_p(X, X_0 \cup X_1)$, we have that

$$\alpha \cap f_*(B) = f_*(f^*(\alpha) \cap B). \tag{A.1}$$

In particular for a map $f : X \to Y$ and a class $\alpha \in C^k(Y)$, a diagram of the form

$$\begin{array}{ccc} C_*(X) & \xrightarrow{f_*} & C_*(Y) \\ {\scriptstyle f^*(\alpha)\cap}\downarrow & & \downarrow{\scriptstyle \alpha\cap} \\ C_{*-k}(X) & \xrightarrow{f_*} & C_{*-k}(Y) \end{array}$$

always commutes.

Slightly more generally, if $D_* \le C_*(X)$ is a subcomplex satisfying that for any $A \in D_p$, and any $q < p$, the restriction $A|_{\sigma_q}$ to the front q–face of σ_p is in D_q. Then the cap product $\cap\colon C^q(X, X_0) \otimes C_p(X, X_0) \to C_{p-q}(X)$ descends to a product

$$\cap\colon C^q(X, X_0) \otimes C_p(X, X_0)/D_p \to C_{p-q}(X)/D_{p-q}$$

where $C_p(X, X_0)/D_p$ is by definition $C_p(X)/(C_p(X_0) + D_p)$. The standard relative cap product described above is the case when $D_* = C_*(X_1)$ with $C_*(X_0)+C_*(X_1) = C_*(X_0\cup X_1)$, an assumption that we now see can be dropped by replacing $C_*(X, X_0 \cup X_1)$ by $C_*(X)/\big(C_*(X_0) + C_*(X_1)\big)$ in the definition of the relative cap product.

In the paper, we will use the standard case, and the case $D_* = C_*(X_1) + C_*(X_2)$ for $X_1, X_2 \subset X$ that do not necessarily satisfy $C_*(X_0) + C_*(X_1) = C_*(X_0 \cup X_1)$. Because this generalized cap product is defined as a quotient of the classical cap product, it has the same naturality properties.

To spaces $U_0 \subset U_1 \subset X$ such that the interiors of U_0^c and U_1 cover X, we can associate the chain complex $C_*^{\mathfrak{U}}(X)$ of *small simplices with respect to* $\mathfrak{U} = \{U_0^c, U_1\}$, whose p–chains are formal sums of maps $\sigma_p \to X$ whose image lies entirely inside either U_0^c or U_1. The inclusion $C_*^{\mathfrak{U}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence, with explicit chain homolopy inverse

$$\rho\colon C_*(X) \longrightarrow C^{\mathfrak{U}}(X)$$

given eg. in [20, Prop 2.21], using subdivisions.

Now given a class $\alpha \in C^k(U_1, U_0^c)$, we will write, by abuse of notation,

$$\alpha \cap = \alpha \cap - \ : C_*(X) \longrightarrow C_{*-k}(U_1)$$

for the following composition of maps

$$C_*(X) \twoheadrightarrow C_*(X, U_0^c) \xrightarrow{\rho} C_*^{\mathfrak{U}}(X, U_0^c) \twoheadrightarrow C_*(U_1, U_0^c) \xrightarrow{\alpha\cap} C_{*-k}(U_1) \tag{A.2}$$

where the middel two maps give an explicit chain inverse for excision, the map ρ being as above and the following map projecting away the simplices not included in U_1.

Given spaces $V_0 \subset V_1 \subset Y$ with the same property, a map $f\colon (X, U_0, U_1) \to (Y, V_0, V_1)$, and a class $\alpha \in C^k(V_1, V_0^c)$, naturality of the cap product extends to give a commutative diagram

$$\begin{array}{ccc} C_*(X) & \xrightarrow{f_*} & C_*(Y) \\ {\scriptstyle f^*(\alpha)\cap}\downarrow & & \downarrow{\scriptstyle \alpha\cap} \\ C_{*-k}(U_1) & \xrightarrow{f_*} & C_{*-k}(V_1) \end{array}$$

and likewise for chains relative to subcomplexes compatible under f.

We give now an example of how such a product will appear in the present paper.

