ON THE INVARIANCE OF THE STRING TOPOLOGY COPRODUCT

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Abstract. We give a variant of Naef's formula for the failure of invariance of the string topology coproduct under homotopy equivalences, using an obstruction class build from the higher homotopy data one can associate to a homotopy equivalence as well as the "fake diagonal". The vanishing of our obstruction class can be seen as a way to measure a form of smallness for homotopy equivalences. We show that the same obstruction rules the failure of invariance for a generalisation of the coproduct to higher dimensional loops.

Introduction

For a closed oriented manifold M, the Chas-Sullivan product [4] is a product on the homology of its free loop space $\Lambda M = \mathrm{Maps}(S^1, M)$ of the form

$$\wedge : H_p(\Lambda M) \otimes H_q(\Lambda M) \longrightarrow H_{p+q-n}(\Lambda M),$$

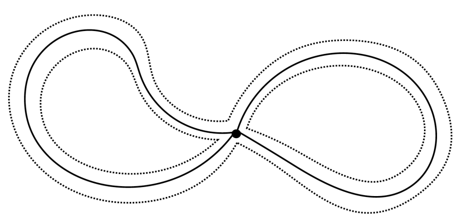
where $H_*(-) = H_*(-;\mathbb{Z})$ denotes singular homology with integral coefficients and $n = \dim M$. Goresky and Hingston defined in [11] a "dual" product in cohomology relative to the constant loops

$$\circledast : H^p(\Lambda M, M) \otimes H^q(\Lambda M, M) \to H^{p+q+n-1}(\Lambda M, M).$$

The associated homology coproduct

$$\vee : H_p(\Lambda M, M) \longrightarrow H_{p-n+1}(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M).^{1}$$

was defined by Sullivan [27]. The idea of the Chas-Sullivan product is to intersect the chains of basepoints of two chains of loops and concatenate the loops at the common basepoints. The homology coproduct looks instead for self-intersections at the basepoint within a single chain of loops and cuts. Both structures are represented geometrically by the same figure, either thought of as two loops intersecting at their basepoint,



or as a single loop having a self-intersection at its basepoint. For the coproduct, it is essential to look for self-intersections with the basepoint at all times t along the loop, including $t = 0$ and $t = 1$ (where all loops tautologically have a self-intersection!)—Tamanoi showed that the coproduct that only looks for self-intersections at $t = \frac{1}{2}$ is trivial [28]. In contrast, $H^*(\Lambda S^n, S^n)$ for an odd sphere S^n is generated by just 4 classes as an algebra with the cohomology product $\circledast$ [11, 15.3]. The domain $[0, 1]$ of the time variable t is the reason why the coproduct naturally is an operation in relative homology: it has non-trivial boundary terms coming from the boundary of the interval. This operation can be associated to a 1-parameter family of surfaces in the harmonic compactification of the moduli space of Riemann surfaces, making it a so-called *compactified operation*.

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[1] As we work with homology with integral coefficients, we do not in general have a Künneth isomorphism, which is why the coproduct in integral homology has target $H_*(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M)$ rather than $H_*(\Lambda M, M) \otimes H_*(\Lambda M, M)$.

This coproduct detects *intersection multiplicity* of homology classes in the loop space, the largest number k such that any representing chain of a given homology class in $H_*(\Lambda M)$ necessarily has a loop with a k-fold self-intersection at its basepoint. (See [14, Thm 3.10].)

It has been shown by several authors that the Chas-Sullivan product is homotopy invariant, in the sense that a homotopy equivalence $f : M_1 \to M_2$ induces an isomorphism $\Lambda f : H_*(\Lambda M_1) \to H_*(\Lambda M_2)$ of algebras with respect to the Chas-Sullivan product (see [7, Thm 1],[12, Prop 23],[8, Thm 3.7]). In contrast, a coproduct of the same flavour was used in [5, Sec 2.1] to define the differential in *string homology*, a knot invariant that detects the unknot. Also the paper [23] found indications of non-homotopy invariance in the S^1–equivariant version of the coproduct. And indeed, more recently Naef showed through a computation with lens spaces that the coproduct is not homotopy invariant [20]! (See also [21, Sec 2.3-4] for an overview and generalization of Naef's example.) Note though that, over the rational and for simply-connected manifolds, Rivera-Wang showed that the Goresky-Hingston coproduct is homotopy invariant using a Hochschild homology model [26].

Naef suggested in [20] the following transformation formula: given a homotopy equivalence $f : M_1 \to M_2$ and writing $\vee_1$ and $\vee_2$ for the string coproducts of M_1 and M_2 respectively, for $A \in H_*(\Lambda M_1)$,

$$\vee_2(\Lambda f_*(A)) - (\Lambda f \times \Lambda f)_*(\vee_1 A) = \mathrm{diag}(\tau_{THH}(f)) \wedge_2 \Lambda f_*(A) - \Lambda f_*(A) \wedge_2 \mathrm{diag}(\tau_{THH}(f))$$

for $\tau_{THH}(f) \in H_1(\Lambda_2, M_2)$ the Dennis trace of the Whitehead torsion of f, mapped via the antidiagonal

$$\mathrm{diag} : \Lambda M_2 \to \Lambda M_2 \times \Lambda M_2$$

taking a loop γ to the pair (γ, γ^{-1}), and where $\wedge_2$ denotes the Chas-Sullivan product in M_2. The formula was recently proved by Naef-Safronov [22], with an alternative proof given by Kenigsberg-Porcelli in [16] when $\pi_2(M) = 0$. We prove here a version of this formula, with potentially different obstruction class:

Theorem A. *Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence between closed oriented manifolds and denote by $\vee_1$ and $\vee_2$ the string coproducts of the manifolds M_1 and M_2. The map induced by $\Lambda f : \Lambda M_1 \to \Lambda M_2$ in homology respects the coproduct, up to an error term given by the following formula: for $A \in H_*(\Lambda M_1)$,*

$$\vee_2(\Lambda f_*(A)) - (\Lambda f \times \Lambda f)_*(\vee_1 A) = \mathrm{diag}(T_f) \wedge_2 \Lambda f_*(A) - \Lambda f_*(A) \wedge_2 \mathrm{diag}(T_f)$$

where $\mathrm{diag}(T_f) \in H_1(\Lambda M_2 \times \Lambda M_2, M_2 \times M_2)$ *is the anti-diagonal of a class* $T_f \in H_1(\Lambda M_2, M_2)$ *defined in Section 4.5.*

The class T_f has two main ingredients:

(1) a "transverse representative" of the *fake diagonal* $\Delta_1^f = \{(m, m') \in M_1 \times M_1 \mid f(m) = f('m)\}$ (see Section 4.6),
(2) a family of paths in M_2 build from a choice of *higher homotopy data* (f, g, h_1, h_2, K) for f, where $g : M_2 \to M_1$ is a homotopy inverse for f, with $g \circ f \simeq_{h_1} \mathrm{id}_{M_1}$ and $f \circ g \simeq_{h_2} \mathrm{id}_{M_2}$ and $K : M_1 \times I \times I \to M_2$ is a homotopy between $h_2 \circ (f \times I)$ and $f \circ h_1$ (see Section 3.5, and in particular Lemma 3.15).

The fake diagonal and the non-triviality of the higher homotopy K can be seen as measurements of how far away a homotopy equivalence is from being a homeomorphism. It follows directly from this description of the obstruction class T_f that is vanishes under a *boundedness* assumption for the homotopy f (see Definition 4.11 and Corollary 4.12) and also when the fake diagonal is "diagonal" (see Corollary 4.7).

Kenigsberg and Porcelli actually also have a version of this formula in [16], with instead an obstruction class coming from the work of Geoghegan-Nicas [10]. This latter obstruction class is known to agree with $\tau_{THH}(f)$ at least when $\pi_2(M) = 0$. It is natural to conjecture that the three obstruction classes appearing in the papers [22, 16] and here all agree, at least when $\pi_2(M) = 0$, but this does not follow directly from the above statement and its analogue in [22, 16]. Such a comparison would result in a new description of the trace of torsion.

We use in this paper the Cohen-Jones approach to string topology [6], with the products and coproducts described above being defined as lifts to the loop space ΛM of the intersection product on $H_*(M)$, using a Thom-Pontryagin construction for the intersection product. In the spirit of e.g.; [12] and [21], we will consider more general *intersection products*: to $\mathcal{R} \to \mathcal{E} \to M \times M$, as [21], we associate a map

$$int_M : C_*(\mathcal{E}, \mathcal{R}) \xrightarrow{int_{U_M}} C_{*-n}(\mathcal{E}|_{U_M}, \mathcal{R}|_{U_M}) \xrightarrow{r} C_{*-n}(\mathcal{E}|_M, \mathcal{R}|_M)$$

where the first map comes from capping with the pull-back of the Thom class of the diagonal embedding $M \hookrightarrow M \times M$, and the second is induced by the retraction $U_M \to M$ of a tubular neighborhood of this embedding onto its 0-section, and is well-defined e.g.; when $\mathcal{E}|_{U_M} \to U_M$ and $\mathcal{R}|_{U_M} \to \mathcal{E}|_{U_M} \to U_M$ are fibrations. See Section 2 for more details. This chosen level of generality for intersection products is forced upon us by the proof of Theorem A, where several "exotic" intersection products appear.

We consider in this paper the question of whether a map $F = (F, F_0) : (\mathcal{E}_1, \mathcal{R}_1) \longrightarrow (\mathcal{E}_2, \mathcal{R}_2)$ preserves the intersection product in two types of situations, as depicted in the following diagram:

$$\begin{tikzcd}
 & \mathcal{R}_1 \arrow[rr, dash, "F_0"] \arrow[dr] \arrow[ddr] & & \mathcal{R}_2 \arrow[dr] \arrow[ddr, dotted] & \\
 & & \mathcal{E}_1 \arrow[rr, "F"] \arrow[d, "p_1"] & & \mathcal{E}_2 \arrow[d, "p_2"] \\
\text{(A)} & & M_1 \times M_2 \arrow[rr, "f\times f"] & & M_2 \times M_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{R}_1 \arrow[rr, dash, "F_0"] \arrow[dr] \arrow[dd] & & \mathcal{R}_2 \arrow[dr] \arrow[dd, dotted] & \\
 & & \mathcal{E}_1 \arrow[rr, "F"] \arrow[dd, "p_1"] & & \mathcal{E}_2 \arrow[dd, "p_2"] \\
 & U_{M_1} \arrow[rr, dotted, dash] \arrow[dr] & & U_{M_2} \arrow[dr] & \\
\text{(B)} & & M_1 \times M_2 \arrow[rr, "f\times f"] & & M_2 \times M_2
\end{tikzcd}$$

In situation (A), we assume that both $\mathcal{E}_i$ and $\mathcal{R}_i$ are fibrations over $M_i \times M_i$, while in situation (B), $\mathcal{R}_i$ is only assumed to be a fibration over a neighborhood U_{M_i} of the diagonal. While the intersection product relevant to Theorem A will be as in situation (B), where we will see that there is an obstruction to invariance, the proof will use that there is no obstruction to homotopy invariance for intersection products in situation (A):

Theorem B. *(Theorem 3.9) Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence and $F : (\mathcal{E}_1, \mathcal{R}_1) \to (\mathcal{E}_2, \mathcal{R}_2)$ a map of pairs of fibrations over the map $f \times f : M_1 \times M_1 \to M_2 \times M_2$ (as situation (A) in the above diagram). Then there is a chain homotopy*

$$F \circ int_{M_1} \simeq_H int_{M_2} \circ F : C_*(\mathcal{E}_1, \mathcal{R}_1) \longrightarrow C_{*-n}(\mathcal{E}_2|_{M_2}, \mathcal{R}_2|_{M_2}).$$

The above result is essentially stated as Theorem 4.11 in [21], where a sketch proof is given. We give here a full proof as it is an important ingredient for the proof of Theorem A.

The idea of the proof of Theorem A is to use the description of the string coproduct as a relative version of the trivial coproduct mentioned above, the coproduct looking only for self-intersections at time $\frac{1}{2}$, see Section 4.1. The trivial coproduct satisfies the assumptions of Theorem B, and thus the failure of invariance of the string coproduct can be described in terms of the failure of the homotopy for the invariance of the trivial coproduct to preserve the relative part. (See Lemma 3.13.) Here the trivial coproduct will be considered relative to (almost) half-constant loops, a situation of type (B) since the restriction of the evaluation map $\mathbf{ev}_{0,\frac{1}{2}} : \Lambda M \to M \times M$ to almost half-constant loops is a fibration over U_M.

For a map of pairs $F : (\mathcal{E}_1, \mathcal{R}_1) \to (\mathcal{E}_2, \mathcal{R}_2)$ over $f \times f$ as in situation (B), Theorem B gives that $F_* : C_*(\mathcal{E}_1) \to C_*(\mathcal{E}_2)$ respects the intersection products up to homotopy, but not necessarily relatively to $\mathcal{R}_1$ and $\mathcal{R}_2$. A first step in the proof of Theorem A is general computation of the failure of $F_* : H_*(\mathcal{E}_1, \mathcal{R}_1) \to H_*(\mathcal{E}_2, \mathcal{R}_2)$ to preserve the intersection products in situtation (B): for $A \in H_*(\mathcal{E}_1, \mathcal{R}_1)$, Theorem 3.14 gives that

$$\big(F \circ int_{M_1} - int_{M_2} \circ F\big)(A) = int_{U_1}(L_{N_2}(\alpha(\partial A) \times I)), \tag{0.1}$$

where $\partial : H_*(\mathcal{E}_1, \mathcal{R}_1) \to H_{*-1}(\mathcal{R}_1)$ is the boundary map in the long exact sequence in homology, α is a map with the property that it takes the diagonal of M_1 to the fake diagonal Δ_1^f (see Proposition 4.6), and L_{N_2} is a homotopy equivalence, whose existence is given by Proposition 3.3, and for which we give an explicit construction in Section 3.5 in the case of mapping spaces using the higher homotopy data (f, g, h_1, h_2, K) described above.

The formula given in Theorem A is obtained from (0.1) by first noting that, in the case of the string coproduct, for a class $A \in H_*(\Lambda M_1)$, the relevant boundary that needs to be considered has the form

$\partial(A \times I) = A \times \partial I$, and can be described in half-constant loops as $[M_1] \wedge_1 A - A \wedge_1 [M_1]$, for $\wedge_1$ the string product of M_1, and $[M_1]$ the fundamental class of M_1 considered as a class of constant loops. One then shows that the operations $[M_1] \wedge_1 -$ and $- \wedge_1 [M_1]$ can be "pulled outside" in the right hand side of (0.1), at the price of replacing the class of constant loops $[M_1]$ with the class $diag(T_f)$. This latter class is essentially the result of applying the sequence of maps on the right hand side of (0.1) to $[M_1]$. The final form of the formula also uses Theorem B a certain relative Chas-Sullivan product, to replace the string product $\wedge_1$ computed in M_1 by the product $\wedge_2$ of M_2.

Higher dimensional loops. One generalization of the string coproduct to higher dimensional loops that fits the set-up of the present paper is as follows: Denoting $\Lambda^r M := \mathrm{Maps}(S^r, M) = \mathrm{Maps}((I^r/\partial I^r), M)$, there is an evaluation map

$$\mathbf{ev_{0,t}} : \Lambda^r M \times I^r \to M \times M$$

evaluating the loops at the basepoint $\mathbf{0}$ and at time $\mathbf{t} = (t_1, \ldots, t_r) \in I^r$. Consider the operation

$$\vee^r : H_*(\Lambda^r M) \xrightarrow{\times I^r} H_{*+r}(\Lambda^r M \times I^r, \Lambda^r M \times \partial I^r) \xrightarrow{int_M} H_{*+r-n}(\mathbf{ev}^{-1}_{\mathbf{0,t}}(\Delta M), \Lambda^r M \times \partial I^r)$$
$$\xrightarrow{\text{reread}} H_{*+r-n}(\mathrm{Maps}(S^1 \times S^{r-1}, M), \mathcal{R}^r)$$

where the last map is a "reread" map, taking a pair $(\gamma, \mathbf{t})$ to the composition

$$S^1 \times S^{r-1} \cong S^1 \times \partial I^r \longrightarrow (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{c}=(\frac{1}{2},\ldots,\frac{1}{2})} \xrightarrow{\theta_{\mathbf{t}}} (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{t}} \xrightarrow{\gamma} M$$

where the first map uses that the line between $\mathbf{t} \in \partial I^r$ and $\mathbf{c}$ is mapped to a circle in the target, and where $\theta_{\mathbf{t}}$ is a reparametrization map defined by piecewise linear scaling in each of the r directions. The reread map takes $\Lambda^r \times \partial I^r$ to the subspace $\mathcal{R}$ of half-constant maps. (See Section 4.2 for more details.)

The above map specializes to the string coproduct $\vee$ when $r = 1$ after appropriately identifying $\mathrm{Maps}(S^1 \times S^0, M)$ with $\Lambda M \times \Lambda M$.

The obstruction class $T_f \in H_1(\Lambda M_2, M_2)$ defines a class $s^{r-1}T_f \in H_1(\mathrm{Maps}(S^1 \times S^{r-1}, M_2), M_2)$ by precomposing with the projection $S^1 \times S^{r-1} \to S^1$ on the first factor. In the case $r = 1$, the class $s^0 T_f$ recovers the anti-diagonal $\mathrm{diag}(T_f)$ appearing in Theorem A.

Theorem A generalizes to the following:

Theorem C. *Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence between closed oriented manifolds and denote by $\vee_1^r$ and $\vee_2^r$ the higher string coproducts of the manifolds M_1 and M_2 defined as above. The map induced by $\Lambda^r f : \Lambda^r M_1 \to \Lambda^r M_2$ in homology respects these coproducts, up to an error term given by the following formula: for $A \in H_*(\Lambda^r M_1)$,*

$$\vee_2^r\big(\Lambda^r f_*(A)\big) \;-\; \mathrm{Maps}(S^1 \times S^{r-1}, f)_*\big(\vee_1^r A\big) = s^{r-1}T_f \wedge^2_{\partial I^r} \Lambda^r f_*(A)$$

in $H_{*+1-n}(\mathrm{Maps}(S^1 \times S^{r-1}, M), \mathcal{R})$, where

$$\wedge^2_{\partial I^r} : H_*\big(\mathrm{Maps}(S^1 \times S^{r-1}, M_2) \times \Lambda^r M_2\big) \xrightarrow{int_{M_2}} H_{*-n}\big(\mathrm{Maps}(S^1 \times S^{r-1}, M_2) \times_{M_2} \Lambda^r M_2\big)$$
$$\xrightarrow{[-,-]} H_{*+r-1-n}\big(\mathrm{Maps}(S^1 \times S^{r-1}, M_2)\big)$$

is an intersection product composed with a form of Browder bracket, that takes value in half-constant maps $\mathcal{R}_2^r$ when the left input is in constant maps (see Section 4.2).

In particular the failure of invariance for such operations is still ruled by the same obstruction class T_f as in the case of single loop spaces ΛM in Theorem A. Both results will be proved together, Theorem A being the case $r = 1$ in Theorem C.

Organization of the paper. In Section 1, we give a proof of the homotopy invariance of the homology intersection product (Proposition 1.7) using only its definition in terms of capping with a Thom class. This proof is not at all efficient, but the intermediate results proved in that section will be used later in the paper. In Section 2, we define lifted intersection products, in the generality needed in the paper. We treat there the invariance property of intersection products lifted from a fixed manifold. Section 3 considers the question of invariance of intersection products over a homotopy equivalence. Theorems 3.9 (Theorem B) treates the case where $\mathcal{E}_i \to M_i \times M_i$ and $\mathcal{R} \to M_i \times M_i$ are fibrations (situation (A)), and Theorem 3.14 the case where the fibration $\mathcal{E}_i \to M_i \times M_i$ restricts on the relative part to a fibration $\mathcal{R}_i \to U_{M_i}$ instead (situation (B)). Section 4.5 then builds on Theorem 3.14 and the explicit homotopy given in Proposition 3.17 to prove Theorems A and C. Finally Appendix A details the relationship between the tubular neighborhoods of two composable embedding and the tubular neighborhood of their composition.

Acknowledgements. This paper originates in a joint work with Nancy Hingston, and the spirit of our join paper [14] is still very much present here. The first version of this paper contained a "proof" of homotopy invariance of the coproduct that was contradicted by Naef's example [20]. The error in the original version of the paper originated in a bad choice of relative term making a suspension "isomorphism" not an isomorphism anymore. We have here corrected that relative term to revive the isomorphism part of the suspension isomorphism, and this makes the less standard map denoted $[\![\bar{e}_I^*\overline{\tau}_e\cap]\!]$ in e.g.; Diagram (3.5) not necessarily an isomorphism, a fact tightly related to the failure of the homotopy called L_{N_2} in e.g. in Theorem 3.14 or Proposition 3.17, to be a relative homotopy equivalence. We thank Florian Naef for discussing his example with us early on, as well as for extended discussions on the topic of invariance over the years. The revision of this paper has also been influenced by exchanges with Alexandru Oancea and Kai Cieliebak, and we thank Maxime Ramzi for helpfull discussions about homotopy equivalences. In the course of this work, the author was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) and the Copenhagen Centre for Geometry and Topology (DNRF151), the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 772960), the Max Planck Institute in Bonn, the Heilbronn Institute for Mathematical Research, and would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Homotopy Harnessing Higher Structures (EPSRC grant numbers EP/K032208/1 and EP/R014604/1).

1. Homotopy invariance of the intersection product

The easiest way to prove the homotopy invariance of the intersection product on the homology of closed manifolds is to use its definition as Poincaré dual of the cup product, and use the homotopy invariance of the cup product. In fact, the compatibility is easily checked with any degree d map, up to appropriate scaling, see e.g., [1, VI, Prop 14.2]. We will present here a proof of the homotopy invariance using instead the definition of the intersection product via capping with a Thom class, as a warm-up, and partial preparation, for the invariance of the string topology coproduct.

Let M be a closed oriented Riemannian manifold of injectivity radius ρ, and let TM denote its ε–tangent bundle, where ε is small compared to ρ. (To be precise, the paper [14] whose constructions we use assumes $\varepsilon < \frac{\rho}{14}$.) We will also assume that $\varepsilon < \frac{1}{2}$.

The normal bundle of the diagonal embedding $M \hookrightarrow M \times M$ is isomorphic to TM, and we can choose as tubular embedding the map $\nu_M : TM \to M \times M$ given by $\nu_M(m, V) = (m, \exp_m V)$. (By an *embedding*, we will in this paper always mean a C^1–map that is an immersion and a homeomorphism onto its image.) Note that ν_M has image the ε–neighborhood of the diagonal

$$U = \{(m, m') \in M \times M \mid |m - m'| < \varepsilon\} \subset M \times M.$$

Let $\tau_M \in C^n(M \times M, (\Delta M)^c)$ be a chosen Thom class for this tubular neighborhood, in cochains relative to the complement of the diagonal $\Delta : M \to M \times M$, and $r\colon U \to M$ the associated retraction. After going through "small simplices" to get a chain inverse ρ to excision, capping with the Thom class defines a map

$$[\tau_M\cap]\colon C_{p+q}(M \times M, (\Delta M)^c) \xrightarrow{\rho} C_*(U, (\Delta M)^c) \xrightarrow{\tau_M\cap} C_{*-n}(U).$$

(See eg., [14, A.2] for a detailed description.)

Up to sign, the intersection product on a closed manifold M can be defined as the composition

$$C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \longrightarrow C_{p+q}(M \times M, (\Delta M)^c) \xrightarrow{[\tau_M\cap]} C_{p+q-n}(U) \xrightarrow{r} C_{p+q-n}(M)$$

(See eg., [14, App B].)

Let $f : M_1 \to M_2$ be a homotopy equivalence between two closed oriented Riemannian manifolds M_1 and M_2. In particular f has degree ± 1, that is $f[M_1] = \pm[M_2]$. Let U_i be the ε_i–tubular neighborhoods of the diagonal in $M_i \times M_i$ as above, with Thom class τ_{M_i} and $r_i : U_i \to M_i$ the associated retraction. After changing f in its homotopy class, we may assume that

- f is smooth;
- ε_1 is small enough so that $f(U_1) \subset U_2$,

where the last property uses that M_1 is assumed closed and hence is compact. Leaving out the cross-product, which is well-known to be natural, we want to show that the diagram

$$\begin{tikzcd}
H_{p+q}(M_1 \times M_1) \arrow[r] \arrow[d, "f\times f"'] & H_{p+q}(M_1 \times M_1, (\Delta M_1)^c) \arrow[r, "{[\tau_{M_1}\cap]}"] & H_{p+q-n}(U_1) \arrow[r, "r_1"] \arrow[d, "f\times f"] & H_{p+q-n}(M_1) \arrow[d, "f"] \\
H_{p+q}(M_2 \times M_2) \arrow[r] & H_{p+q}(M_2 \times M_2, (\Delta M_2)^c) \arrow[r, "{[\tau_{M_2}\cap]}"] & H_{p+q-n}(U_2) \arrow[r, "r_2"] & H_{p+q-n}(M_2)
\end{tikzcd}$$

commutes. One difficulty is that we cannot assume that $f \times f$ takes the complement of the diagonal in $M_1 \times M_1$ to the corresponding complement in $M_2 \times M_2$, so that we do not have a vertical map in the second column in the diagram. For the same reason, we cannot use f directly to compare the Thom classes τ_{M_1} and τ_{M_2}. To make up for this, we will replace f by an embedding and use instead the larger diagram (1.4) below.

Pick an embedding $e : M_1 \hookrightarrow D^k$, for some k large. Then the map $(f, e)\colon M_1 \to M_2 \times D^k$ is an embedding too and we can consider the composition of embeddings

(1.1)
$$\begin{tikzcd}
M_1 \arrow[r, hook, "\Delta_1"] \arrow[dr, hook, "{(\mathrm{id},f,e)}"'] & M_1 \times M_1 \arrow[d, hook, "{id\times(f,e)}"] \\
& M_1 \times M_2 \times D^k.
\end{tikzcd}$$

To this situation, we can associate a diagram

$$\begin{tikzcd}
TM_1 \arrow[d] \arrow[drr, hook, "\nu_1=\nu_{M_1}"] & & N_2 = M_1 \times N(f,e) \arrow[d, "p_{N_2}"'] \arrow[drr, hook, "\nu_2=id\times\nu_e"] & & \\
M_1 \arrow[rr, hook, "\Delta_1"] & & M_1 \times M_1 \arrow[rr, hook, "{id\times(f,e)}"] & & M_1 \times M_2 \times D^k,
\end{tikzcd}$$

where $\nu_1 := \nu_{M_1} : TM_1 \to M_1 \times M_1$ is the tubular neighborhood of ΔM_1 defined above, and $N(f,e)$ is the normal bundle of (f,e) with $\nu_e : N(f,e) \to M_2 \times D^k$ a chosen tubular embedding; we choose here, as in [2, Thm 12.11], to define ν_e using the restriction of the exponential map of $M_2 \times D^k$ to the subbundle $N(f,e) \le TM_2 \oplus \mathbb{R}^k$ of vectors normal to $d(f,e)TM_1$. Pick a Thom class $\tau_e \in C^k(M_2 \times D^k, ((f,e)M_1)^c)$ for ν_e. Then

(1.2)
$$\overline{\tau}_e := 1 \times \tau_e \in C^k(M_1 \times M_2 \times D^k, M_1 \times ((f,e)M_1)^c)$$

is a Thom class of ν_2. Applying Proposition A.1, we now have the following:

Proposition 1.1. *Consider the sequence of embeddings* $M_1 \overset{\Delta_1}{\hookrightarrow} M_1 \times M_1 \overset{id\times(f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$ *and let* ν_1, ν_2 *and* $\tau_1 = \tau_{M_1}, \tau_2 = \overline{\tau}_e$ *be the embeddings and Thom classes chosen above. Then the following hold:*

1. *The bundle* $N_3 := TM_1 \oplus N(f,e)$ *is isomorphic to the normal bundle of the composed embedding* $(id, f, e) : M_1 \to M_1 \times M_2 \times D^k$.
2. *There is an isomorphism* $\hat{\nu}_1 : N_3 = TM_1 \oplus N(f,e) \xrightarrow{\cong} N_2|_{U_1} = (M_1 \times N(f,e))|_{U_1}$*, as bundles over* $TM_1 \cong U_1$*, with the property that*
$$\nu_3 := \nu_2 \circ \hat{\nu}_1 : N_3 \longrightarrow M_1 \times M_2 \times D^k$$
is a tubular embedding for (id, f, e)*. Explicitly,*
$$\nu_3(m, V, W) = \nu_2(\nu_1(m, V), W') = (m, \nu_e(\exp_m V, W'))$$
for $W' \in (N(f,e))_{\nu_1(m,V)}$ *the parallel transport of* W *along the path* $\nu_1(m, tV)$.

(3) The class

$$\tau_3 := p_{N_2}^* \tau_{M_1} \cup \overline{\tau}_e \in C^{n+k}(M_1 \times M_2 \times D^k, M_1^c)$$

is a Thom class for the tubular embedding ν_3, and M_1 is identified with its image under the map (id, f, e), and the diagram

$$\begin{tikzcd}
H_{*+k}(M_1 \times M_2 \times D^k) \arrow[r, "\llbracket \tau_e \cap \rrbracket"] \arrow[dr, "\llbracket \tau_3 \cap \rrbracket"'] & H_*(M_1 \times M_1) \arrow[d, "\llbracket \tau_{M_1} \cap \rrbracket"] \\
& H_{*-n}(M_1)
\end{tikzcd}$$

commutes, where we use the notation $\llbracket \tau \cap \rrbracket \colon C_*(X) \to C_*(Y)$ for maps that are compositions of the form

$$\llbracket \tau \cap \rrbracket : C_*(X) \to C_*(X, Y^c) \xrightarrow{[\tau \cap]} C_*(U) \xrightarrow{r} C_*(Y)$$

for U a tubular neighborhood of X inside Y with associated Thom class τ and r the associated retraction.

The embedding $(\mathrm{id}, f, e) : M_1 \to M_1 \times M_2 \times D^k$ is also equal to the composition of embeddings

$$\begin{tikzcd}
M_1 \arrow[r, hook, "(\mathrm{id}, f)"] \arrow[dr, hook, "(\mathrm{id}, f, e)"'] & M_1 \times M_2 \arrow[d, hook, "(\mathrm{id}, e\circ\pi)"] \\
& M_1 \times M_2 \times D^k.
\end{tikzcd} \tag{1.3}$$

for $(\mathrm{id}, e \circ \pi) : M_1 \times M_2 \to M_1 \times M_2 \times D^k$ taking (m, n) to $(m, n, e(m))$. This gives another associated diagram of tubular embeddings, namely

$$\begin{tikzcd}
f^*TM_2 \arrow[d] \arrow[dr, hook, "\bar{\nu}_1"] & \bar{N}_2 = M_1 \times M_2 \times \mathbb{R}^k \arrow[d, "p_{\bar{N}_2}"'] \arrow[dr, hook, "\bar{\nu}_2"] & \\
M_1 \arrow[r, hook, "(\mathrm{id}, f)"] & M_1 \times M_2 \arrow[r, hook, "(\mathrm{id}, e\circ\pi)"] & M_1 \times M_2 \times D^k,
\end{tikzcd}$$

where we choose the explicit tubular embedding

$$\bar{\nu}_1(m, V) = (m, \exp_{f(m)} V) \quad \text{ and } \quad \bar{\nu}_2(m, n, x) = (m, n, e(m) + \ell(x))$$

with ℓ a scaling map so that $\bar{\nu}_2$ is well-defined, recalling also that TM_2 denotes the ε_2–tangent bundle of M_2 so that $\bar{\nu}_1$ is well-defined. Note that $\bar{\nu}_1$ has image

$$U_{1,2} := \{(m, n) \in M_1 \times M_2 \mid |f(m) - n| < \varepsilon_2\} = (f \times 1)^{-1}(U_2)$$

for $U_2 \subset M_2 \times M_2$ the ε_2–neighborhood of the diagonal as above.

Lemma 1.2. *The map $f \times 1$ induces a map of pairs $f \times 1 : (M_1 \times M_2, M_1^c) \to (M_2 \times M_2, (\Delta M_2)^c)$ and, if f is a homotopy equivalence, the class $(f \times 1)^* \tau_{M_2} \in C^n(M_1 \times M_2, M_1^c)$ is a Thom class for the tubular neighborhood $(f^*TM_2, \bar{\nu}_1)$. (Here $M_1 \cong (\mathrm{id}, f)M_1 \in M_1 \times M_2$.)*

Proof. The Thom class $\tau \in C^n(M_1 \times M_2, M_1^c)$ is characterised by the property $\tau \cap [M_1 \times M_2] = [(1 \times f)M_1]$. We are interested in the class $\tau = (f \times 1)^* \tau_{M_2}$. By the naturality of the cap product,

$$\begin{aligned}
(f \times 1)_*((f \times 1)^* \tau_{M_2} \cap [M_1 \times M_2]) &= \tau_{M_2} \cap (f \times 1)_*[M_1 \times M_2] \\
&= \deg(f) \tau_{M_2} \cap [M_2 \times M_2] = \deg(f)[\Delta M_2].
\end{aligned}$$

Now $(f \times 1)_*[(1 \times f)M_1] = \Delta f[\Delta M_1] = \deg(f)[\Delta M_2]$ and the result follows from the fact that $(f \times 1)_*$ is invertible. $\square$

For the tubular neighborhood $(M_1 \times M_2 \times \mathbb{R}^k, \bar\nu_2)$, the inclusion $M_1 \times M_2 \times \partial D^k \hookrightarrow (M_1 \times M_2)^c \subset M_1 \times M_2 \times D^k$, where the second inclusion takes (m,n) to $(m,n,e(m))$, induces a quasi-isomorphism

$$\bar{i}^* : C^*(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \xrightarrow{\simeq} C^*(M_1 \times M_2 \times D^k, (M_1 \times M_2)^c)$$

Let

$$\overline{\eta} := \bar{i}^*(1 \times \eta)$$

for $\eta \in C^k(D^k, \partial D^k)$ a chosen representative of the dual of the fundamental class. Then $\overline{\eta}$ is a Thom class for $\bar N_2$ as $\bar{i}_*(\overline{\eta} \cap [M_1 \times M_2 \times D^k]) = (1 \times \eta) \cap \bar{i}_*[M_1 \times M_2 \times D^k] = [M_1 \times M_2 \times \{0\}]$ is homologous to the image of the zero-section of $\bar N_2$ under $\bar{i}_*$. Indeed, the embedding $(\mathrm{id}, e \circ \pi)$ is isotopic to the embedding $(\mathrm{id}, 0)$, replacing e by the constant map at $0 \in D^k$, through an isotopy only affecting the disc.

Applying Proposition A.1 and Lemma 1.2 to this new factorisation of the embedding (id, f, e), we get

Proposition 1.3. *Consider the sequence of embeddings* $M_1 \overset{(\mathrm{id},f)}{\hookrightarrow} M_1 \times M_2 \overset{(\mathrm{id},e\circ\pi)}{\hookrightarrow} M_1 \times M_2 \times D^k$ *and let* $\bar\nu_1, \bar\nu_2$ *and* $\bar\tau_1 = (f \times 1)^*\tau_{M_2}, \bar\tau_2 = \overline{\eta}$ *be the embeddings and associated Thom classes chosen above. Assume that* f *is a homotopy equivalence. Then the following hold:*

1. *The bundle* $\bar N_3 := f^*TM_2 \oplus \mathbb{R}^k$ *is isomorphic to the normal bundle of the composed embedding* $(id, f, e) : M_1 \longrightarrow M_1 \times M_2 \times D^k$.
2. *There is an isomorphism* $\tilde\nu_1 : \bar N_3 = f^*TM_2 \oplus \mathbb{R}^k \xrightarrow{\cong} \bar N_2|_{U_{1,2}} = M_1 \times M_2 \times \mathbb{R}^k|_{U_{1,2}}$ *as bundles over* $f^*TM_2 \cong U_{1,2}$, *with the property that*
$$\bar\nu_3 := \bar\nu_2 \circ \tilde\nu_1 : \bar N_3 \longrightarrow M_1 \times M_2 \times D^k$$
is a tubular embedding for (id, f, e). *Explicitly,*
$$\bar\nu_3(m, V, W) = \bar\nu_2(\bar\nu_1(m, V), W) = (m, \exp_{f(m)} V, e(m) + W)$$
for $W \in \mathbb{R}^k_{(m,f(m))} \equiv \mathbb{R}^k_{(m,\exp_{f(m)} V)}$.
3. *The class*
$$\bar\tau_3 := p^*_{\bar N_2}(f \times 1)^*\tau_{M_2} \cup \overline{\eta} \in C^{n+k}(M_1 \times M_2 \times D^k, M_1^c)$$
is a Thom class for the tubular embedding $\bar\nu_3$ *and the diagram*
$$\begin{tikzcd} H_{*+k}(M_1 \times M_2 \times D^k) \arrow[r, "{[\![\overline{\eta}\cap]\!]}"] \arrow[dr, "{[\![\bar\tau_3\cap]\!]}"'] & H_*(M_1 \times M_2) \arrow[d, "{[\![(f\times 1)^*\tau_{M_2}\cap]\!]}"] \\ & H_{*-n}(M_1) \end{tikzcd}$$
commutes, where the notation $[\![\tau\cap]\!]$ *is as in Proposition 1.1.*

We have given two explicit tubular neighborhoods (N_3, ν_3) and $(\bar N_3, \bar\nu_3)$ of the embedding (id, f, e), coming from two different ways of considering it as a composition of embeddings. The bundles N_3 and $\bar N_3$ are isomorphic, as they are both isomorphic to the normal bundle of the embedding. More than that, the uniqueness theorem for tubular neighborhoods implies that the embeddings are necessarily isotopic, and the following result holds:

Proposition 1.4. *Let* $(N_3, \nu_3), \tau_3$ *and* $(\bar N_3, \bar\nu_3), \bar\tau_3$ *be the tubular neighborhoods and Thom classes for the embedding* (id, f, e) *obtained in Propositions 1.1 and 1.3. There exists a bundle isomorphism* $\phi : N_3 \to \bar N_3$ *and a diffeomorphism* $h = (\mathrm{id} \times \tilde h) : M_1 \times M_2 \times D^k \longrightarrow M_1 \times M_2 \times D^k$, *fibered over the projection to* M_1, *such that*

1. h *restricts to the identity on* $M_1 \overset{(\mathrm{id},f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$ *and outside* $U_{1,2} \times \mathring{D}^k$;
2. h *is isotopic over* M_1 *to the identity relative to the same subspaces;*
3. $h \circ \nu_3 = \bar\nu_3 \circ \phi : \delta N_3 \longrightarrow M_1 \times M_2 \times D^k$, *for* δN_3 *(resp.* $\delta \bar N_3 = \phi(\delta N_3)$*) an appropriately chosen isomorphic subbundle.*
4. $[\bar\tau_3] = [h^*\bar\tau_3] = [\tau_3] \in H^{n+k}(M_1 \times M_2 \times D^k, M_1^c)$.

Note that (1) implies that $h(M_1^c) = M_1^c$ and (3) that $h(\nu_3\delta N_3) = \bar\nu_3\delta\bar N_3$.

Proof. Uniqueness of tubular neighborhoods (see e.g. [15, Chap 4, Thm 5.3]) gives that the tubular neighborhoods (N_3, ν_3) and $(\bar{N}_3, \bar{\nu}_3)$ are isotopic, and this is what we will use to produce the diffeomorphism h. To obtain detailed properties of the resulting h in our particular case, we follow the actual construction of the isotopy in the proof of this theorem in [15]. The first step in the proof is to replace N_3 by an isomorphic subbundle δN_3 such that $\nu_3(\delta N_3) \subset \bar{\nu}_3(\bar{N}_3)$. Then one considers the map $g = \bar{\nu}_3^{-1} \circ \nu_3 : \delta N_3 \to \bar{N}_3$. In our case, because ν_3 and $\bar{\nu}_3$ both restrict to the identity on M_1 in the first factor, having the form $\nu_3(m, V, W) = (m, \nu_e(\exp_m V, W'))$ and $\bar{\nu}_3(m, X, Y) = (m, \exp_{f(m)} X, e(m) + Y)$, we have that g has the form $g(m, V, W) = (m, X, Y)$ sends fibers to fibers. Now the map g is fiberwise homotopic to its fiber derivative $T_{M_1} g : \delta N_3 \to \bar{N}_3$ through the homotopy

$$H(m, V, W, t) = \begin{cases} t^{-1} g(m, tV, tW) & t \in (0, 1] \\ T_{M_1} g(m, V, W) & t = 0 \end{cases}$$

We set $\phi = T_{M_1} g$ and obtained the desired isotopy $F : \delta N_3 \times I \to M_1 \times M_2 \times D^k$ from $\nu_3 = \bar{\nu}_3 \circ g$ to $\bar{\nu}_3 \circ \phi$ by setting

$$F(m, V, W, 1 - t) = \bar{\nu}_3(H(m, V, W, t)).$$

By the isotopy extension theorem (see eg. [15, Chap 8, Thm 1.3]), there exists an isotopy of diffeomorphisms

$$\mathbf{F} : (U_{1,2} \times D^k) \times I \longrightarrow U_{1,2} \times D^k$$

such that $\mathbf{F}(-, 0)$ is the identity on $U_{1,2} \times D^k$ and such that $\mathbf{F}$ restricts to F on $\nu_3(\delta N_3) \times I$. As $\mathbf{F}$ is only non-trivial in a neighborhood of the image of F, can also be chosen to be fiberwise, and F has support inside $\bar{N}_3 \cong TM_2 \oplus \mathbb{R}^k$, $\mathbf{F}$ can be constructed so that it has support within the slices $\exp_{(m, f(m), e(m))} TM_2 \oplus \mathbb{R}^k \subset \{m\} \times M_2 \times D^k$. It also fixes $(\mathrm{id}, f, e)M_1$ because F fixes it. Hence by construction, $h := \mathbf{F}(-, 1)$ is a diffeomorphism of $U_{1,2} \times D^k$ of the form $h = \mathrm{id} \times \tilde{h}$, that extends to $M_1 \times M_2 \times D^k$ via the identity, and satisfies (1)–(3) in the statement.

Because h is a diffeomorphism compatible with the tubular embeddings, it pulls back the Thom class to a Thom class. Unicity of the Thom class in cohomology gives the last part of the statement. □

Convention 1. Recall that $N_3 \cong N_2|_{U_1}$ and $\bar{N}_3 \cong \bar{N}_2|_{U_{1,2}}$. We replace N_3 by the smaller subbundle δN_3, and $\bar{N}_3$ by $\delta\bar{N}_3 = \phi(\delta N_3)$, as defined in the above proof, scaling U_1, N_2, $U_{1,2}$, U_2 and $\bar{N}_2$ accordingly.