Example A.1. Fix $\varepsilon > \varepsilon_0 > 0$ and let $X = M^2$ with $U_0 = U_{M,\varepsilon_0}$ and $U_1 = U_{M,\varepsilon}$ for

$$U_{M,\varepsilon} = \{(x, y) \in M^2 \mid |x - y| < \varepsilon\} \ \subset \ M^2$$

as above, and the same for ε_0. Recall from Section 1.3 that we have picked a Thom class u_M for our manifold M such that $\tau_M := \kappa_M^* u_m = 0$ on chains supported on the complement of U_{M,ε_0}. In other words, $\tau_M \in C^n(M^2, U^c_{M,\varepsilon_0})$. We will also write

$$\tau_M \in C^n(U_{M,\varepsilon}, U^c_{M,\varepsilon_0})$$

for its restriction to $U_{M,\varepsilon}$. Taking $(X, U_0, U_1) = (M^2, U_{M,\varepsilon_0}, U_{M,\varepsilon})$ and $\alpha = \tau_M$, the sequence of maps (A.2) defines a map

$$\tau_M\cap\colon C_*(M^2) \longrightarrow C_{*-n}(U_{M,\varepsilon}).$$

Consider now also the space $Y = \Lambda^2$, with

$$V_0 = U_{CS,\varepsilon_0} := \{(\gamma,\lambda) \in \Lambda^2 \mid |\gamma(0) - \lambda(0)| < \varepsilon_0\} \;\subset\; \Lambda^2$$

and likewise $V_1 = U_{CS,\varepsilon}$. Evaluating each loop at 0 defines a map

$$e \times e\colon (\Lambda^2, V_0, V_1) \longrightarrow (M^2, U_0, U_1)$$

and pulling back the class τ_M along this map defines a class

$$\tau_{CS} := (e\times e)^* \tau_M \in C^n(\Lambda^2, U^c_{CS,\varepsilon_0})$$

which again can be restricted to $U_{CS,\varepsilon}$. Applying (A.2) to this case gives a map

$$\tau_{CS}\cap\colon C_*(\Lambda^2) \longrightarrow C_*(U_{CS,\varepsilon}).$$

Moreover, the naturality of the construction gives that the diagram

$$\begin{array}{ccc} C_*(\Lambda^2) & \xrightarrow{\ e\times e\ } & C_*(M^2) \\ {\scriptstyle \tau_{CS}\cap}\downarrow & & \downarrow{\scriptstyle \kappa_M^*\tau_M\cap} \\ C_{*-k}(U_{CS,\varepsilon}) & \xrightarrow{\ e\times e\ } & C_{*-k}(U_{M,\varepsilon}). \end{array}$$

Appendix B. Signs in the intersection product

The intersection product $H_*(M)\otimes H_*(M) \to H_{*-n}(M)$ can be defined using Poincaré duality, or using the above Thom collapse map κ_M. (See the diagram below.) There is a sign difference between the two definitions, and only the definition using Poincaré duality directly yields an associative product. If one uses the Thom collapse map, one needs to correct the resulting product by a sign to make it associative. The sign comes from the fact that the following diagram only commutes up to sign:

$$\begin{array}{ccccccc} H_p(M)\otimes H_q(M) & \xrightarrow{\ \times\ } & H_{p+q}(M^2) & \xrightarrow{\ \tau_M\cap\ } & H_{p+q}(U_\varepsilon) & \xleftarrow[\cong]{\ \Delta_*\ } & H_{p+q}(M) \\ {\scriptstyle \cap[M]\otimes\cap[M]}\uparrow & & \uparrow{\scriptstyle \cap[M^2]} & & & & \uparrow{\scriptstyle \cap[M]} \\ H^{n-p}(M)\otimes H^{n-q}(M) & \xrightarrow{\ \times\ } & H^{2n-p-q}(M^2) & & \xrightarrow{\ \Delta^*\ } & & H^{n-p-q}(M). \end{array}$$

Here the intersection product would be obtained by inverting the first vertical map and going along the bottom of the diagram, while the top of the diagram gives an (up to sign) "Thom-Pontrjagin" definition of it. To get the precise sign, we see that the left square commutes up to a sign $(-1)^{n(n-q)} = (-1)^{n-nq}$ as

$$(a\cap[M])\times(b\cap[M]) = (-1)^{n\deg(b)}(a\times b)\cap[M\times M].$$

For the right square we have that τ_M is Poincaré dual to $\Delta_*[M]$ inside M^2, so using the naturality of the cap product we get

$$\begin{aligned} \Delta_*(\Delta^* a\cap[M]) = a\cap\Delta_*[M] &= a\cap(\tau_M\cap[M^2]) = (a\cup\tau_M)\cap[M^2] \\ &= (-1)^{\deg(a)n}(\tau_M\cup a)\cap[M^2] = (-1)^{\deg(a)n}\tau_M\cap(a\cap[M^2]) \end{aligned}$$

for any $a\in H^*(M^2)$. This gives a sign $(-1)^{n(2n-p-q)} = (-1)^{np+nq}$. So the whole diagram commutes up to a sign $(-1)^{n-np}$. (See also [12, Prop 4].)

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