Consider now the diagram

$$\tag{1.4} \begin{tikzcd} H_*(M_1 \times M_1) \arrow[rr, "{[\![\tau_{M_1} \cap]\!]}"] \arrow[dd, "1 \times f_*"'] & & H_{*-n}(M_1) \arrow[dd, equal] \\ (1) & H_{*+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \arrow[ul, Rightarrow, "{[\![\overline{\tau}_e \cap]\!]}", "\cong"'] \arrow[ur, "{[\![\tau_3 \cap]\!]}"'] \arrow[dl, Rightarrow, "{[\![\overline{\eta} \cap]\!] = [(1 \times \eta) \cap]}"', "\cong"] \arrow[dr, "{[\![\bar{\tau}_3 \cap]\!]}"] & (2) \\ H_*(M_1 \times M_2) \arrow[rr, "{[\![(f \times 1)^* \tau_{M_2} \cap]\!]}"] \arrow[d, "f_* \times 1"'] & & H_{*-n}(M_1) \arrow[d, "f_*"] \\ H_*(M_2 \times M_2) \arrow[rr, "{[\![\tau_{M_2} \cap]\!]}"] & & H_{*-n}(M_2) \end{tikzcd}$$

Invariance of the intersection product will follow from the commutativity of this diagram once we have also shown that the maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are isomorphisms. This last fact is a consequence of the homology suspension theorem for $[\![\overline{\eta} \cap]\!]$ as it identifies with the map $[(1 \times \eta) \cap]$, and, for $[\![\overline{\tau}_e \cap]\!]$, will follow from the commutativity of (1) in the diagram and the fact that f is a homology isomorphism. [2]

We analyse the diagram step by step. The top triangle commutes by Proposition 1.1(3), and the middle triangle by Proposition 1.3(3). The bottom square commutes by the naturality of the cap product. So what we need to show is that the triangles labeled (1) and (2) commute. To prove commutativity of (1), we use the following lemma:

[2] Note that the map $[\![\tau_e \cap]\!]$ in the diagram is not directly a case of a Thom isomorphism because the source of the map is not correct, and thus we cannot deduce that it is an isomorphism from the Thom isomorphism theorem.

Lemma 1.5. *Let $i : (M_2 \times D^k, M_2 \times \partial D^k) \longrightarrow (M_2 \times D^k, ((f,e)M_1)^c)$ be the inclusion of pairs. If f is a degree d map, then*

$$i^*[\tau_e] = d[1 \times \eta] \in H^k(M_2 \times D^k, M_2 \times \partial D^k)$$

for τ_e the Thom class of $(N(f,e),\nu_e)$ and $\eta \in C^k(D^k,\partial D^k)$ a representative of the dual of $[D^n]$ as above.

Proof. Identifying M_1 with its image under (f,e) in $M_2 \times D^k$, consider the diagram

$$\begin{tikzcd}
H^*(N(f,e),M_1^c) \arrow[d, "\cap\,[N(f,e)]"'] & H^*(M_2 \times D^k, M_1^c) \arrow[l, "\nu_e^*"', "\cong"] \arrow[r, "i^*"] \arrow[d, "\cap\,[M_2\times D^k]"'] & H^*(M_2 \times D^k, M_2 \times \partial D^k) \arrow[d, "\cap\,[M_2\times D^k]"] \\
H_{n+k-*}(N(f,e)) \arrow[r, "\nu_{e*}"] & H_{n+k-*}(M_2 \times D^k) & H_{n+k-*}(M_2 \times D^k) \arrow[l, "i_*=id"']
\end{tikzcd}$$

where the vertical maps cap with the fundamental class in each case and ν_e^* is an isomorphism by excision. The two squares commute by naturality of the cap product, given that $[M_2 \times D^k] = i_*[M_2 \times D^k] = \nu_{e*}[N(f,e)] \in H^*(M_2 \times D^k, M_1^c)$. Hence in homology we have a commuting diagram

$$\begin{tikzcd}
{[\tau_e] \in} & H^k(M_2 \times D^k, M_1^c) \arrow[r, "i^*"] \arrow[d, "(\cap\,[N(e,f)])\circ\nu_e^*"'] & H^k(M_2 \times D^k, M_2 \times \partial D^k) \arrow[d, "\cap[M_2\times D^k]"] & \ni \deg(f)[1\times\eta] \\
& H_n(N(f,e)) \arrow[r, "\nu_{e*}"] & H_n(M_2 \times D^k) \arrow[d, "\cong"] & \\
{[M_1] \in} & H_n(M_1) \arrow[u, "\cong"'] \arrow[r, "f"] & H_n(M_2) & \ni \deg(f)[M_2]
\end{tikzcd}$$

where the first square commutes by the commutativity of the previous diagram, and the bottom square commutes as it comes from a commuting square in the category of spaces. Now by definition of the Thom classes, $[\tau_e]$ is taken to $[M_1]$ along the left vertical maps, and $[1 \times \eta]$ to $[M_2]$ along the right vertical maps as $[1 \times \eta]$ is a Thom class for the trivial D^k–bundle on M_2. On the other hand f takes $[M_1]$ to $d[M_2]$ along the bottom horizontal map, from which the result follows. $\square$

Proposition 1.6. *Assume that f is a degree 1 map. The subdiagram labeled (1) in Diagram (1.4) commutes. Moreover, the maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are both isomorphisms.*

Proof. Recall that $N_2 = M_1 \times N(f,e)$. Spelling out the maps, the diagram becomes the following:

$$\begin{tikzcd}
H_*(M_1 \times M_1) \arrow[d, "1\times f_*"'] & H_*(N_2) \arrow[l, "\cong"'] \arrow[d, "\nu_{2*}"'] & H_{*+k}(N_2,(M_1\times M_1)^c) \arrow[l, "\overline{\tau}_e\cap"', "\cong"] \arrow[r, "\cong"] \arrow[dl, "\overline{\tau}_e\cap"'] \arrow[dr, phantom, "(a)"] & H_{*+k}(M_1\times M_2\times D^k,(M_1\times M_1)^c) \\
H_*(M_1 \times M_2) & H_*(M_1 \times M_2 \times D^k) \arrow[l, "\cong"'] & & H_{*+k}(M_1\times M_2\times D^k, M_1\times M_2\times\partial D^k) \arrow[ll, "(1\times\eta)\cap"', "\cong"] \arrow[u, "1\times i_*"'] \arrow[d, "\cong"', "\overline{i}_*"] \arrow[ull, "\overline{\eta}\cap", phantom] \\
& H_*(\bar{N}_2) \arrow[ul, "\cong"] \arrow[u, "\bar{\nu}_{2*}", "\cong"'] & H_{*+k}(\bar{N}_2,(M_1\times M_2)^c) \arrow[l, "\overline{\eta}\cap"', "\cong"] \arrow[r, "\cong"] \arrow[ul, "\overline{\eta}\cap"] \arrow[ur, phantom, "(b)"] & H_{*+k}(M_1\times M_2\times D^k,(M_1\times M_2)^c)
\end{tikzcd}$$

The subdiagram labeled (a) commutes by naturality of the cap product since $i^*[\overline{\tau}_e] = [1\times\eta]$ by Lemma 1.5 as f has degree 1. Likewise subdiagram (b) commutes as $\overline{\eta} = \overline{i}^*(1\times\eta)$ by definition. The rest commutes on the chain or space level by definition of the maps.

We have marked in the diagrams the maps that are isomorphisms because of the Thom isomorphism, excision, or homotopy equivalences coming either from contracting a disc or a bundle, or from the equivalence $\bar{N}_2 \xrightarrow{\simeq} M_1 \times \mathcal{M}_2 \times D^k$ and likewise for the complements. From this we see that the map $[\![\overline{\eta}\cap]\!]$ is a composition of isomorphisms, and hence is itself an isomorphism. And because $1 \times f_*$ is an isomorphism, the commutativity of the diagram gives that also $[\![\overline{\tau}_e \cap]\!]$ is an isomorphism. $\square$

Note that the top part of the diagram in the proof shows that the left-hand map $1 \times f_*$ is invertible if and only if the right-hand map $1 \times i_*$ is, and these two maps are, up the a twisted suspension, inverses of each other.

Proposition 1.7. *Let $f : M_1 \to M_2$ be a homotopy equivalence between closed manifolds. Then f induces a ring map $H_*(M_1) \to H_*(M_2)$, with ring structure given by the intersection product, up to the sign* $\deg(f)$*, that is the diagram*

$$\begin{tikzcd}
H_*(M_1 \times M_1) \arrow[r, "{[\![\tau_{M_1} \cap]\!]}"] \arrow[d, "f_* \times f_*"'] & H_{*-n}(M_1) \arrow[d, "f_*"] \\
H_*(M_2 \times M_2) \arrow[r, "{[\![\tau_{M_2} \cap]\!]}"] & H_{*-n}(M_2)
\end{tikzcd}$$

commutes up to the sign $\deg(f) = \pm 1$.

We repeat here that this is *not* the optimal result for invariance of the intersection product, and absolutely not the optimal proof! See eg., [1, VI, Prop 14.2] for the stronger statement for any degree d map as a direct consequence of Poincaré duality. The proof given here has though the advantage of being liftable to the loop space.

Proof. Given that f is a homotopy equivalence, it has degree ± 1. If $\deg(f) = -1$, changing the orientation of M_2, so that the degree of f becomes 1, changes the intersection product of M_2 by a sign (-1). Hence it is enough to consider the case where $\deg(f) = 1$.

We need to show that Diagram (1.4) commutes, and that the two maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are isomorphisms. We have already seen that the top and middle triangles commute by Propositions 1.1(3) and 1.3(3), and the bottom square by the naturality of the cap product. The commutativity of the triangle labeled (1) and the two required isomorphisms are given by Proposition 1.6. For the triangle labeled (2), commutativity follows from Proposition 1.4. Indeed, the map h in the proposition gives a commutative diagram

$$\begin{tikzcd}
H_{*+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \arrow[r] \arrow[dr] & H_{*+k}(M_1 \times M_2 \times D^k, M_1^c) \arrow[r, "{[\tau_3 \cap]}"] \arrow[d, "h"] & H_{*-n}(N_3) \arrow[r] \arrow[d, "h"] & H_{*-n}(M_1) \arrow[d, equal] \\
& H_{*+k}(M_1 \times M_2 \times D^k, M_1^c) \arrow[r, "{[\bar{\tau}_3 \cap]}"] & H_{*-n}(\bar{N}_3) \arrow[r] & H_{*-n}(M_1)
\end{tikzcd}$$

because h is isotopic to the identity relative to $M_1 \times M_2 \times \partial D^k$ (for the first triangle), pulls back $\bar{\tau}_3$ to τ_3 (for the middle square), and fixes M_1 (for the last square). Hence Diagram (1.4) commutes, which finishes the proof. ☐

2. Lifted intersection products

In this section, following [21, Sec 4.1] we lift the intersection product along maps $p : \mathcal{E} \to M \times M$, give examples that will be relevant to us, and study its first basic invariance property.

Recall the Thom class $\tau_M \in C^n(U_M, M^c) \simeq C^n(M \times M, M^c)$ from the previous section. Pulling back along a map $p : \mathcal{E} \to M \times M$ defines a class $p^*\tau_M \in C^n(\mathcal{E}|_{U_M}, (\mathcal{E}|_M)^c)$ that can be used to define a lift of the intersection product of M using essentially the same definition:

Definition 2.1. Given a map $p : \mathcal{E} \to M \times M$, we define the *short intersection product* as the composition

$$int_{U_M} : C_*(\mathcal{E}) \longrightarrow C_*(\mathcal{E}, \mathcal{E}|_{M^c}) \xrightarrow{\rho} C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{M^c}) \xrightarrow{p^*\tau_M \cap} C_*(\mathcal{E}|_{U_M}). \tag{2.1}$$

where ρ is a choice of homotopy inverse for excision. When p is a fibration, we get moreover an associated *intersection product* int_M defined as the composition

$$int_M : C_*(\mathcal{E}) \xrightarrow{int_{U_M}} C_*(\mathcal{E}|_{U_M}) \xrightarrow{\simeq} C_{*-n}(\mathcal{E}|_M) \tag{2.2}$$

where the second is induced by the equivalence $M \xrightarrow{\simeq} U_M$ using that $p|_{U_M}$ is a fibration.

Let $\Lambda M = \mathrm{Maps}(S^1, M)$ denote the free loop space. Evaluation at any point $t \in S^1$ defines a fibration $\mathbf{ev}_t : \Lambda M \to M$. This will be our main source of examples of fibrations.

Example 2.2. Though not originally introduced this way, a widely studied example of such a lifted intersection product is the *Chas-Sullivan product* on $H_*(\Lambda M)$. This product can be obtained by applying the above construction to the fibration $\mathbf{ev}_0 \times \mathbf{ev}_0 : \Lambda M \times \Lambda M \to M \times M$, followed by loop concatenation:

$$C_*(\Lambda M \times \Lambda M) \xrightarrow{int_M} C_{*-n}(\Lambda M \times_M \Lambda M) \xrightarrow{concat} C_{*-n}(\Lambda M).$$

Here we have identified $(\Lambda M \times \Lambda M)|_M$ with the figure eight space $\Lambda M \times_M \Lambda M$. (See [14, Sec 2] for an account of how this definition relates to earlier definitions of [4, 6].)

Another example is the so-called *trivial coproduct*, obtained in a similar fashion using the fibration $\mathbf{ev}_{0,\frac{1}{2}} : \Lambda M \to M \times M$, evaluating a loop at both $t = 0$ and $t = \frac{1}{2}$, followed by the cut map:

$$C_*(\Lambda M) \xrightarrow{int_M} C_{*-n}(\Lambda M \times_M \Lambda M) \xrightarrow{cut} C_{*-n}(\Lambda M \times \Lambda M).$$

Here again $\Lambda M|_M$ identifies with the figure eight space $\Lambda M \times_M \Lambda M$. This coproduct was shown to be essentially trivial by Tamanoi, see [28, Thm B], hence its name.

We give now two "exotic" examples of such intersection products that will be used in the proof of the invariance Theorem 3.9.

Example 2.3. Recall from Section 1 the tubular neighborhoods N_2 (resp. N_3) of $M_1 \times M_1$ (resp. M_1) in $M_1 \times M_2 \times D^k$:

$$\begin{tikzcd}
 & N_3 \arrow[r, hook] \arrow[d] & N_2 = M_1 \times N(f,e) \arrow[r, hook, "\nu_2"] \arrow[d, "p_{N_2}"'] & M_1 \times M_2 \times D^k \\
M_1 \arrow[r, hook] & U_1 \arrow[r, hook] & M_1 \times M_1 \arrow[ur, hook] &
\end{tikzcd}$$

where $N_3 \cong N_2|_{U_1}$, see Proposition 1.1. Given a map $p : \mathcal{E}_2 \to M_2 \times M_2$, consider the pull-backs

$$\begin{tikzcd}
\mathcal{E}_{N_3} \arrow[r] \arrow[d, "p"'] & \mathcal{E}_{N_2} \arrow[r] \arrow[d, "p"'] & \mathcal{E}_{1,2} \times D^k \arrow[r] \arrow[d, "p"] & \mathcal{E}_{1,2} \arrow[r] \arrow[d, "p"] & \mathcal{E}_2 \arrow[d, "p"] \\
N_3 \arrow[r, hook] & N_2 \arrow[r, hook] & M_1 \times M_2 \times D^k \arrow[r, two heads] & M_1 \times M_2 \arrow[r, "f \times \mathrm{id}"] & M_2 \times M_2
\end{tikzcd}$$

Now the above construction associates to the composition

$$\mathcal{E}_{N_2} \xrightarrow{p} N_2 \xrightarrow{p_{N_2}} M_1 \times M_1$$

a short intersection product of the form

$$int_{U_1} : C_*(\mathcal{E}_{N_2}) \longrightarrow C_{*-n}(\mathcal{E}_{N_2}|_{U_1}) \cong C_{*-n}(\mathcal{E}_{N_3}).$$

Example 2.4. Considering now the tubular neighborhoods $\bar{N}_2$ (resp. $\bar{N}_3$) of $M_1 \times M_2$ (resp. M_1) in $M_1 \times M_2 \times D^k$,

$$\begin{tikzcd}
 & \bar{N}_3 \arrow[r, hook] \arrow[d] & \bar{N}_2 = M_1 \times M_2 \times \mathbb{R}^k \arrow[r, hook, "\bar{\nu}_2"] \arrow[d, "p_{\bar{N}_2}"] & M_1 \times M_2 \times D^k \\
M_1 \arrow[r, hook] \arrow[d, "f"'] & U_{1,2} \arrow[r, hook] \arrow[d] & M_1 \times M_2 \arrow[ur, hook] \arrow[d, "f \times \mathrm{id}"] & \\
M_2 \arrow[r, hook] & U_2 \arrow[r, hook] & M_2 \times M_2 &
\end{tikzcd}$$

where $\bar{N}_3 \cong \bar{N}_2|_{U_{1,2}} \cong \bar{N}_2|_{U_2}$, see Proposition 1.3. The composition

$$\mathcal{E}_{\bar{N}_2} \longrightarrow \bar{N}_2 \xrightarrow{p_{\bar{N}_2}} M_1 \times M_2 \xrightarrow{f \times \mathrm{id}} M_2 \times M_2$$

is not necessarily a fibration even if p was one, but we still have a short intersection product as in the previous example:

$$int_{U_2} : C_*(\mathcal{E}_{\bar{N}_2}) \longrightarrow C_{*-n}(\mathcal{E}_{\bar{N}_2}|_{U_2}) \cong C_{*-n}(\mathcal{E}_{\bar{N}_3}).$$

2.1. **Relative intersection products.** Given a map $\mathcal{E} \to M \times M$ as above, and in addition a map $\mathcal{R} \to \mathcal{E}$, we get a relative short intersection product

$$int_{U_M} : C_*(\mathcal{E}, \mathcal{R}) \longrightarrow C_*(\mathcal{E}, \mathcal{E}|_{M^c} \cup \mathcal{R}) \xrightarrow{\rho} C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{M^c} \cup \mathcal{R}|_{U_M}) \xrightarrow{p^*\tau_M \cap} C_*(\mathcal{E}|_{U_M}, \mathcal{R}|_{U_M}). \tag{2.3}$$

This can further be composed with a retraction map

$$C_*(\mathcal{E}|_{U_M}, \mathcal{R}|_{U_M}) \xrightarrow{r} C_{*-n}(\mathcal{E}|_M, \mathcal{R}|_M)$$

for example when both $\mathcal{E}|_{U_M} \to U_M$ and the composition $\mathcal{R}|_{U_M} \to \mathcal{E}|_{U_M} \to U_M$ are fibrations, giving a relative intersection product

$$int_M : C_*(\mathcal{E}, \mathcal{R}) \xrightarrow{int_{U_M}} C_*(\mathcal{E}|_{U_M}, \mathcal{R}) \xrightarrow{r} C_{*-n}(\mathcal{E}|_M, \mathcal{R}). \tag{2.4}$$

Example 2.5 (The relative coproduct)**.** Let $\mathcal{E} = \Lambda M$ and consider again the fibration $\mathbf{ev}_{0,\frac{1}{2}} : \Lambda M \to M \times M$ of Example 2.2. Let

$$\mathcal{R} = \big\{\gamma \in \Lambda M \mid \gamma(t) = \gamma(0)\ \forall t \in [0, \frac{1}{2}] \text{ or } \forall t \in [\frac{1}{2}, 1]\big\}$$

be the subspace of ΛM of half-constant loops. The map $\mathbf{ev}_{0,\frac{1}{2}}$ restricts to a map $\mathbf{ev}_{0,\frac{1}{2}} : \mathcal{R} \to M \subset M \times M$ with value on the diagonal. Then $\mathcal{R}|_{U_M} = \mathcal{R}|_M$ and we get a relative intersection product

$$int_M : C_*(\Lambda M, \mathcal{R}) \longrightarrow C_{*-n}(\Lambda M|_M, \mathcal{R}).$$

Now $\Lambda M|_M \cong \Lambda M \times_M \Lambda M$ identifies with the figure eight space and this intersection product can further be composed with a cut map to define a coproduct:

$$C_*(\Lambda M, \mathcal{R}) \xrightarrow{int_M} C_{*-n}(\Lambda M|_M, \mathcal{R}) \xrightarrow{cut} C_{*-n}(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M).$$

As already mentioned in Example 2.2, the non-relative version of this coproduct is almost completely trivial in homology, but it follows from the computations of Goresky-Hingston [11, Thm 15.3] for spheres and Naef [20] for lens spaces, that this relative version of the coproduct is highly non-trivial.

2.2. **Invariance of intersection products over a fixed manifold.** Suppose that we are given a map, or relative map over $M \times M$ (taking $\mathcal{R}_i$ empty in the non-relative case):

$$\begin{tikzcd}
\mathcal{R} \arrow[r, dash, "F_0"] \arrow[dr] \arrow[ddrr, bend right, "p_1"'] & \mathcal{R}' \arrow[dr] \arrow[ddr, dotted] & \\
& \mathcal{E} \arrow[r, "F"] \arrow[dr] & \mathcal{E}' \arrow[d, "p_2"] \\
& & M \times M
\end{tikzcd} \tag{2.5}$$

We give here an enhanced version of [21, Prop 4.6], showing that such maps respects the (short) intersection products over M.

Proposition 2.6. *Let $(F, F_0) : (\mathcal{E}, \mathcal{R}) \to (\mathcal{E}', \mathcal{R}')$ be a map of spaces over $M \times M$ as in Diagram (2.5) (with $\mathcal{R}, \mathcal{R}'$ possibly empty). Then there is a chain homotopy*

$$int_{U_M} \circ (F, F_0)_* \simeq (F, F_0)_* \circ int_{U_M} : C_*(\mathcal{E}, \mathcal{R}) \longrightarrow C_{*-n}(\mathcal{E}'|_{U_M}, \mathcal{R}'|_{U_M}).$$

If the restrictions $p|_{U_M} : \mathcal{E}|_{U_M} \to U_M$ and $p'|_{U_M} : \mathcal{E}'|_{U_M} \to U_M$ are fibrations (and also for $\mathcal{R}$, $\mathcal{R}'$ in the relative case), then the same holds for the intersection product int_M.

Proof. We consider first the short intersection product. So suppose that we are given a diagram of maps over $M \times M$. We need to check that the following diagram commutes (where in the non-relative case, $\mathcal{R}$ and $\mathcal{R}'$ should just be ignored):

$$\begin{tikzcd}
C_*(\mathcal{E}, \mathcal{R}) \arrow[r] \arrow[d, "F"] & C_*(\mathcal{E}, \mathcal{E}|_{M^c} \cup \mathcal{R}) \arrow[d, "F"] & C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{M^c} \cup \mathcal{R}|_{U_M}) \arrow[l, "\simeq"'] \arrow[r, "p^*\tau_M \cap"] \arrow[d, "F"] & C_*(\mathcal{E}|_{U_M}, \mathcal{R}|_{U_M}) \arrow[d, "F"] \\
C_*(\mathcal{E}', \mathcal{R}') \arrow[r] & C_*(\mathcal{E}', \mathcal{E}'|_{M^c} \cup \mathcal{R}') & C_*(\mathcal{E}'|_{U_M}, \mathcal{E}'|_{M^c} \cup \mathcal{R}'|_{U_M}) \arrow[l, "\simeq"'] \arrow[r, "(p')^*\tau_M \cap"] & C_*(\mathcal{E}'|_{U_M}, \mathcal{R}'|_{U_M})
\end{tikzcd}$$

The first square commutes already on the space level, the second by naturality of the excision isomorphism and the last by naturality of the cap product as $F^*(p')^*\tau_M = p^*\tau_M$ since we assumed that $p' \circ F = p$. The first part of the result follows after picking a homotopy inverse to excision via small simplices.

When the maps are moreover fibrations over U_M, we use the equivalence $U_M \xleftarrow{\simeq} M$ to get a further diagram

$$\begin{tikzcd}
C_*(\mathcal{E}|_{U_M}, \mathcal{R}|_{U_M}) \arrow[d, "F"] & C_{*-n}(\mathcal{E}|_M, \mathcal{R}|_M) \arrow[l, "\simeq"'] \arrow[d, "F"] \\
C_*(\mathcal{E}'|_{U_M}, \mathcal{R}'|_{U_M}) & C_{*-n}(\mathcal{E}'|_M, \mathcal{R}'|_M) \arrow[l, "\simeq"']
\end{tikzcd}$$

that commutes as it commutes already on the space level. The fact that F (resp. (F, F_0) respects the full intersection product int_M up to homotopy follows. $\square$

3. Invariance of intersection products along a homotopy equivalence

Let $f : M_1 \to M_2$ be a smooth map and suppose now that we have fibrations $p_1 : \mathcal{E}_1 \to M_1 \times M_1$ and $p_2 : \mathcal{E}_2 \to M_2 \times M_2$ together with a map $F : \mathcal{E}_1 \to \mathcal{E}_2$ over $f \times f$. Such a map always factorizes through the pull-back $\mathcal{E}_{11,2} := (f \times f)^*\mathcal{E}_2$:

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r] \arrow[d, "p_1"'] & (f \times f)^*\mathcal{E}_2 \arrow[r] \arrow[dl] & \mathcal{E}_2 \arrow[d, "p_2"] \\
M_1 \times M_1 \arrow[rr, "f \times f"] & & M_2 \times M_2.
\end{tikzcd} \tag{3.1}$$

If we consider the pull-back $(f \times f)^*\mathcal{E}_2$ as a space over $M_1 \times M_1$, Proposition 2.6 already tells us that the map $\mathcal{E}_1 \to (f \times f)^*\mathcal{E}_2$ respects the intersection product int_{M_1}. If we instead consider it as a space over $M_2 \times M_2$, the same result tells us that the map $(f \times f)^*\mathcal{E}_2 \to \mathcal{E}_2$ respects the intersection product int_{M_2}. But we have so far no way of comparing the intersection products int_{M_1} and int_{M_2} on $(f \times f)^*\mathcal{E}_2$. To compare them, we will follow the pattern of argument given in the case of the homology intersection product in Section 1, working also with the intermediate spaces obtained from pulling back $\mathcal{E}_2$ to $M_1 \times M_2$ and $M_1 \times M_2 \times D^k$ and various normal bundles considered in that section.

We will in what follows assume that we have fibrations $\mathcal{E}_i \to M_i \times M_i$, and we will treat the following two types of relative situations:

$$\begin{tikzcd}
\mathcal{R}_1 \arrow[rr, dash, "F_0"] \arrow[dr] \arrow[ddr] & & \mathcal{R}_2 \arrow[dr] \arrow[ddr, dotted] & \\
& \mathcal{E}_1 \arrow[rr, "F"] \arrow[d, "p_1"] & & \mathcal{E}_2 \arrow[d, "p_2"] \\
\text{(A)} & M_1 \times M_2 \arrow[rr, "f \times f"] & & M_2 \times M_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}_1 \arrow[rr, dash, "F_0"] \arrow[dr] \arrow[dd] & & \mathcal{R}_2 \arrow[dr] \arrow[dd, dotted] & \\
& \mathcal{E}_1 \arrow[rr, "F"] \arrow[dd, "p_1"] & & \mathcal{E}_2 \arrow[dd, "p_2"] \\
U_1 \arrow[rr, dotted, dash] \arrow[dr] & & U_2 \arrow[dr] & \\
\text{(B)} & M_1 \times M_2 \arrow[rr, "f \times f"] & & M_2 \times M_2
\end{tikzcd} \tag{3.2}$$

Assumption (A). We have a commuting diagram as in (3.2)(A) with the maps $p_i : \mathcal{E}_i \to M_i \times M_i$ and $\mathcal{R}_i \to \mathcal{E}_i \to M_i \times M_i$ for $i = 1, 2$ being fibrations. (The case $\mathcal{R}_i = \emptyset$ gives the non-relative case.)

Assumption (B). We have a commuting diagram as in (3.2)(B) with the maps $\mathcal{E}_i \to M_i \times M_i$ and $\mathcal{R}_i \to U_i$ being fibrations.

In Section 3.2 below, we will construct a diagram of the following shape under either assumption:

$$\begin{tikzcd}
C_*(\mathcal{E}_1, \mathcal{R}_1) \arrow[r, "int_{U_1}"] \arrow[d, dashed, leftrightarrow] \arrow[d, bend right=60, "{(F,F_0)}"'] & C_{*-n}(\mathcal{E}_1|_{U_1}, \mathcal{R}_1|_{U_1}) \arrow[d, "{[\hat{h}]}"] & C_{*-n}C_*(\mathcal{E}_1|_{M_1}, \mathcal{R}_1|_{M_1}) \arrow[l, "\sim"'] \arrow[d, "{(F,F_0)}"] \\
C_*(\mathcal{E}_2, \mathcal{R}_2) \arrow[r, "int_{U_2}"'] & C_{*-n}(\mathcal{E}_2|_{U_2}, \mathcal{R}_2|_{U_2}) & C_{*-n}C_*(\mathcal{E}_2|_{M_2}, \mathcal{R}_2|_{M_2}) \arrow[l, "\sim"']
\end{tikzcd}$$

where the dashed vertical arrow is a zig-zag, and the arrow labeled $[\hat{h}]$ is induced by a lift $\hat{h}$ of the diffeomorphism $h : N_3 \to \bar{N}_3$ of Proposition 1.4. We will use the above diagram to prove invariance of the intersection product under Assumption (A) in Section 3.3, and describe in Section 3.4 an obstruction to invariance when only Assumption (B) holds.

Remark 3.1. Strictly speaking, in both cases we only need to assume that $\mathcal{E}_1, \mathcal{R}_1$ are fibration over U_1, since we only need to apply Proposition 2.6 to those spaces by the remarks at the beginning of this section. Also, using the equivalence $U_M \simeq M$, one can required in Assumption (B) that $\mathcal{R}_i \to M_i$ is a fibration, instead of $\mathcal{R}_i \to U_i$, as would be the case in the standard definition of the string coproduct coming from a coproduct relative to half-constant loops.

3.1. **Intermediate spaces.** To construct the above diagrams, we will use the pull-backs $\mathcal{E}_{N_2}$, $\mathcal{E}_{1,2}$ and $\mathcal{E}_{\bar{N}_2}$ of $\mathcal{E}_2$ as in the following diagram:

$$
\begin{tikzcd}
\mathcal{E}_1 \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{E}_{1,2} \times D^k \arrow[rr] \arrow[dd] & & \mathcal{E}_{1,2} \arrow[r] \arrow[dd] \arrow[ddr] & \mathcal{E}_2 \arrow[dd] \\
& \mathcal{E}_{N_2} \arrow[ur, hook] \arrow[dl] \arrow[dd, dashed] & & \mathcal{E}_{\bar{N}_2} \arrow[ul, hook'] \arrow[ur] \arrow[ddr] \arrow[dd, dashed] & & \\
M_1 \times M_1 \arrow[rr] & & M_1 \times M_2 \times D^k \arrow[rr] & & M_1 \times M_2 \arrow[r] & M_2 \times M_2 \\
& N_2 \arrow[ul, dashed] \arrow[ur, hook] & & \bar{N}_2 \arrow[ul, hook'] \arrow[ur, dashed] & &
\end{tikzcd}
$$

with a compatible diagram with spaces $\mathcal{R}_{1,2}, \mathcal{R}_{N_2}, \mathcal{R}_{N_3}$ and $\mathcal{R}_{\bar{N}_3}$ similarly as the pull-backs of $\mathcal{R}_2$ along the same maps.

Note that under Assumption (B), the spaces $\mathcal{R}_{1,2}, \mathcal{R}_{N_2}, \mathcal{R}_{N_3}$ and $\mathcal{R}_{\bar{N}_3}$ will live over the following subspaces, pulled-back from U_2:

$$
\begin{tikzcd}
U_1 \arrow[rr] \arrow[dd, hook] & & U_{1,2} \times D^k \arrow[rr] \arrow[dd, hook] & & U_{1,2} \arrow[r] \arrow[dd, hook] & U_2 \arrow[dd, hook] \\
& N_2 \cap U_{1,2} \times D^k \arrow[ul, dashed] \arrow[ur, hook] \arrow[dd, hook] & & U_{1,2} \times \mathbb{R}^k \arrow[ul, hook'] \arrow[ur, dashed] \arrow[dd, hook] & & \\
M_1 \times M_1 \arrow[rr] & & M_1 \times M_2 \times D^k \arrow[rr] & & M_1 \times M_2 \arrow[r] & M_2 \times M_2 \\
& N_2 \arrow[ul, dashed] \arrow[ur, hook] & & \bar{N}_2 \arrow[ul, hook'] \arrow[ur, dashed] & &
\end{tikzcd}
$$

where $U_{1,2} = \{(m, n) \in M_1 \times M_2 \mid |f(m) - n| < \varepsilon\}$.

The crucial difference between assumptions (A) and (B) is actually visible in this diagram already: if $f : M_1 \to M_2$ is a homotopy equivalence, then the spaces $M_1 \times M_1$, $M_1 \times M_2$ and $M_2 \times M_2$ are all homotopic. Likewise their subspaces U_1, $U_{1,2}$ and U_2 are homotopic, being all homotopic to $M_1 \simeq M_2$, but the subspace $N_2 \cap U_{1,2} \times D^k$ can have a different homotopy type. This last space can be interpreted as a neighborhood of the fake diagonal discussed in Section 4.6. What always holds is instead the following:

Lemma 3.2. *When f is a homotopy equivalence, the map of pairs*

$$(\nu_2, \mathrm{id}) : (N_2, M_1) \hookrightarrow (M_1 \times M_2 \times D^k, M_1)$$

is a relative homotopy equivalence.

Proof. The map ν_2 is homotopic to the composition of homotopy equivalences

$$N_2 = M_1 \times N(f, e) \xrightarrow{p_2} M_1 \times M_1 \xrightarrow{\mathrm{id} \times f} M_1 \times M_2 \xrightarrow{\simeq} M_1 \times M_2 \times D^k.$$

Hence ν_2 is a homotopy equivalence. Moreover ν_2 restricts to the identity on the diagonal M_1. Hence it induces a homotopy equivalence of pairs since the pairs are good pairs. (See e.g. [19, Chap 6, Sec 5].) $\square$

Let $j : (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \to (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)$ denote the map identifying the source as a pull-back of the target along the map $N_2 \to M_1 \times M_2 \times D^k$. The above discussion has the following consequence:

Proposition 3.3. *Suppose $f : M_1 \to M_2$ is a homotopy equivalence. Then the following holds.*

(i) If Assumption (A) holds, the map $j : (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \to (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)$ is a relative homotopy equivalence.

(ii) If only Assumption (B) holds, $j : (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}|_{M_1}) \to (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k|_{M_1})$ is a relative homotopy equivalence.

(iii) If Assumption (B) holds and $N_2 \cap (U_{1,2} \times D^k) \simeq U_{1,2} \times D^k$, then map $j : (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \to (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)$ is a relative homotopy equivalence.

Proof. The map j fits in a pull-back diagram

$$\begin{tikzcd}
\mathcal{E}_{N_2} \arrow[r, "j"] \arrow[d] & \mathcal{E}_{1,2} \times D^k \arrow[r] \arrow[d, "p"] & \mathcal{E}_2 \arrow[d, "p_2"] \\
N_2 \arrow[r, "\nu_2"] & M_1 \times M_2 \times D^k \arrow[r] & M_2 \times M_2
\end{tikzcd}$$

and similarly for the relative terms. The result then follows from the fact that $\nu_2 : (N_2, M_1) \to (M_1 \times M_2 \times D^k, M_1)$ is a relative homotopy equivalence when f is a homotopy equivalence, by Lemma 3.2. □

3.2. **Comparing the intersection products of M_1 and M_2.** Recall from Examples 2.3 and 2.4 the intersection products int_{U_1} on $\mathcal{E}_{N_2}$ and int_{U_2} on $\mathcal{E}_{\bar{N}_2}$. The goal of the section is to construct a homotopy commuting diagram:

(3.3)

$$\begin{tikzcd}
C_*(\mathcal{E}_1, \mathcal{R}_1) \arrow[rr, "int_{U_1}"] \arrow[d, "F_{N_2}"] \arrow[dddd, bend right=60, "{(F,F_0)}"'] \arrow[dddd, phantom, bend right=30, "\boxed{\text{Prop 3.8}}"] \arrow[drr, phantom, "\boxed{\text{Prop 2.6}}"] & & C_{*-n}(\mathcal{E}_1|_{U_1}, \mathcal{R}_1|_{U_1}) \arrow[d, "F_{N_2}"] \arrow[dddd, bend left=60, "{(F,F_0)}"] \arrow[dddd, phantom, bend left=30, "\boxed{\text{Prop 3.6}}"] \\
C_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \arrow[rr, "int_{U_1}"] \arrow[ddrr, phantom, "\boxed{\text{Prop 3.5}}"] & & C_{*-n}(\mathcal{E}_{N_3}, \mathcal{R}_{N_3}) \arrow[dd, "\hat{h}"'] \\
C_{*+k}(\mathcal{E}_{1,2} \times D^k, \mathcal{E}_{1,2} \times \partial D^k \cup \mathcal{R}_{1,2} \times D^k) \arrow[u, dashed, "{[\![\overline{p}^* \overline{\tau}_e \cap]\!]}"'] \arrow[d, "{[\![\overline{p}^* \overline{\eta} \cap]\!]}", "\simeq"'] & & \\
C_*(\mathcal{E}_{\bar{N}_2}, \mathcal{R}_{\bar{N}_2}) \arrow[rr, "int_{U_2}"] \arrow[d] \arrow[drr, phantom, "\boxed{\text{Prop 2.6}}"] & & C_{*-n}(\mathcal{E}_{\bar{N}_3}, \mathcal{R}_{\bar{N}_3}) \arrow[d] \\
C_*(\mathcal{E}_2, \mathcal{R}_2) \arrow[rr, "int_{U_2}"] & & C_{*-n}(\mathcal{E}_2|_{U_2}, \mathcal{R}_2|_{U_2})
\end{tikzcd}$$

where we already know that the top and bottom squares homotopy commute as a direct application of Proposition 2.6. In this section, we will construct the middle square and check that the three remaining pieces of the diagram commute. In the diagram, the second vertical map on the left hand side is dashed to emphasize that it goes the "wrong way", an issue that we will take care of in Sections 3.3 and 3.4 in case (A) and (B) respectively. We start by defining the map $\hat{h}$.

Recall from Proposition 1.4 the diffeomorphism $h : M_1 \times M_2 \times D^k \longrightarrow M_1 \times M_2 \times D^k$, isotopic to the identity, that identifies the tubular neighborhoods $\nu_3 N_3$ and $\bar{\nu}_3 \bar{N}_3$ fixing their 0-section M_1, and that is the identity outside $U_{1,2} \times \mathring{D}^k$.

Proposition 3.4. *Suppose that $(F, F_0) : (\mathcal{E}_1, \mathcal{R}_1) \to (\mathcal{E}_2, \mathcal{R}_2)$ is as in Assumption (A) or (B). Then there is a pair of continuous maps*

$$(\hat{h}, \hat{h}_0) \colon (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k) \longrightarrow (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)$$

such that

(1) The map $p : \mathcal{E}_{1,2} \times D^k \to M_1 \times M_2 \times D^k$ intertwines the maps $\hat{h}$ and h, for h the diffeomorphism of Proposition 1.4. Likewise, the map $\mathcal{R}_{1,2} \times D^k \to M_1 \times M_2 \times D^k$ (composing with the map $U_{1,2}\times D^k \to M_1\times M_2\times D^k$ in case (B)) intertwines the maps $\hat{h}_0$ and h. In particular, $\hat{h}(\mathcal{E}_{N_3}) \subset \mathcal{E}_{\bar{N}_3}$ and $\hat{h}((\mathcal{E}_{N_3})^c) \subset (\mathcal{E}_{\bar{N}_3})^c$ and $\hat{h}_0(\mathcal{R}_{N_3}) \subset \mathcal{R}_{\bar{N}_3}$;

(2) The map $\hat{h}$ fixes $\mathcal{E}_{1,2} \times D^k|_{M_1}$ and $\mathcal{E}_{1,2} \times D^k|_{(U_{1,2}\times \mathring{D}^k)^c}$;

(3) The pair $(\hat{h}, \hat{h}_0)$ is homotopic to the identity relative to $\mathcal{E}_{1,2} \times D^k|_{M_1}$ and $\mathcal{E}_{1,2} \times D^k|_{(U_{1,2}\times \mathring{D}^k)^c}$.

Proof. From Proposition 1.4 we have that $h = \mathrm{id} \times \tilde{h} : M_1 \times M_2 \times D^k \to M_1 \times M_2 \times D^k$ restricting to the identity outside $U_{1,2}\times\mathring{D}^k$ by (1) in the proposition. Write $h(m, n, x) = (m, n', x')$. If $(m, n, x) \neq (m, n', x')$, which can only happen if they are in $U_{1,2} \times D^k$, let $\mathrm{id} \times\lambda$ in $U_{1,2} \times D^k$ be the geodesic between these two points, keeping the M_1-coordinate fixed. (The M_2 component of λ is a straight line in the ball of radius ε around $f(m)$ in M_2, and its D^k component is the straight line from x to x' in D^k.) Define

$$\underline{h} : U_{1,2} \times D^k \times I \to U_{1,2} \times D^k$$

by $\underline{h}(m, n, x, t) = (m, \lambda(t))$, with λ the above defined path. This is a continuous map, and it extends to the a map $\underline{h} : M_1 \times M_2 \times D^k \times I \to M_1 \times M_2 \times D^k$ by defining it to be the projection outside $U_{1,2} \times D^k$.

Recall that h restricts to the identity on $M_1 \xrightarrow{(\mathrm{id},f,e)} M_1 \times M_2 \times D^k$ as well as outside $U_{1,2} \times \mathring{D}^k$. Let $\underline{p} = p \times \mathrm{id} : \mathcal{E}_{1,2} \times D^k \times I \to M_1 \times \mathcal{M}_2 \times D^k \times I$ denote the pulled back fibration and consider the lifting problem

$$\begin{tikzcd}
(\mathcal{E}_{1,2} \times D^k) \times \{0\} \cup \big((\mathcal{E}_{1,2} \times D^k)|_{M_1} \cup (\mathcal{E}_{1,2} \times D^k|_{U^c_{1,2}\times D^k})\big) \times I \arrow[r, "\mathrm{id}"] \arrow[d, hook] & \mathcal{E}_{1,2} \times D^k \arrow[d, "p"] \\
(\mathcal{E}_{1,2} \times D^k) \times I \arrow[r, "\underline{h}\circ\underline{p}"] \arrow[ur, dashed, "\widehat{\underline{h}\circ\underline{p}}"] & M_1 \times M_2 \times D^k
\end{tikzcd}$$

and similarly for $\mathcal{R}_{1,2} \times D^k$, though restricting to the subspace $U_{1,2} \times D^k$ in case (B). We define $\hat{h}$ by setting $\hat{h}(m, \gamma, x) = \widehat{\underline{h} \circ \underline{p}}(m, \gamma, x, 1)$ after solving the resulting relative lifting problem. All the desired properties then hold by construction. □

We now show that the middle square in Diagram 3.3 commutes.

Proposition 3.5. *Suppose that $(F, F_0) : (\mathcal{E}_1, \mathcal{R}_1) \to (\mathcal{E}_2, \mathcal{R}_2)$ is a map as in Assumption (A) or (B). Then the middle square in Diagram (3.3)*

$$\begin{tikzcd}
C_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \arrow[d, "int_{U_1}"'] & C_{*+k}(\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k \cup \mathcal{E}_{1,2} \times \partial D^k) \arrow[l, "{[[p^*\bar{\tau}_e\cap]]}"'] \arrow[r, "{[p^*\bar{\eta}\cap]}"] \arrow[dl, dashed, "{[[p^*\tau_3\cap]]}"'] \arrow[dr, dashed, "{[[p^*\bar{\tau}_3\cap]]}"] & C_*(\mathcal{E}_{\bar{N}_2}, \mathcal{R}_{\bar{N}_2}) \arrow[d, "int_{U_2}"] \\
C_{*-n}(\mathcal{E}_{N_3}, \mathcal{R}_{N_3}) \arrow[rr, "\hat{h}"] & & C_{*-n}(\mathcal{E}_{\bar{N}_3}, \mathcal{R}_{\bar{N}_3})
\end{tikzcd}$$

commutes up to chain homotopy. Moreover, the maps $[p^*\eta\cap]$ and $\hat{h}$ are homology isomorphisms.

Proof. Recall from Propositions 1.1 and 1.3 that the classes $\tau_3 \in C^{n+k}(M_1 \times M_2 \times D^k, \nu_3(N_3)^c)$ and $\bar{\tau}_3 \in C^{n+k}(M_1\times M_2\times D^k, \nu_3(\bar{N}_3)^c)$ are Thom classes for the bundles N_3 and $\bar{N}_3$. The left and right triangles in the diagrams are the lift to $\mathcal{E}_{1,2}\times D^k$ of the triangles occurring in Propositions 1.1(3) and 1.3(3), with the left diagonal map capping with $p^*\tau_3$ and right diagonal map capping with $p^*\bar{\tau}_3$. The commutativity of the diagrams follows from Proposition A.1 using the relations $\tau_3 := p_2^*\tau_{M_1} \cup \bar{\tau}_e$ and $\bar{\tau}_3 := \bar{p}_2^*(f \times 1)^*\tau_{M_2} \cup \bar{\eta}$. Finally the middle triangle commutes by Proposition 3.4 (3), that gives in particular that $(\hat{h}, \hat{h}_0)$ is homotopic to the identity relative to $\mathcal{E}_{1,2} \times \partial D^k$, and Proposition 1.4(4), that gives the compatibility between capping with τ_3 and with $\bar{\tau}_3$. □

The following result gives the compatibility between the right-hand vertical composition $[\hat{h}]$ induced by $\hat{h}$ in Diagram 3.3 and the map induced by (F, F_0) on the fibrations restricted to the neighborhoods of the diagonals, or equivalently to the diagonal itself:

Proposition 3.6. *The diagram of pairs of spaces*

$$\begin{tikzcd}
(\mathcal{E}_1|_{U_1}, \mathcal{R}_1|_{U_1}) \arrow[d] & & (\mathcal{E}_1|_{M_1}, \mathcal{R}_1|_{M_1}) \arrow[ll, hook', "\simeq"'] \arrow[d] \arrow[ddd, bend left, "{(F,F_0)}"] \\
(\mathcal{E}_{N_3}, \mathcal{R}_{N_3}) = (\mathcal{E}_{N_2}|_{U_1}, \mathcal{R}_{N_2}|_{U_1}) \arrow[d, "{(\hat{h},\hat{h}_0)}"'] & & \mathcal{E}_{N_2}|_{M_1} \arrow[ll, hook'] \arrow[dll] \arrow[dd] \\
(\mathcal{E}_{\bar{N}_3}, \mathcal{R}_{\bar{N}_3}) = (\mathcal{E}_{\bar{N}_2}|_{U_1}, \mathcal{R}_{\bar{N}_2}|_{U_1}) \arrow[d] & & \\
(\mathcal{E}_2|_{U_2}, \mathcal{R}_2|_{U_2}) & & (\mathcal{E}_2|_{M_2}, \mathcal{R}_2|_{M_2}) \arrow[ll, hook', "\simeq"']
\end{tikzcd}$$

commutes. Moreover, under Assumptions (A) and (B), the horizontal maps are homotopy equivalences, and hence induce an isomorphism in homology. In particular, the right hand side of Diagram 3.3 commutes.

Proof. The first statement follows from the fact that $(\hat{h}, \hat{h}_0)$ is isotopic to the identity relative to $\mathcal{E}_{N_2}|_{M_1} = \mathcal{E}_{1,2} \times D^k|_{M_1}$ by Proposition 3.4 (3), giving the commutativity of the middle triangle, and the second follows from the fact that $\mathcal{E}_i, \mathcal{R}_i \to U_i$ are fibrations, and $M_i \to U_i$ are homotopy equivalences, for $i = 1, 2$. $\square$

Finally we show that the left vertical composition in Diagram 3.3 agrees with the map induced by (F, F_0) in homology.

Let

$$\begin{aligned}
&\iota\colon \big(\mathcal{E}_{1,2} \times D^k, (\mathcal{E}_{1,2} \times \partial D^k) \cup \mathcal{R}_{1,2} \times D^k\big) \longrightarrow \big(\mathcal{E}_{1,2} \times D^k, (\mathcal{E}_{N_2})^c \cup \mathcal{R}_{1,2} \times D^k\big)\\
&j\colon (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \hookrightarrow (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)
\end{aligned}$$

denote the maps of pairs, and similarly for $\bar{\iota}$ and $\bar{j}$.

Lemma 3.7. *If f is a degree 1 map, then the following diagram commutes up to homotopy:*

(3.4)
$$\begin{tikzcd}
C_{*+k}(\mathcal{E}_{1,2} \times D^k, (\mathcal{E}_{N_2})^c \cup \mathcal{R}_{1,2} \times D^k) \arrow[r, dashed, "{[p^*\overline{\tau}_e \cap]}"] & C_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \arrow[d, "j"] \\
C_{*+k}(\mathcal{E}_{1,2} \times D^k, \mathcal{E}_{1,2} \times \partial D^k \cup \mathcal{R}_{1,2} \times D^k) \arrow[u, dashed, "\iota"] \arrow[d, dashed, "\bar{\iota}"'] \arrow[r, dashed, "p^*(\bar{\eta})\cap"] \arrow[ur, "{[\![p^*\overline{\tau}_e \cap]\!]}"] \arrow[dr, "{[\![p^*\overline{\eta} \cap]\!]}"'] & C_*(\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k) \\
C_{*+k}(\mathcal{E}_{1,2} \times D^k, (\mathcal{E}_{\bar{N}_2})^c \cup \mathcal{R}_{1,2} \times D^k) \arrow[r, dashed, "{[p^*\overline{\eta} \cap]}"'] & C_*(\mathcal{E}_{\bar{N}_2}, \mathcal{R}_{\bar{N}_2}) \arrow[u, "\bar{j}"']
\end{tikzcd}$$

where the dashed arrows are added for clarity; by definition the two triangles on the left side of the diagram commute. Moreover, the maps $[\![p^*\overline{\eta}\cap]\!]$ and $\bar{j}$ in the diagram are isomorphism, and hence

$$[\![p^*\overline{\tau}_e\cap]\!] : H_{*+k}(\mathcal{E}_{1,2} \times D^k, \mathcal{E}_{1,2} \times \partial D^k \cup \mathcal{R}_{1,2} \times D^k) \longrightarrow H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2})$$

is an isomorphism if and only if the map of pairs

$$j : (\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \to (\mathcal{E}_{1,2} \times D^k, \mathcal{R}_{1,2} \times D^k)$$

induces an isomorphism in homology.

Proof. Lemma 1.5 together with the commutativity of

$$\begin{tikzcd}
(\mathcal{E}_{1,2} \times D^k, \mathcal{E}_{1,2} \times \partial D^k) \arrow[r, "\iota"] \arrow[d, "p"'] & (\mathcal{E}_{1,2} \times D^k, (\mathcal{E}_{N_2})^c) \arrow[d, "p"] \\
(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \arrow[r, "\mathrm{id} \times i"] & (M_1 \times M_2 \times D^k, \nu_2(N_2)^c)
\end{tikzcd}$$

and naturality of the cap product gives that the top square in the statement commutes up to a homotopy relative to the half-constant loops (coming from chosen chain inverses to excision). The commutativity of the bottom square follows similarly by definition of $\overline{\eta}$.

The map $\bar{j}$ induces an isomorphism in homology since $\mathcal{E}_{\bar{N}_2} \cong \mathcal{E}_{1,2} \times \mathbb{R}^k$ and $\mathcal{R}_{\bar{N}_2} \cong \mathcal{R}_{1,2} \times \mathbb{R}^k$ and the map $\bar{j}$ is just a linear map on the last coordinates. The map $p^*(1\times\eta)\cap$ is an isomorphism in homology because it is the relative version of the suspension isomorphism for $\mathcal{E}_{1,2} \times D^k$ and $\mathcal{R}_{1,2} \times D^k$. This proves the result. □

Proposition 3.8. *The left hand side of Diagram* (3.3) *commutes.*

Proof. By the lemma, we can replace the zig-zag capping with τ_e and η with the maps j and $\bar{j}$. Now we just observe that these maps fit in a commuting diagram

$$\begin{tikzcd}
H_*(\mathcal{E}_1,\mathcal{R}_1) \arrow[rr] \arrow[dd, "{(F,F_0)}"'] \arrow[ddrr, bend right] & & H_*(\mathcal{E}_{N_2},\mathcal{R}_{N_2}) \arrow[d, "j"'] & \\
& & H_*(\mathcal{E}_{1,2}\times D^k,\mathcal{R}_{1,2}\times D^k) & H_{*+k}(\mathcal{E}_{1,2}\times D^k,\mathcal{R}_{1,2}\times D^k) \arrow[l, "{[p^*\bar{\eta}\cap]}"'] \arrow[ul, "{[[p^*\bar{\tau}_e\cap]]}"'] \arrow[dl, "{[[p^*\bar{\eta}\cap]]}"] \\
H_*(\mathcal{E}_2,\mathcal{R}_2) & & H_*(\mathcal{E}_{\bar{N}_2},\mathcal{R}_{\bar{N}_2}) \arrow[ll] \arrow[u, "\bar{j}"] &
\end{tikzcd}$$

where the right part of the diagram commutes by the lemma, and where going around the left and right hand sides of the diagram gives the maps appearing in Diagram 3.3. □

3.3. **Homotopy invariance of a (relative) intersection product under Assumption (A).** In this section, we use the results we have proved so far to give a proof of the following:

Theorem 3.9. *Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence and $(F,F_0) : (\mathcal{E}_1,\mathcal{R}_1) \to (\mathcal{E}_2,\mathcal{R}_2)$ be as in Assumption (A). Then there is a chain homotopy*

$$(F,F_0)\circ int_{M_1} \simeq_H int_{M_2}\circ (F,F_0) : C_*(\mathcal{E}_1,\mathcal{R}_1) \longrightarrow C_{*-n}(\mathcal{E}_2|_{M_2},\mathcal{R}_2|_{M_2}).$$

The same holds under Assumption (B) if also $N_2 \cap (U_{1,2}\times D^k) \simeq U_{1,2}\times D^k$.

The above result is closely related to [21, Thm 4.11]. We give here a complete proof for completeness.

Corollary 3.10 (Homotopy invariance of the Chas-Sullivan product and a relative version)**.** *Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence. Then the induced map $F = \Lambda f : \Lambda_1 \to \Lambda_2$ respects the Chas-Sullivan product, also when considered relative to "half-constant loops": the diagram*

$$\begin{tikzcd}
H_*(\Lambda_1\times\Lambda_1, M_1\times\Lambda_1\cup\Lambda_1\times M_1) \arrow[r, "int_{M_1}"] \arrow[d] & H_{*_n}(\Lambda_1\times_{M_1}\Lambda_1, M_1\times_{M_1}\Lambda_1\cup\Lambda_1\times_{M_1} M_1) \arrow[d] \\
H_*(\Lambda_2\times\Lambda_2, M_2\times\Lambda_2\cup\Lambda_2\times M_2) \arrow[r, "int_{M_2}"] & H_{*_n}(\Lambda_2\times_{M_2}\Lambda_2, M_2\times_{M_2}\Lambda_2\cup\Lambda_2\times_{M_2} M_2)
\end{tikzcd}$$

commutes. The non-relative version of this diagram also commutes. Composed with the concatenation map, it gives the homotopy invariance of the Chas-Sullivan product, as first proved in [7].

Proof. Apply Theorem 3.9 to the fibrations $\mathbf{ev}_0\times\mathbf{ev}_0 : \mathcal{E}_i = \Lambda_i\times\Lambda_i \to M_i\times M_i$ for $i = 1,2$ and their restriction to $\mathcal{R}_i = M_i\times\Lambda_i\cup\Lambda_i\times M_i$, with the map $F = \Lambda f\times\Lambda f$. □

Example 3.11 (Products for mapping spaces)**.** Given spaces X_1, X_2 with chosen good basepoints x_1, x_2 (so the evaluation maps are fibrations), and a map $X_3 \to X_1\cup_{x_1\sim x_2} X_2$, one can mimmic the definition of the Chas-Sullivan product and get an intersection product

$$H_*(\mathrm{Map}(X_1,M)\times\mathrm{Map}(X_2,M)) \xrightarrow{int_M} H_{*-n}(\mathrm{Map}(X_1,M)\times_M\mathrm{Map}(X_2,M)) \longrightarrow H_{*-n}(\mathrm{Map}(X_3,M))$$

that is homotopy invariant by Theorem 3.9. For example, one could take $X_1 = X_2 = X_3 = S^n$ with a chosen map

$$X^3 = S^n \to S^n/S^{n-1} \xrightarrow{\simeq} S^n\vee S^n = X_1\cup_{x_1\sim x_2} X_2$$

induced by collapsing an equatorial sphere.

Example 3.12 (String coproducts)**.** The homotopy invariance of the non-relative (and almost completely trivial) coproduct follows from the above theorem, using the fibrations $\mathbf{ev}_{0,\frac{1}{2}}:\Lambda_i\to M_i\times M_i$ with $F=\Lambda f$.

Consider now the subspace $\mathcal{R}_i\subset\Lambda_i$ of *half-constant loops*, i.e. loops γ such that γ is constant on $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$. The same coproduct on the pair $(\Lambda_i,\mathcal{R}_i)$ does not satisfy the assumptions of the theorem anymore, since in this case the restriction $\mathbf{ev}_{0,\frac{1}{2}}:\mathcal{R}_i\to M_i\times M_i$ is only a fibration over the diagonal.

Another non-example would be the S^1–parametrized coproduct associated to the map

$$\mathbf{ev}_{0,t}:\Lambda_i\times S^1\to M_i\times M_i$$

evaluating the loop at time 0 and $t\in S^1$, as this map is not a fibration over $M_i\times M_i$.

Proof of Theorem 3.9. The results proved so far can be summarized in the following diagram
(3.5)

$$\begin{tikzcd}
& C_*(\mathcal{E}_1,\mathcal{R}_1) \arrow[rr, "int_{M_1}"] \arrow[d, "F_{N_2}"] \arrow[drr, phantom, "\boxed{\text{Prop } 2.6}"] & & C_{*-n}(\mathcal{E}_1|_{M_1},\mathcal{R}_1|_{M_1}) \arrow[d] \\
& C_*(\mathcal{E}_{N_2},\mathcal{R}_{N_2}) \arrow[rr, "int_{M_1}"'] \arrow[dl, "j"'] \arrow[drr, phantom, "\boxed{\text{Prop } 3.5}"] & & C_{*-n}(\mathcal{E}_{N_3},\mathcal{R}_{N_3}) \arrow[dd, "\hat{h}"] \\
C_*(\mathcal{E}_{1,2}\times D^k,\mathcal{R}) \arrow[dr, shift left, "\cong"] \arrow[rr, phantom, "\boxed{\text{Lem } 3.7}"] & & C_{*+k}(\mathcal{E}_{1,2}\times D^k,\partial D^k\cup\mathcal{R}) \arrow[ul, dashed, "{[\![\overline{p}^*\overline{\tau}_e\cap]\!]}"'] \arrow[dl, shift right, "\cong"'] & \\
& C_*(\mathcal{E}_{\bar{N}_2},\mathcal{R}_{\bar{N}_2}) \arrow[ul, shift left, "\bar{j}"] \arrow[ur, shift right, "{[\![\overline{p}^*\overline{\eta}\cap]\!]}"'] \arrow[rr, "int_{M_2}"] \arrow[d] \arrow[drr, phantom, "\boxed{\text{Prop } 2.6}"] & & C_{*-n}(\mathcal{E}_{\bar{N}_3},\mathcal{R}_{\bar{N}_3}) \arrow[d] \\
& C_*(\mathcal{E}_2,\mathcal{R}_2) \arrow[rr, "int_{M_2}"'] & & C_{*-n}(\mathcal{E}_2|_{M_2},\mathcal{R}_2|_{M_2})
\end{tikzcd}$$

where $\mathcal{R}=\mathcal{R}_{1,2}\times D^k$, and where the left-hand and right-hand vertical compositions agree with the map F in homology by Propositions 3.8 and 3.6 respectively. The diagram homotopy commutes under both assumptions (A) and (B) when f is a degree 1 homotopy equivalence, but commutativity of the diagram only allows us to conclude if we know that the map $[\![\overline{p}^*\overline{\tau}_e\cap]\!]$ is homotopy invertible, or equivalently that the map j is homotopy invertible. Now, this follows from Proposition 3.3 under Assumption (A), and under Assumption (B) if we additionally have that $N_2\cap(U_{1,2}\times D^k)\simeq U_{1,2}\times D^k$, since R_{N_2} is the pullback fibration over the first of these two spaces of the fibration $R_{1,2}\times D^k\longrightarrow U_{1,2}\times D^k$. ☐

3.4. **Quantifying the failure of invariance of relative intersection products under Assumption (B).** Suppose that the map $(F,F_0):(\mathcal{E}_1,\mathcal{R}_1)\to(\mathcal{E}_2,\mathcal{R}_2)$ only satisfies the weaker Assumption (B), that is $\mathcal{E}_i\to M_i\times M_i$ are fibrations but $\mathcal{R}_i\to M_i\times M_i$ are only fibrations when restricted to U_i. Forgetting about the relative parts $\mathcal{R}_1$ and $\mathcal{R}_2$, Theorem 3.9 implies that the map $F:\mathcal{E}_1\to\mathcal{E}_2$ respects the intersection products when f is a degree 1 homotopy equivalence. This can be used to obtain an obstruction of invariance of the relative map (F,F_0) using the following algebraic lemma:

Lemma 3.13. *Suppose that $\phi,\psi:(C_*,D_*)\to(E_*,F_*)$ are relative chain maps that are non-relatively chain homotopic: $\phi\simeq\psi:C_*\to E_*$. Let $H:C_*\to E_{*+1}$ be a chain homotopy between ϕ and ψ. Then for a relative cycle $a\in C_*$ with $\partial a\in D_{*-1}$, we have that*

$$\phi(a)-\psi(a)=H(\partial a)\in H_*(E_*,F_*).$$

We want to apply the lemma to the maps $\phi=F\circ int_{M_1}$ and $\psi=int_{M_2}\circ F$ from $(C_*,D_*)=C_*(\mathcal{E}_1,\mathcal{R}_1)$ to $(E_*,F_*)=C_*(\mathcal{E}_2,\mathcal{R}_2)$, with the homotopy H coming from the homotopy commutativity of Diagram (3.5), as given by Theorem 3.9 in the non-relative case.

As already mentioned above, a key difference between Assumptions (A) and (B) is that Proposition 3.3, saying that the map denoted j in Diagram (3.5) is homotopy invertible, holds in the first case but not necessarily in the second. A consequence of this is that the wrong way map $[\![\overline{p}^*\overline{\tau}_e\cap]\!]$ in the diagram can only be assumed invertible up to a homotopy that need not preserve the relative part. The following result makes precise how this affects the invariance property of intersection products.

Let

$$\alpha = [[p^*\overline{\tau}_e\cap]] \circ (\times D^k) \circ \overline{j}^{-1} \circ j : H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}) \longrightarrow H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2})$$

denote the "monodromy" around the left square in Diagram (3.5), and let

$$L_{N_2} H_*(\mathcal{R}_{N_2}) \longrightarrow H_{*+1}(\mathcal{E}_{N_2}, \mathcal{R}_{N_2})$$

denote the map induced by the homotopy $L_{N_2} : \mathcal{E}_{N_2} \times I \to \mathcal{E}_{N_2}$ between the identity and $j^{-1} \circ j : H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}|_{M_1}) \to H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}|_{M_1})$, as given by Proposition 3.3.

Theorem 3.14. *Let $F : (\mathcal{E}_1, \mathcal{R}_1) \to (\mathcal{E}_2, \mathcal{R}_2)$ be as in Assumption (B). The difference $F \circ int_{M_1} - int_{M_2} \circ F$ in homology is given by the composition*

$$H_*(\mathcal{E}_1, \mathcal{R}_1) \xrightarrow{\partial} H_{*-1}(\mathcal{R}_1) \xrightarrow{F_{N_2}} H_{*-1}(\mathcal{R}_{N_2}) \xrightarrow{\alpha} H_{*-1}(\mathcal{R}_{N_2}) \xrightarrow{L_{N_2}} H_*(\mathcal{E}_{N_2}, \mathcal{R}_{N_2})$$
$$\xrightarrow{int_{U_1}} H_{*-n}(\mathcal{E}_{N_3}, \mathcal{R}_{N_3}) \xrightarrow{\hat{h}} C_*(\mathcal{E}_{\bar{N}_3}, \mathcal{R}_{\bar{N}_3}) \to C_{*-n}(\mathcal{E}_2|_{M_2}, \mathcal{R}_2|_{M_2})$$

That is for $A \in H_*(\mathcal{E}_1, \mathcal{R}_1)$, we have

$$\big(F \circ int_{M_1} - int_{M_2} \circ F\big)(A) = int_{U_1}(L_{N_2}(\alpha(\partial_{N_2} A) \times I)),$$

where $\partial_{N_2} = F_{N_2} \circ \partial$ denotes the image of the boundary in $\mathcal{R}_{N_2}$ and where we have supressed the last two maps from the notation.

Proof. We start as in the proof of Theorem 3.9 with Diagram (3.5). We need to compare the outside compositions $F \circ int_{M_1}$ and $int_{M_2} \circ F$ in relative homology and we know that the diagram commutes in non-relative homology. To apply Lemma 3.13, we need a chain homotopy that gives the commutativity in homology. We pick as homotopy inverse to the wrong way map $[\![\overline{p}^*\overline{\tau}_e\cap]\!]$ the composition $\beta = (\times D^k) \circ \overline{j}^{-1} \circ j$ that makes the square to its left commute. With this choice, the pentagone on its right does not commute anymore, and looking at Proposition 3.5, we see more precisely that what does not commute anymore is a triangle

$$\begin{tikzcd}
C_*(\mathcal{E}_{N_2}) \arrow[rr, "int_{U_1}"] \arrow[dr, "\beta = (\times D^k) \circ \overline{j}^{-1} \circ j"'] & & C_{*-n}(\mathcal{E}_{N_3}) \\
& C_{*+k}(\mathcal{E}_{1,2} \times D^k, \mathcal{E}_{1,2} \times \partial D^k) \arrow[ur, "{[[p^*\tau_3\cap]]}"'] &
\end{tikzcd}$$

where we know that the triangle with the original arrow commutes, i.e. that $int_{M_1} \circ [\![\overline{p}^*\overline{\tau}_e\cap]\!] \simeq [[p^*\tau_3\cap]]$. Denoting that homotopy H_1, which we know by Proposition 3.5 exists respecting the relative terms, we thus have

$$[[p^*\tau_3\cap]] \circ \beta \simeq_{H_1} int_{U_1} \circ [\![\overline{p}^*\overline{\tau}_e\cap]\!] \circ \beta = int_{U_1} \circ \alpha$$

where we see that $[\![\overline{p}^*\overline{\tau}_e\cap]\!] \circ \beta$ is indeed the definition of the map α. So left is to give a homotopy witnessing that the map α is (non-relatively) homotopic to the identity. We can get such a homotopy in terms of the homotopy L_{N_2} as follows:

$$\begin{aligned}
\alpha = [[p^*\overline{\tau}_e\cap]] \circ (\times D^k) \circ \overline{j}^{-1} \circ j &\simeq_{L_{N_2} \circ \alpha} j^{-1} \circ j \circ [[p^*\overline{\tau}_e\cap]] \circ (\times D^k) \circ \overline{j}^{-1} \circ j \\
&\simeq j^{-1} \circ \overline{j} \circ [[p^*\overline{\eta}\cap]] \circ (\times D^k) \circ \overline{j}^{-1} \circ j \\
&\simeq j^{-1} \circ \overline{j} \circ \overline{j}^{-1} \circ j \simeq j^{-1} \circ j \simeq_{\overline{L_{N_2}}} \mathrm{id}
\end{aligned}$$

where $\overline{L_{N_2}}$ is the homotopy L_{N_2} in reverse. So in relative homology, the homotopy is the sum $H = (L_{N_2} \circ \alpha) + \overline{L_{N_2}}$.

By Lemma 3.13, the failure of invariance on a class $A \in H_*(\mathcal{E}_1, \mathcal{R}_1)$ is computed by the resulting homotopy applied to its boundary $\partial A \in H_{*-1}(\mathcal{R}_1)$. Putting together our computation back into Diagram (3.5) with the map $[\![\overline{p}^*\overline{\tau}_e\cap]\!]$ replaced by the chosen inverse β, we see that this failure is given as

$$G(int_{U_1}(H(F_{N_2}(\partial A))) \times I))) = G(int_{M_1}(L_{N_2}(\alpha(\partial_{N_2}A) \times I)) + G(int_{M_1}(\overline{L_{N_2}}(\partial_{N_2}A \times I)),$$

where only the first term is non-trivial, since $\partial_{N_2}A = F_{N_2}(\partial A)$ lies inside $\mathcal{R}_{N_2}|_{M_1}$. □

3.5. **Construction of the main homotopy in the case of mapping spaces.** We give here an explicit choice for the homotopy L_{N_2} in the special case relevant to the string coproduct, and more generally for certain mapping spaces. This choice will have additional properties that will be useful for the proofs of Theorems A and C. The construction will use "higher homotopy data" one can associate to a homotopy equivalence, as we describe now.

Given a homotopy equivalence $f : M_1 \to M_2$, we make the following choices. Let g be a homotopy inverse for f. We choose $h_1 : M_1 \times I \to M_1$ a homotopy $g \circ f \simeq \mathrm{id}$, $h_2 : M_2 \times I \to M_2$ a homotopy $f \circ g \simeq \mathrm{id}$ and $K : M_1 \times I \times I \to M_2$ a homotopy $f \circ h_1 \simeq h_2 \circ (f \times \mathrm{id})$ relative to $M_1 \times \partial I$, where both maps agree. This is possible by Vogt's lemma [29], and a contractible choice by e.g.; Riehl-Verity [25, Prop 4.4.7].

Let $\mathcal{P}M_2 = \mathrm{Maps}(I, M_2)$ denote the path space of M_2. Evaluation at 0 and 1 defines a fibration

$$\mathbf{ev}_{0,1} : \mathcal{P}M_2 \longrightarrow M_2 \times M_2.$$

Using the same convention as before, we denote $\mathbf{ev}_{0,1} : \mathcal{P}_{N_2} \to N_2$ the pulled-back fibration over $N_2 = M_1 \times N(f, e)$. Recall that elements of N_2 can be written as tuples $(m, m', (V, W))$ with $m, m' \in M_1$ and $(V, W) \in T_*(M_2 \times D^k)$, where we identify $N(f, e)$ with the subbundle of $T(M_2 \times D^k) = TM_2 \oplus \mathbb{R}^k$ of vectors orthogonal to $(f, e)_*TM_1$.

The intersection $N_2 \cap (\mathrm{id}, f)M_1 \times D^k$ is a version of the fake diagonal Δ_1^f, since for $(m, m', (V, W)) \in N_2$ to also lie in $(\mathrm{id}, f)M_1 \times D^k$, we need to have $f(m) = \exp_{f(m')} V$, and $\exp_{f(m')} V$ is canonically close to $f(m')$. For such $m, m' \in M_1$, the homotopy h_1 defines paths $f \circ h_1(m, -)$ and $f \circ h_1(m', -)$ in M_2, between $f \circ g \circ f(m)$ and $f(m)$ (resp. $f \circ g \circ f(m')$ and $f(m')$). Even when $f(m)$ is assumed close to $f(m')$, these two paths have no reason to stay close to each other. A way to formulate the following result is to say that the higher homotopy K can be used to give a homotopy between these two paths, naturally in m and m', and staying in N_2.

Lemma 3.15. *The homotopy data (f, g, h_1, h_2, K) defines a map*

$$\underline{K} = \big(h_1 \circ (\pi \times \mathrm{id}), K_{\mathcal{P}}, e \circ h_1 \circ (\pi' \times \mathrm{id})\big) : \quad \big(N_2 \cap ((\mathrm{id}, f)M_1 \times D^k)\big) \times I \ \longrightarrow \ \mathcal{P}_{N_2}$$

for $(\pi, \pi') : N_2 = M_1 \times N(f, e) \to M_1 \times M_1$ the two projections to M_1, with the property that

$$K_{\mathcal{P}}(m, m', (V, W), t)(t') = \begin{cases} f \circ h_1(m, t) & t' = 0 \\ f \circ h_1(m', t) & t' = 1. \end{cases}$$

Proof. Let

$$K_{\mathcal{P}}(m, m' + (V, W), t) := \ K(m, t, -) \, * \, h_2(\exp^{-1}_{f(m')}((1 - (-))V, t) \, * \, K(m', t, 1 - (-)).$$

This concatenated path is well-defined since $K(m, t, -)$ is a path from $f \circ h_1(m, t)$ to $h_2(f(m), t) = h_2(\exp^{-1}_{f(m')}(V), t)$, since we assumed $(m, m', V, W) \in (\mathrm{id}, f)M_1 \times D^k$, then $h_2(\exp^{-1}_{f(m')}((1 - (-))V, t)$ is a path from there to $h_2(f(m'), t)$, which is connected by the last part $K(m', t, 1 - (-))$ to $f \circ h_1(m', t)$, as needed. The evaluation map takes these thin loops to the 0-section of N_2 by construction. □

Let S be a topological space, and $s_0, s_1 \in S$ two points such that $(S, \{s_0, s_1\})$ is an NDR-pair. This means in particular that S comes equipped with a map

$$u : S \cong S \times \{1\} \hookrightarrow S \times I \longrightarrow S \times \{0\} \cup \{s_0, s_1\} \times I$$

where the second map is a retraction, so that $u(s_i) = (s_i, 1)$ for $i = 0, 1$. We will here consider evaluation maps on mapping spaces of the form

$$\mathcal{E} = \mathrm{Maps}(S, M) \xrightarrow{p := \mathbf{ev}_{s_0, s_1}} M \times M.$$

The map p_i can be seen to be a fibration using the map u just defined. The map u also defines a preferred image of $\mathcal{P}M$ in $\mathcal{E}$:

Definition 3.16. For (S, s_0, s_1) as above, consider the map

$$\pi : S \xrightarrow{u} S \times \{0\} \cup \{s_0, s_1\} \times I \twoheadrightarrow \{s_0\} \times I \cup_{(s_0,0)\sim(s_1,0)} \{s_1\} \times I \cong I$$

where the second map is the projection. Define

$$\mathcal{E}^{thin} := \pi^*(\mathcal{P}M) = \{\gamma : S \to M \mid \gamma = \gamma_0 \circ \pi \text{ for } \gamma_0 \in \mathcal{P}M\} \subset \mathcal{E}$$

to be the image of the path space under precomposition with π.

Of particular interest to us will be subspaces $\mathcal{R}_i \subset \mathcal{E}_i$ of the following form: let $P = \{P_a\}_{a\in A}$ be a collection of subspaces of S, all containing s_0 and s_1 and with each $P_a \hookrightarrow S$ a cofibration. Define

$$\mathcal{R}_i := \cup_{a\in A}\{\gamma \in \mathrm{Maps}(S, M_i) \mid \gamma(s) = \gamma(s_0)\ \forall s \in P_a\}$$
$$\mathcal{R}_i^{\varepsilon} := \cup_{a\in A}\{\gamma \in \mathrm{Maps}(S, M_i) \mid |\gamma(s) - \gamma(s_0)| < \varepsilon\ \forall s \in P_a\}$$

Then $\mathcal{R}_i^\varepsilon$ is a fibration over U_i, so $(\mathcal{E}_i, \mathcal{R}_i^\varepsilon)$ is a pair as in Assumption (B). The stricter version $\mathcal{R}_i$, that is only a fibration over the diagonal $M_i \subset M_i \times M_i$, will play a role too.

Our main examples will be of the following form: let $S = I^r/\partial I^r$ with $s_0 = \mathbf{0}$ and $s_1 = \mathbf{c} = (\frac{1}{2}, \dots, \frac{1}{2})$, and with $P = \{I^{j-1} \times [0, \frac{1}{2}] \times I^{n-j}\}_{j=1}^r \cup \{I^{j-1} \times [\frac{1}{2}, 1] \times I^{n-j}\}_{j=1}^r$. Then $\mathcal{E}_i = \Lambda^r M_i$ is the r-loop space and

$$\mathcal{R}_i = \cup_{j=1}^r\{\gamma : (I^r/\partial I^r) \to M \mid \gamma|_{I^{j-1}\times[0,\frac{1}{2}]\times I^{n-j}} \text{ or } \gamma|_{I^{j-1}\times[\frac{1}{2},1]\times I^{n-j}} \text{ is constant}\} \subset \Lambda^r M,$$

is a subspace of half-constant loops.

We will denote $\mathcal{E}_i^0 := \{\gamma \in \mathrm{Maps}(S, M_i) \mid \gamma \text{ is constant}\} \cong M_i$ the subspace of constant maps, and likewise for $\mathcal{E}_{N_2}^0$ and $\mathcal{E}_{1,2}^0$. Note that

$$\mathcal{E}_{N_2}^0 \cong N_2 \cap ((\mathrm{id}, f)M_1 \times D^k),$$

where the right hand space is the space appearing in the source of the map $\underline{K}$ above.

Recall that $j : \mathcal{E}_{N_2} \to \mathcal{E}_{1,2} \times D^k$ is the inclusion. By Proposition 3.3, we know that j is a homotopy equivalence relative to the subspace $\mathcal{R}_{N_2}|_{M_1}$ sitting over the diagonal $M_1 \hookrightarrow M_1 \times M_2 \times D^k$. We make here the following choice of homotopy inverse for j in the present case of mapping spaces:

Proposition 3.17. *Let $(\mathcal{E}_2 = \mathrm{Maps}(S, M_2), \mathcal{R}_2)$ be mapping spaces associated to triples $(S, \{s_0, s_1\}, P = \{P_a\}_{a\in A})$ as above. Then the map $j^{-1} : \mathcal{E}_{1,2} \times D^k \longrightarrow \mathcal{E}_{N_2}$ defined by*

$$j^{-1}(m, \gamma, x) = (g \circ f(m), f \circ g \circ \gamma, e \circ g(\gamma(s_1)))$$

is a homotopy inverse to the inclusion $j : \big(\mathcal{E}_{N_2}, \mathcal{R}_{N_2}|_{M_1}\big) \to \big(\mathcal{E}_{1,2} \times D^k, (\mathcal{R}_{1,2} \times D^k)|_{M_1}\big)$, with the following properties:

(i) The map j^{-1} preserves the constant and partially constant maps: $j^{-1}(\mathcal{E}_{1,2}^0 \times D^k) \subset \mathcal{E}_{N_2}^0$ and $j^{-1}(\mathcal{R}_{1,2} \times D^k) \subset \mathcal{R}_{N_2}$.

We can choose the homotopy $L_{N_2} : \mathcal{E}_{N_2} \times I \to \mathcal{E}_{N_2}$ between $j^{-1} \circ j$ and the identity, relative to $\mathcal{R}_{N_2}|_{M_1}$ as well as its subspace $\mathcal{E}_{N_2}^0|_{M_1}$, and so that

(ii) $\mathbf{ev}_0 \circ L_{N_2} = h_1(\mathbf{ev}_0 \times \phi) : \mathcal{E}_{N_2} \times I \to M_1$ for $\phi : I \to I$ a monotone reparametrization.

(iii) $L_{N_2}(\mathcal{E}_{N_2}^0 \times \partial I) \subset \mathcal{E}_{N_2}^0$ and $L_{N_2}(\mathcal{E}_{N_2}^0 \times I) \subset \mathcal{E}_{N_2}^{thin}$. Moreover, under the identification $\mathcal{E}_{N_2}^0 \cong N_2 \cap (\mathrm{id}, f)M_1 \times D^k$, we have that $L_{N_2}|_{\mathcal{E}_{N_2}^0 \times I} \simeq \pi^*\underline{K}$ for $\underline{K}$ the map of Lemma 3.15 and $\pi^* : \mathcal{P}_{N_2} \to \mathcal{E}_{N_2}^{thin}$ the map induced by π.

Proof. By construction, the map j^{-1} has image inside $\mathcal{E}_{N_2}$, landing in the subspace of maps evaluating to the 0-section of N_2. The map j^{-1} preserves constant loops and partially constant loops since it only changes the loop by postcomposing with the map $f \circ g$, and it preserves $(\mathcal{E}_{1,2} \times D^k)|_{M_1} = \mathcal{E}_{N_2}|_{M_1}$ since if (m, γ, x) satisfies that $\gamma(s_1) = f(m)$ and $x = e(m)$, then $(g \circ f(m), f \circ g \circ \gamma, e \circ g(\gamma(s_1)))$ also satisfies that $f \circ g \circ \gamma(s_1) = f(g \circ f(m))$ and $e \circ g(\gamma(s_1)) = e(g \circ f(m))$. Hence it also preserves $(\mathcal{R}_{1,2} \times D^k)|_{M_1} = \mathcal{E}_{N_2}|_{M_1} \cap \mathcal{R}_{1,2} \times D^k = \mathcal{R}_{N_2}|_{M_1}$ and $(\mathcal{E}_{1,2}^0 \times D^k)|_{M_1} = \mathcal{E}_{N_2}^0|_{M_1}$. So property (i) in the statement holds.

We will check that j^{-1} is a relative homotopy inverse by writing down explicit homotopies $L_{N_2} : \mathcal{E}_{N_2} \times I \to \mathcal{E}_{N_2}$ and $L_{1,2} : \mathcal{E}_{1,2} \times D^k \times I \to \mathcal{E}_{1,2} \times D^k$ between $j^{-1} \circ j$ (resp. $j \circ j^{-1}$) and the identity. The homotopy L_{N_2} will be constructed in such a way that (ii) and (iii) in the statement will be satisfied.

We start by defining the homotopy $L_{1,2}$. Let $(m, \gamma, x) \in \mathcal{E}_{1,2} \times D^k$. Then

$$j \circ j^{-1}(m, \gamma, x) = (g \circ f(m), f \circ g \circ \gamma, e \circ g(\gamma(s_1))).$$

We can use the homotopy h_1 on the first component, to get a homotopy between m and $g \circ f(m)$, and the homotopy h_2 will give a homotopy between γ and $f \circ g \circ \gamma$. However the pair $(h_1(m,t), h_2(\gamma,t)))$ does not live in $\mathcal{E}_{1,2}$ since the map $h_2(\gamma, t) : S \to M_2$, evaluated at s_0 as t varies, is the path $h_2(f(m), t)$ instead of $f(h_1(m,t))$ as would be required. To match our choice of homotopy on the first and second component, we can use the higher homotopy K, which is exactly connecting such points.

Let $S^\uparrow := S \times \{0\} \cup \{s_0, s_1\} \times I$ denote the "spiky version" of S and recall from above that we have a map $u : S \to S^\uparrow$ taking s_i to $(s_i, 1)$ for $i = 0, 1$. Define first $L^\uparrow_{1,2} : \mathcal{E}_{1,2} \times D^k \times I \to \mathcal{E}_{1,2} \times D^k$ by

$$L^\uparrow_{1,2}(m, \gamma, x, t) = (h_1(m, t), \gamma^K_t, \ell(x, t))$$

for $\ell(x, -)$ is the straight line from x to $e(g(\gamma(s_1)))$ and $\gamma^K_t : S \xrightarrow{u} S^\uparrow \longrightarrow M_2$ defined on $S^\uparrow$ by

$$\begin{aligned} \gamma^K_t(s) &= h_2(\gamma(s), t) \\ \gamma^K_t(s_0, t') &= K(m, t, 1 - t') \\ \gamma^K_t(s_1, t') &= h_2(\gamma(s_1), t), \end{aligned}$$

that is we use the path defined by $K(m, t, -) : I \to M_2$ from $f(h_1(m,t))$ to $h_2(f(m), t)$ in reverse on the spike $s_0 \times I$, and otherwise we use $h_2(\gamma, t)$. See Figure 1 for an illustration in the case where $S = S^1$ with s_0, s_1 two antipodal points.

As K is a homotopy relative to $M_1 \times \partial I$, we have that $K(m, t, -)$ is a trivial path when $t = 0$ or 1, so that $L^\uparrow_{1,2}(m, \gamma, x, 0) = (m, \gamma, x)$ and $L^\uparrow_{1,2}(m, \gamma, x, 1) = (g \circ f(m), f \circ g \circ \gamma, e(g(\gamma(s_1))))$, so that $L^\uparrow_{1,2}$ is indeed a homotopy between the identity and $j \circ j^{-1}$.

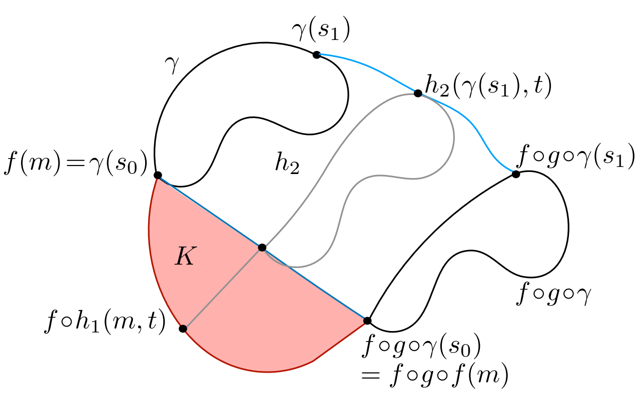

FIGURE 1. The loop component of $L_{1,2}(m, \gamma, -)$.

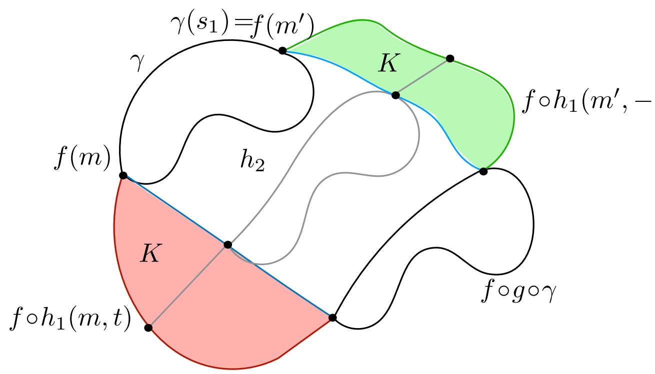

FIGURE 2. The loop component of $L_{N_2}(m, \gamma, x, -)$.

The homotopy $L^\uparrow_{1,2}$ does not yet preserve $\mathcal{R}_{N_2}|_{M_1}$ though, since the constant part of such loops γ will go through the family of paths $K(m, t, -)$ between $f(h_1(m,t))$ and $h_2(\gamma(s_0), t)$, and hence not stay constant. Also the disc component will need to be modified. We modify $L^\uparrow_{1,2}$ as follows: Let $v_{m,t} : M_2 \to M_2$ be

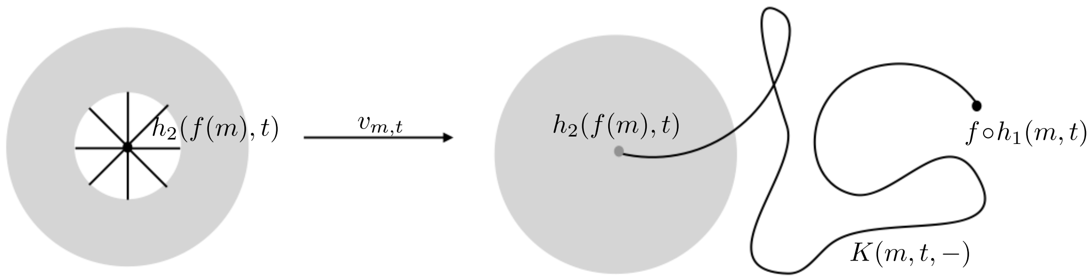

FIGURE 3. Deformation map $v_{m,t} : M_2 \to M_2$

defined by

$$v_{m,t}(n) = \begin{cases} K(m,t,1-\frac{2|n-h_2(f(m),t)|}{\varepsilon_t}) & |n-h_2(f(m),t)| \le \varepsilon_t \\ \exp_{f(m)}(2(|n-h_2(f(m),t)|-\frac{1}{2})\exp^{-1}_{h_2(f(m),t)}(n)) & \varepsilon_t \le |n-h_2(f(m),t)| \le 2\varepsilon_t \\ n & |n-h_2(f(m),t)| \ge 2\varepsilon_t \end{cases}$$

where $\varepsilon_t = \min(t\varepsilon,(1-t)\varepsilon)$ is 0 at times $t=0$ and $t=1$ and $0<\varepsilon_t \le \frac{\varepsilon}{2}$ when $t \ne 0,1$. The map $v_{m,t}$ is illustrated in Figure 3. It has the following properties:

- it takes the points at distance ε_t of $h_2(f(m),t)$ to the path defined by $K(m,t,-)$ between that point and $f\circ h_1(m,t)$;
- it stretches the points at distance between ε_t and $2\varepsilon_t$ to the ball of radius $2\varepsilon_t$ around $h_2(f(m),t)$;
- it fixes the rest of M_2;
- it is the identity at the times $t=0$ and 1 where $\varepsilon_t=0$ and the path $K(m,t,-)$ is a constant path.

Now we define $L_{1,2} : \mathcal{E}_{1,2}\times D^k \times I \to \mathcal{E}_{1,2}\times D^k$ via the formula:

$$L_{1,2}(m,\gamma,x,t)(s) = (h_1(m,t),\bar{\gamma}^K_t,\tilde{\ell}(x,t)), \quad \text{where} \quad \bar{\gamma}^K_t(s) = v_{m,t}\circ h_2(\gamma(s),t)$$

which essentially a reparametrization of $L^{\uparrow}_{1,2}$ with the extra property that the map $\bar{\gamma}^K_t : S \to M_2$ now satisfies that $\bar{\gamma}^K_t(s) = \bar{\gamma}^K_t(s_0) = f(h_1(m,t))$ for all $s \in S$ such that $\gamma(s)=\gamma(s_0)$, since for such points, $\gamma^K_t(s) = h_2(\gamma(s),t) = h_2(f(m),t)$. In particular, $L_{1,2}$ does preserve $R_{1,2}$. The path $\tilde{\ell}(x,-)$ equals $\ell(m,-)$ outside $\mathcal{E}_{N_2}$, is defined to be $e\circ h_1(m',-)$ on points (m,γ,x) such that $(\gamma(s_1),x) = (f(m'),e(m'))$, and is the linear interpolation of these two maps on the rest of $\mathcal{E}_{N_2} \subset \mathcal{E}_{1,2}\times D^k$. Hence $L_{1,2}$ preserves $(\mathcal{R}_{1,2}\times D^k)|_{M_1}$.

We are left to define L_{N_2}, a homotopy between the identity and $j^{-1}\circ j$. We have again that $j^{-1}\circ j(m,\gamma,x) = (g\circ f(m), f\circ g(\gamma), e(g\circ\gamma(s_1)))$.

We will start by defining L_{N_2} on the subspace $\mathcal{E}_{N_2}|_{M_1}$. On this subspace of $\mathcal{E}_{N_2}$, we can define L_{N_2} with the same definition as $L_{1,2}$: Let $L_{N_2}|_{M_1} : \mathcal{E}_{N_2}|_{M_1}\times I \to \mathcal{E}_{N_2}|_{M_1}$ be defined by

$$L_{N_2}|_{M_1}(m,\gamma,e(m),t)(s) = (h_1(m,t),\bar{\gamma}^K_t, e(h_1(m,t))), \quad \text{where} \quad \bar{\gamma}^K_t(s) = v_{m,t}\circ h_2(\gamma(s),t).$$

Just as for $L_{1,2}$ above, this maps respects the subspace $\mathcal{R}_{N_2}|_{M_1}$ since $\gamma^K_t(s) = \gamma^K_t(s_0)$ whenever the same holds for γ. It thus also respect $\mathcal{E}^0_{N_2}|_{M_1}$. (Note that this homotopy only works on the restriction of $\mathcal{E}_{N_2}$ to the actual diagonal M_1 and cannot easily be modified to work for a point $(m,\gamma,e(m'))$ living instead over the fake diagonal Δ^f_1.)

We now extend the definition to $\mathcal{E}_{N_2}\times I$. Given $(m,\gamma,x)\in\mathcal{E}_{N_2}$, write $(\gamma(s_1),x) = m' + (V,W) \in N(f,e)$, where we identify $N(f,e)$ with a subbundle of $T(M_2\times D^k)$ as above. The map-component of the homotopy will be defined in two parts: first pushing the map γ along $\exp_{f(m')}(1-t)V$ around s_1, so it goes through $f(m')$ at s_1, and then homotoping further just as in the case of $L_{1,2}$ using the added sticks along K both at s_0 and s_1. For elements not in $\mathcal{R}_{N_2}|_{M_1}$, it is enough to work with "spiky maps" $L^{\uparrow}_{N_2}:\mathcal{E}_{N_2}\times D^k\times I\to\mathcal{E}^{\uparrow}_{N_2}\times D^k$. Set

$$L^{\uparrow}_{N_2}(m,\gamma,x,t)=\begin{cases}(m,\gamma^{N_2}_{2t},e(m')+(1-2t)W) & 0\le t\le\frac12\\ (h_1(m,2t-1),(\gamma^{N_2}_1)^{K,K}_{2t-1},e(h_1(m',2t-1))) & \frac12\le t\le 1.\end{cases}$$

The map $\gamma^{N_2}_t:S\to M_2$ is defined by

$$\gamma^{N_2}_t(s)=h\big(\gamma(s_1),\exp_{f(m')}(u_1(s)(1-t)V))\big)(\gamma(s))$$

for $h:U_{M_2}\to\mathrm{Diff}(M_2)$ the push-map of [14, Lem 2.1], here taking $\gamma(s_1)$ to $\exp_{f(m')}((1-t)V)$ and only affecting the map γ at $s\in S$ close to s_1, where $u_1:S\xrightarrow{u}S^{\uparrow}\twoheadrightarrow I$ is the projection to $\{s_1\}\times I$ so that $u_1(s)=t$ if $u(s)=(s_1,t)$ and $u_1(s)=0$ else. The map $\gamma^{K,K}_t:S\xrightarrow{u}S^{\uparrow}\to M_2$ is defined on $S^{\uparrow}$ by

$$\begin{aligned}\gamma^{K,K}_t(s)&=h_2(\gamma(s),t)\\ \gamma^{K,K}_t(s_0,t')&=K(m,t,1-t')\\ \gamma^{K,K}_t(s_1,t')&=K(m',t,1-t').\end{aligned}$$

Finally we define L_{N_2} by gluing together the maps $L_{N_2}|_{M_1}$ and $L^{\uparrow}_{N_2}$ using the interpolation parameter: let $\delta:\mathcal{E}_{N_2}\to[0,1]$ be defined by

$$\delta(m,\gamma,x)=\min(\frac{|\gamma(s_1)-f(m)|}{\varepsilon}+\frac{|x-e(m)|}{\varepsilon},1).$$

Then $\delta^{-1}(0)=\mathcal{E}_{N_2}|_{M_1}$ is the subspace of tuples (m,γ,x) satisfying that $\gamma(s_1)=f(m)$ and $x=e(m)$. Define

$$L_{N_2}(m,\gamma,x,t)=\begin{cases}(m,\gamma^{N_2}_{\frac{2t}{\delta}},e(m')+(1-\frac{2t}{\delta})W) & 0\le t\le\frac{\delta}{2}\\ (h_1(m,\frac{2t-\delta}{2-\delta}),\delta(\gamma^{N_2}_1)^{K,K}_{\frac{2t-\delta}{2-\delta}}+(1-\delta)\overline{(\gamma^{N_2}_1)}^{K}_{\frac{2t-\delta}{2-\delta}},e(h_1(m',\frac{2t-\delta}{2-\delta}))) & \frac{\delta}{2}\le t\le 1\end{cases}$$

for $\delta=\delta(m,\gamma,x)$, and where the interpolation $\delta\gamma^{K,K}_t+(1-\delta)\bar{\gamma}^K_t$ is defined pointwise in M_2, and is well-defined for points with $\delta<1$.

One checks that (ii) holds with the reparametrization $\phi:I\to I$ collapsing the interval $[0,\frac{\delta}{2}]$. For (iii), the fact that $L_{N_2}(\mathcal{E}^0_{N_2}\times\partial I)\subset\mathcal{E}^0_{N_2}$ is in fact a consequence of (i). More generally, $L_{N_2}(\mathcal{E}^0_{N_2}\times I)\subset\mathcal{E}^{thin}_{N_2}$ since $\gamma^{N_2}_t$, $\overline{(\gamma^{N_2}_1)}^K_t$, and $(\gamma^{N_2}_1)^{K,K}_t$ are thin maps by definitions, and interpolation preserves the property of being a thin map. Explicitly, for $(m,\gamma,x)\in\mathcal{E}^0_{N_2}$, with $\gamma=[n]$ the constant map at $n\in M_2$, and $(n,x)=(m',(V,W))\in N(f,e)$, we have that $\gamma^{N_2}_t$ is the thin loop non-trivial only on $\{s_1\}\times I$ where it goes back and forth along $\exp_{f(m')}(rV)$ for $1\ge r\ge 1-t$. The map $(\gamma^{N_2}_1)^{K,K}_t$ then adds the paths $K(m,t,-)$ and $K(m',t,-)$ to $h_2(\gamma^{N_2}_1,t)$, reconstructing the map $K_{\mathcal{E}}$ up to a reparametrization. When $\delta<1$, this is interpolated with the homotopic map $\overline{(\gamma^{N_2}_1)}^K_t$. Hence L_{N_2} is indeed homotopic to $\underline{K}$ on this subspace. □

We will see below that the non-trivial homotopy comes from the fact that the "fake" half-constant loops, those that are half-constant but that do no live over M_1, are not necessarily preserved by the homotopy L_{N_2}.

4. The string coproduct and the proof of Theorems A and C

We now explain how to deduce the formula in Theorem A from applying Theorem 3.14 and C in the case of the string coproduct, and more generally for the higher coproducts.

4.1. **The string coproduct in terms of the trivial coproduct.** Let $\mathbf{ev}=\mathbf{ev}_{0,\frac12}:\Lambda M\to M\times M$ be the evaluation map defined by $\mathbf{ev}(\gamma)=(\gamma(0),\gamma(\frac12))$. Then

$$\Lambda M|_M=\{\gamma\in\Lambda M\mid\gamma(\frac12)=\gamma(0)\}$$

identifies with the "figure eight space", seen as the subspace of ΛM of loops γ having a self-intersection $\gamma(0) = \gamma(\frac{1}{2})$. We let $\mathcal{R} \subset \Lambda M|_M$ denote the subspace of half-constant loops:

$$\mathcal{R} = \{\gamma \in \Lambda M \mid \gamma(t) = \gamma(0)\ \forall t \in [0, \frac{1}{2}] \text{ or } \forall t \in [\frac{1}{2}, 1]\},$$

with

$$\mathcal{R}^\varepsilon = \{\gamma \in \Lambda M \mid |\gamma(t) - \gamma(0)| < \varepsilon\ \forall t \in [0, \frac{1}{2}] \text{ or } \forall t \in [\frac{1}{2}, 1]\}$$

the subspace of almost half-constant loops.

Following [11] (see also [14, Thm 2.13], combining with Section 3.2 of that paper, or [21, sec 2.5]), we can compute the string coproduct as the composition

$$\begin{aligned}(4.1)\quad & H_*(\Lambda M, M) \xrightarrow{\times I} H_{*+1}(\Lambda M \times I, \Lambda M \times \partial I \cup M \times I) \xrightarrow{J} H_{*+1}(\Lambda M, \mathcal{R}^\varepsilon) \\ & \xrightarrow{int_{U_M}} H_{*+1-n}(\Lambda M|_{U_M}, \mathcal{R}^\varepsilon) \xrightarrow{r} H_{*+1-n}(\Lambda M|_M, \mathcal{R}) \xrightarrow{cut} H_{*+1-n}(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M)\end{aligned}$$

where

$$J : \Lambda M \times I \to \Lambda M$$

is the reparametrizing map defined by $J(\gamma, s) = \gamma \circ \theta_{\frac{1}{2} \to s}$ for $\theta_{\frac{1}{2} \to s} : [0, 1] \to [0, 1]$ the piecewise linear map that fixes 0 and 1 and takes $\frac{1}{2}$ to s, and $r = Q_1 \circ k_{\frac{1}{2}} : \Lambda M|_{U_M} \to \Lambda M|_M$ is the composition of the retraction map $k_{\frac{1}{2}}$ deforming the loops so the almost self-intersection at time $\frac{1}{2}$ becomes an actual intersection as in [14, Sec 2.4] and the squeezing map Q_1 of [14, Lem 3.5] retracting small balls around $\gamma(0)$ to further retract $\mathcal{R}^\varepsilon$ to $\mathcal{R}$. It has the property that it retracts $\mathcal{R}^\varepsilon$ to $\mathcal{R}$. The papers [11, 14] mainly work with $\mathcal{R}$ rather than $\mathcal{R}^\varepsilon$, but this makes no difference for the resulting composed map, as already used in [14].

Any map $f : M_1 \to M_2$ induces a map $\Lambda f : \Lambda M_1 \to \Lambda M_2$ that respects the constant loops and half-constant loops, as well as their ε version for appropriately chosen $\varepsilon_1, \varepsilon_2$. Also, the maps $\times I$, J and *cut* are all natural in maps of the form Λf. Hence the question of homotopy invariance of the coproduct reduces to the homotopy/non-homotopy invariance of the middle composition

$$int_M : H_*(\Lambda M, \mathcal{R}^\varepsilon) \xrightarrow{int_{U_M}} H_{*-n}(\Lambda M|_{U_M}, \mathcal{R}^\varepsilon) \xrightarrow{r} H_{*-n}(\Lambda M|_M, \mathcal{R})$$

in (4.3), which is a relative version of the *short trivial coproduct*

$$int_M : H_{*-n}(\Lambda M) \longrightarrow H_{*-n}(\Lambda M|_M) \cong H_*(\Lambda M \times_M \Lambda M)$$

(denoted $\vee^{\mathrm{F}}_{\frac{1}{2}}$ in [14]) that looks for self-intersections of the form $\gamma(0) = \gamma(\frac{1}{2})$ and does not cut. This last operation has long been known to be essentially trivial (see [28] and [14, Lem 4.5]), and we also know from Proposition 2.6 that it is homotopy invariant, a property that could also be checked directly in this case. What we are interested in here is how invariant the relative version of this map is. Given that the (almost) half-constant loops $\mathcal{R}^\varepsilon$ define a fibration over U_M, and not $M \times M$, we are in the situation of Assumption (B), and we know from Theorem 3.14 that there is an obstruction to invariance coming from applying certain maps to the boundary of the class of loops considered. We start by analysing what the boundary is like in the case considered here.

Lemma 4.1. *Suppose $A \in C_*(\Lambda_1)$ is a non-relative cylce. Then*

$$\partial(J(A \times I)) = [M_1] \wedge A - A \wedge [M_1]$$

is the left and right half-constant loops versions of A.

Proof. Given that J is a chain map, $\partial(J(A \times I)) = J(A \times \partial I)$. Now $J(\gamma, 0) = \gamma(0) * \gamma$ reparametrizes the loop to become constant on $[0, \frac{1}{2}]$ and similarly $J(\gamma, 1) = \gamma * \gamma(1)$. This is precisely the effect on chain of taking a Chas-Sullivan product with the fundamental class. □

We want to emphasize that the chain $A \times I$ in the above lemma is never a cycle in $C_*(\Lambda_1 \times I)$, as $\partial(J(A \times I)))$ is always non-trivial. The lemma shows that this last cycle identifies with $[M_1] \wedge A - A \wedge [M_1]$, which is a boundary in $C_*(\Lambda_1)$, corresponding to the fact that $[M_1]$ is a unit for the string product. It is however not a boundary in $C_*(\mathcal{R}_1)$, where we will be considering it.

From Theorem 3.14, we have that the failure of invariance for such a cycle A is computed by

$$cut \circ \hat{h} \circ int_{M_1} \circ L_{N_2}\big((\alpha([M_1] \wedge A - A \wedge [M_1])) \times I\big) \tag{4.2}$$

To prove Theorem A, we will "pull out" $\wedge A$ and $A\wedge$ in the above expression.

4.2. **Higher coproducts.** Let $\Lambda^r M = \mathrm{Maps}(S^r, M)$ denote the space of *r-loops* in M, and recall from the introduction the higher coproduct

$$\vee^r : H_*(\Lambda^r M) \xrightarrow{int_M} H_{*+r-n}((\Lambda^r M \times I^r)|_M, \Lambda^r \times \partial I^r) \xrightarrow{\text{reread}} H_{*+r-n}(\mathrm{Maps}(S^1 \times S^{r-1}, M), \mathcal{R}^r)$$

that looks for self-intersections of the form $\gamma(\mathbf{0}) = \gamma(\mathbf{t})$ for some $\mathbf{t} = (t_1, \dots, t_r) \in I^r$, where we identify S^r with $I^r/\partial I^r$, and then "rereads" the loops by precomposing loops $(I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{t}} \longrightarrow M$ with the map

$$S^1 \times S^{r-1} \cong S^1 \times \partial I^r \longrightarrow (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{c}} \xrightarrow{(\theta_{\frac{1}{2}\to t_j})_{j=1}^r} (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{t}}$$

where the first map takes $S^1 \times \{\mathbf{t}\}$ to the line from $\mathbf{t}$ to $\mathbf{c} = (\frac{1}{2}, \dots, \frac{1}{2})$, which is a circle in the target, and the map $\theta_{\frac{1}{2}\to t_j} : [0,1] \to [0,1]$ is the same reparametrization map as in the previous section, applied one coordinate at a time. Here

$$\mathcal{R}^r = \bigcup_{j=1}^r \{\gamma \in \mathrm{Maps}(S^1 \times S^{r-1}, M) \mid \gamma(s, s') = \gamma(0, \mathbf{0}) \ \forall s' \in S^{r-1}_{j,\ell} \text{ or } \forall s' \in S^{r-1}_{j,r}\}$$

where $S^{r-1}_{j,\ell} = S^{r-1} \cap (I^{j-1} \times [0, \frac{1}{2}] \times I^{r-j})$ and $S^{r-1}_{j,r} = S^{r-1} \cap (I^{j-1} \times [\frac{1}{2}, 1] \times I^{r-j})$ are the "left" and "right" hemispheres of S^{r-1} in the jth direction. Note that the reread map does indeed take $\Lambda^r \times \partial I^r$ to $\mathcal{R}^r$ because of the reparametrizations θ.

This operation is a direct generalization of the string coproduct considered above, and just like the string coproduct, it can be computed using a relative version of a "trivial coproduct". Let

$$\mathcal{R}_{\Lambda^r} = \bigcup_{j=1}^r \{\gamma \in \Lambda^r M \mid \gamma(s) = \gamma(\mathbf{0}) \ \forall s \in I^{j-1} \times [0, \frac{1}{2}] \times I^{r-j} \text{ or } \forall s \in I^{j-1} \times [\frac{1}{2}, 1] \times I^{r-j}\}$$

be the subspace of half-constant loops, and likewise for its ε–version $\mathcal{R}^\varepsilon_{\Lambda^r}$. We can compute the higher string coproduct as the composition

$$\begin{aligned} H_*(\Lambda^r M) &\xrightarrow{\times I^r} H_{*+r}(\Lambda M^r \times I^r, \Lambda M \times \partial I^r) \xrightarrow{J^r} H_{*+r}(\Lambda^r M, \mathcal{R}^\varepsilon_{\Lambda^r}) \xrightarrow{int_{U_M}} \\ & H_{*+r-n}(\Lambda^r M|_{U_M}, \mathcal{R}^\varepsilon_{\Lambda^r}) \xrightarrow{r} H_{*+r-n}(\Lambda^r M|_M, \mathcal{R}_{\Lambda^r}) \xrightarrow{\text{reread}} H_{*+r-n}(\mathrm{Maps}(S^1 \times S^{r-1}, M), \mathcal{R}^r) \end{aligned} \tag{4.3}$$

where

$$J^r : \Lambda^r M \times I^r \to \Lambda^r M$$

is the reparametrizing map defined by $J^r(\gamma, \mathbf{t}) = \gamma \circ (\theta_{\frac{1}{2}\to t_j})_{j=1}^r$, and $r = Q_1 \circ k_{\frac{1}{2}} : \Lambda^r M|_{U_M} \to \Lambda^r M|_M$ is the composition of the retraction map $k_{\frac{1}{2}}$ deforming the r-loops so the almost self-intersection at time $\mathbf{c} = (\frac{1}{2}, \dots, \frac{1}{2})$ becomes an actual intersection, just as in [14, Sec 2.4], and the squeezing map Q_1 of [14, Lem 3.5] retracting a small ball around $\gamma(0)$ to further retract $\mathcal{R}^\varepsilon_{\Lambda^r}$ to $\mathcal{R}_{\Lambda^r}$. The reread map is now just precomposing with the map $S^1 \times S^{r-1} \to (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{c}}$ described above.

To give an analogue of Lemma 4.1 in higher dimensions, we need an appropriate higher version of the string product, namely a standard intersection product, followed by a concatenation map *along* ∂I^r. We can do this using the action of the little r-disc operad $\mathcal{D}_r$ on r-loops sharing the same basepoint. In the case $r = 1$, this operation will recover the two terms $[M_1] \wedge A - A \wedge [M_1]$ in one go.

Let

$$[-,-] : C_*(\Lambda^r M \times_M \Lambda^r M) \xrightarrow{-\times b} C_{*+r-1}(\Lambda^r M \times_M \Lambda^r M \times \mathcal{D}_r(2)) \longrightarrow C_{*+r-1}(\Lambda^r M)$$

denote the Browder (or Gerstenhaber) bracket, defined just as for based loop spaces, where b is a chain representative of the generator of $H_{r-1}(\mathcal{D}_r(2))$, the top homology of the little r-discs operad in arity 2, and the second map is the standard little disc action on r-loops, which make sense as long as the loops share the same basepoint. Now define the higher product

$$\wedge_{\partial I^r} : C_*(\Lambda^r M \times \Lambda^r M) \xrightarrow{int_M} C_{*-n}(\Lambda^r M \times_M \Lambda^r M) \xrightarrow{[-,-]} C_{*+r-1-n}(\Lambda^r M). \tag{4.4}$$

Lemma 4.2. *Suppose $A \in C_*(\Lambda^r M_1)$. Then $\partial(J^r(A \times I^r)) \simeq [M_1] \wedge_{\partial I^r} A \in C_{*+r-1}(\mathcal{R}_{\Lambda^r M_1})$.*

Proof. Let $b : \partial I^r \to D_r(2)$ be a choice of representative for $[-,-]$, e.g. as in Figure 4, with the two subdiscs placed along the line through the center containing $\mathbf{t}$, and such that the second disc lies inside a single hemisphere, so that the value of the bracket on $[M_1] \times A$ is indeed within half-constant loops. Note now that the reparametrized loop $J^r(a, \mathbf{t})$ is homotopic to the loop $b(\mathbf{t})(a, a(0))$, and the homotopy can be chosen continuously in $\mathbf{t}$. The result follows from the fact that $[M_1]$ is a unit for the intersection product. $\square$

Just like in the case $r = 1$, the class $\partial(J^r(A \times I^r))$ would be trivial if considered in $H_{*-1}(\Lambda^r M_1)$, but it is in general not trivial in $H_{*-1}(\mathcal{R}_{\Lambda^r M_1})$.

4.3. **Intermediate intersection products.** In what follows, we write

$$\Lambda_i = \Lambda^r M_i$$

for the r-loop space of M_i, for $i = 1, 2$, and $\mathcal{R}_i^\varepsilon, \mathcal{R}_i$ for its subspaces (almost) half-constant loops, and use the notations $\Lambda_{N_2}, \Lambda_{1,2}$, etc. for the pulled-back fibrations from Λ_2 and $\mathcal{R}_2^\varepsilon$ as for general fibrations in the Section 3. We write Λ_i^0 for the subspace of constant loops in a loop space Λ_i, and likewise for Λ_{N_2} etc.

Recall from Section 3.4 the map $\alpha = [\mathbf{ev}^*\tau_e\cap] \circ (\times D^k) \circ \bar{j}^{-1} \circ j : (\Lambda_{N_2}, \mathcal{R}_{N_2}) \to (\Lambda_{N_2}, \mathcal{R}_{N_2})$. This map does not affect the loop component, and thus respects constant loops and half-constant loops. In particular, $\alpha[M_1]$ can be considered in $C_n(\Lambda^0_{N_2})$ and $\alpha([M_1] \wedge^1_{\partial I^r} A)$ in $C_*(\mathcal{R}_{N_2})$.

To prove Theorems A and C, we will need intermediate intersection products, coming from the following spaces over $M_1 \times M_1$:

$$\begin{tikzcd}
\Lambda_1^0 \times \Lambda_1 \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\mathbf{ev}=(\mathbf{ev}_0,\mathbf{ev}_0)"'] & \Lambda^0_{N_2} \times \Lambda_1 \arrow[r, "j\times\mathrm{id}"] \arrow[d] & (\Lambda^0_{1,2} \times D^k) \times \Lambda_1 \arrow[r, "\bar{j}^{-1}\times\mathrm{id}", "\cong"'] \arrow[dl] & (\Lambda^0_{\bar{N}_2}) \times \Lambda_1 \arrow[dll, bend left, "\mathbf{ev}=(\mathbf{ev}_0,\mathbf{ev}_0)"] \\
 & M_1 \times M_1 & &
\end{tikzcd}$$

followed by the concatenation map $[-,-]$ of the previous section. The first and last vertical maps are fibrations, and one could also replace Λ_1^0 (resp. $\Lambda^0_{1,2}$) with Λ_1^ε (resp. $\Lambda^\varepsilon_{1,2}$) in the first factor. The second vertical map is not in general a fibration because of the factor $\Lambda^0_{N_2}$, even if we replace it by its ε–version, but we can none-the-less construct a retraction map

$$r_1 : \Lambda^0_{N_2} \times_{U_1} \Lambda_1 \longrightarrow \Lambda^0_{N_2} \times_{M_1} \Lambda_1$$

simply by adding a stick to the loop in Λ_1. (Adding a stick to an r-loop is formally done using the map $u : S \to S^\uparrow$ of Section 3.5 in the case $S = I^r/\partial I^r$.)

This data gives in particular compatible intersection products

$$\begin{tikzcd}
H_*(\Lambda_1^0 \times \Lambda_1) \arrow[r, "int_{U_1}"] \arrow[d] & H_{*-n}(\Lambda_1^0 \times_{U_1} \Lambda_1) \arrow[r, "r_1"] \arrow[d] & H_{*-n}(\Lambda_1^0 \times_{M_1} \Lambda_1) \arrow[r, "{[-,-]}"] \arrow[d] & H_{*+r-1-n}(\mathcal{R}_1) \arrow[d] \\
H_*(\Lambda^0_{N_2} \times \Lambda_1) \arrow[r, "int_{U_1}"] & H_{*-n}(\Lambda^0_{N_2} \times_{U_1} \Lambda_1) \arrow[r, "r_1"] & H_{*-n}(\Lambda^0_{N_2} \times_{M_1} \Lambda_1) \arrow[r, "{[-,-]}"] & H_{*+r-1-n}(\mathcal{R}_{N_2})
\end{tikzcd} \tag{4.5}$$

and likewise for the other pairs of spaces.

Proposition 4.3. *Let $\alpha : H_*(\mathcal{R}_{N_2}) \to H_*(\mathcal{R}_{N_2})$ be as in Section 3.4. For any cycle $A \in H_*(\Lambda_1)$,*

$$\alpha\big([M_1] \wedge^1_{\partial I^r} A\big) = \alpha[M_1] \wedge^1_{\partial I^r} A \ \in H_*(\mathcal{R}_{N_2}),$$

where the products are respectively the first and second intersection products in (4.5).

Proof. We want to show that the following diagram commutes, where the right hand horizontal maps are retractions, adding sticks on the (potentially long) loops in Λ_1, followed by concatenations, as above.

$$\begin{tikzcd}
H_*(\Lambda_1^0 \times \Lambda_1) \arrow[r, "int_{U_1}"] \arrow[d, "\bar{j}^{-1}\circ j\circ e"'] & H_{*-n}(\Lambda_1^0 \times_{U_1} \Lambda_1) \arrow[r, "{[-,-]\circ r_1}"] \arrow[d] & H_{*+r-1-n}(\mathcal{R}_1) \arrow[d] \\
H_*(\Lambda_{\bar{N}_2}^0 \times \Lambda_1) \arrow[r, "int_{U_1}"] \arrow[d, "\times D^k"'] \arrow[dr, phantom, "\boxed{(-1)^{nk}}"] & H_{*-n}(\Lambda_{\bar{N}_2}^0 \times_{U_1} \Lambda_1)) \arrow[r, "{[-,-]\circ r_1}"] \arrow[d, "\times D^k"] \arrow[dr, phantom, "\boxed{(-1)^{(r-1)k}}"] & H_{*+r-1-n}(\mathcal{R}_{\bar{N}_2}) \arrow[d, "\times D^k"] \\
H_{*+k}(\Lambda_{1,2}^0 \times \Lambda_1 \times D^k, \partial D^k) \arrow[r, "int_{U_1}"'] \arrow[d, "\tau"'] & H_{*+k-n}(\Lambda_{1,2}^0 \times_{U_1} \Lambda_1 \times D^k, \partial D^k) \arrow[r, "{[-,-]\circ r_1}"'] \arrow[d, "\tau"] & H_{*+k+r-1-n}(\mathcal{R}_{1,2} \times D^k, \partial D^k) \arrow[d, "\tau"] \\
H_{*+k}(\Lambda_{1,2}^0 \times D^k \times \Lambda_1, \partial D^k) \arrow[r, "int_{U_1}"] \arrow[d, "{[\mathbf{ev}^*\tau_e\cap]\times 1}"'] \arrow[dr, phantom, "\boxed{(-1)^{nk}}"] & H_{*+k-n}((\Lambda_{1,2}^0 \times D^k) \times_{U_1} \Lambda_1, \partial D^k) \arrow[r, "{[-,-]\circ r_1}"] \arrow[d, "{[\mathbf{ev}^*\tau_e\cap]\times 1}"] \arrow[dr, phantom, "\boxed{(-1)^{(r-1)k}}"] & H_{*+k+r-1-n}(\mathcal{R}_{1,2} \times D^k, \partial D^k) \arrow[d, "{[\mathbf{ev}^*\tau_e\cap]}"] \\
H_*(\Lambda_{N_2}^0 \times \Lambda_1) \arrow[r, "int_{U_1}"'] & H_{*-n}(\Lambda_{N_2}^0 \times_{U_1} \Lambda_1) \arrow[r, "{[-,-]\circ r_1}"'] & H_{*+r-1-n}(\mathcal{R}_2)
\end{tikzcd}$$

In left column, the first and third squares commute by naturality of the intersection product over a fixed manifold (Proposition 2.6), the second from commuting cap and cross product up to a sign $(-1)^{nk}$, and the fourth by commuting two caps, again up to a sign $(-1)^{nk}$. For the right column, there is $\times[S^{r-1}]$ as part of the map $[-,-]$, which creates likewise signs $(-1)^{(r-1)k}$ when commuted with crossing with the disc D^k in the second square, and capping with the pull-back of τ_e in the last square. The other squares commute by naturality. $\square$

4.4. **Enhanced version of the main homotopy.** Recall that the homotopy L_{N_2} constructed in Proposition 3.17 is a homotopy between $j^{-1}\circ j$ and the identity, whose M_1-component is the chosen homotopy $h_1 : M_1 \times I \to M_1$. Let

$$\hat{L}_{N_2} : \Lambda_{N_2} \times I \xrightarrow{(\mathbf{ev}_0\times \mathrm{id}, L_{N_2})} (M_1 \times I) \times_{h_1} \Lambda_{N_2}$$

be the enhanced version of L_{N_2} (generalised trace of the homotopy) that remembers the original M_1 and time components of the loops. Here we have supressed the reparametrization map ϕ of Proposition 3.17 and the evaluation map $\mathbf{ev}_0$ in the notation, that is

$$(M_1 \times I) \times_{h_1} \Lambda_{N_2} := \{(m,t,m',\gamma,x) \in (M_1\times I)\times \Lambda_{N_2} \mid h_1(m,\phi(t)) = m'\}.$$

By Proposition 3.17, the above map induces a chain map

$$\hat{L}_{N_2}(-\times I)\colon C_*(\Lambda^0_{N_2}) \longrightarrow C_{*+1}((M_1\times I)\times_{h_1}\Lambda_{N_2}, (M_1\times I)\times_{h_1}\Lambda^0_{N_2}).$$

Now there is an intersection product associated to the map

$$\mathbf{ev} : ((M_1 \times I)\times_{h_1} \Lambda_{N_2}) \times \Lambda_1 \longrightarrow M_1 \times M_1$$

defined by $\mathbf{ev}(m,t,m',\gamma,x,\lambda) = (m,\lambda(0))$, and likewise for its restriction to constant loops in Λ_{N_2}. It can be followed by the retraction map r_1 and concatenation $[-,-]$ after applying the h_1 the resulting loop. Picking the representative for $[-,-]\in C_*(D_r(2))$ shown in Figure 4, we have that concatenation induces a map

$$[-,h_1(-)] : \Big(((M_1\times I)\times_{h_1}\Lambda_{N_2})\times_{M_1}\Lambda_1, ((M_1\times I)\times_{h_1}\Lambda^0_{N_2})\times_{M_1}\Lambda_1\Big) \to (\Lambda_{N_2}, R_{N_2})$$

where the map takes $(m,t,(m',\gamma,x),\lambda)$ to $(m',[\gamma, f\circ h_1(\lambda,\phi(t))],x)$. Note that this makes sense since:

(i) The concatenated loop agrees with γ at times $\mathbf{0}$ and $\mathbf{c}$, hence the loop is still in Λ_{N_2} since γ was in that space.
(ii) The concatenation with $f\circ h_1(\lambda,t)$ makes sense since we assumed $\lambda(0)=m$ and $m'=h_1(m,\phi(t))$, so in particular $\gamma(0) = f(m') = f\circ h_1(\lambda(0),\phi(t))$.
(iii) If γ is constant, the resulting concatenated loop will be more than half-constant by our choice of bracket.

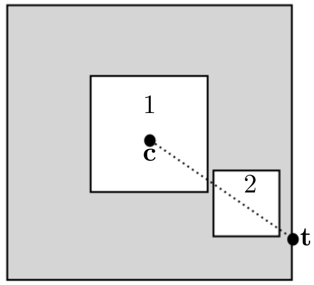

FIGURE 4. Choice of a class in $C_*(D_r(2))$ defining the bracket $[-,-]$

We denote the resulting product

$$\wedge^{h_1}_{\partial I^r} : C_*((M_1 \times I) \times_{h_1} \Lambda_{N_2} \times \Lambda_1, (M_1 \times I) \times_{h_1} \Lambda^0_{N_2} \times \Lambda_1) \to C_{*+r-1-n}(\Lambda_{N_2}, \mathcal{R}_{N_2}).$$

Proposition 4.4. *For any cycle $A \in C_*(\Lambda_1)$ and $B \in C_*(\Lambda^0_{N_2})$, we have that*

$$L_{N_2}(B \wedge^1_{\partial I^r} A) \times I) = \hat{L}_{N_2}(B \times I) \wedge^{h_1}_{\partial I^r} A \quad \in H_*(\Lambda_{N_2}, \mathcal{R}_{N_2})$$

Proof. We need to check that the following diagram commutes:

$$\begin{tikzcd}
C_*(\Lambda^0_{N_2} \times \Lambda_1 \times I, \partial I) \arrow[r, "\tau"] \arrow[d, "{[\mathbf{ev}^*_{0,0}\tau_1 \cap]}"'] & C_*(\Lambda^0_{N_2} \times I \times \Lambda_1, \partial I) \arrow[r, "\hat{L}_{N_2}"] \arrow[d, "{[\mathbf{ev}^*_{0,0}\tau_1 \cap]}"'] & C_*((M_1 \times I) \times_{h_1} \Lambda_{N_2} \times \Lambda_1, \overline{\mathcal{R}}) \arrow[d, "{[\mathbf{ev}^*\tau_1 \cap]}"] \\
C_{*-n}(\Lambda^0_{N_2} \times_{M_1} \Lambda_1 \times I, \partial I) \arrow[r, "\tau"] \arrow[d, "{\times[b]}"'] & C_{*-n}((\Lambda^0_{N_2} \times I) \times_{M_1} \Lambda_1, \partial I) \arrow[r, "\hat{L}_{N_2}"] \arrow[d, "{\times[b]}"] & C_{*-n}(((M_1 \times I) \times_{h_1} \Lambda_{N_2}) \times_{M_1} \Lambda_1, \overline{\mathcal{R}}_{M_1}) \arrow[d, "{\times[b]}"] \\
C_{*+r-1-n}((-) \times D_r(2), \partial I) \arrow[r, "\tau"] \arrow[d] & C_{*+r-1-n}((-) \times D_r(2), \partial I) \arrow[r, "\hat{L}_{N_2}"] & C_{*+r-1-n}((-) \times D_r(2), \overline{\mathcal{R}}_{M_1} \times D_r(2)) \arrow[d] \\
C_{*+r-1-n}(\mathcal{R}_{N_2} \times I, \partial I) \arrow[rr, "L_{N_2}"] & & C_{*+r-1-n}(\Lambda_{N_2}, \mathcal{R}_{N_2})
\end{tikzcd}$$

where $\overline{\mathcal{R}} := (M_1 \times I) \times_{h_1} \Lambda^0_{N_2} \times \Lambda_1$ and $\overline{\mathcal{R}}_{M_1} = ((M_1 \times I) \times_{h_1} \Lambda^0_{N_2}) \times_{M_1} \Lambda_1$. The two top squares commute by the naturality of the cap product, because the evaluation maps are compatible, and the following two squares by naturality of the cross product. The bottom square commutes up to homotopy on the space level. Indeed, starting from $(m, \gamma, x, \lambda, t, \beta)$ in the top left corner of that square, going each half gives a tuple of the form

$$\begin{cases} (m, \beta^{N_2}_t, e(m') + (1-2t)W) & 0 \le t \le \frac{u}{2} \\ (h_1(m, \frac{2t-u}{2-u}), u\beta^{K,K}_{\frac{2t-u}{2-u}} + (1-u)\bar{\beta}^K_{\frac{2t-u}{2-u}}, e(h_1(m', \frac{2t-u}{2-u}))) & \frac{u}{2} \le t \le 1 \end{cases}$$

where $\beta^{N_2}_t = \beta(\gamma^{N_2}_t, f \circ \lambda)$ in both cases, but where $\beta^{K,K}_t$ has the form $\beta(\gamma^{K,K}_t, f \circ h_1(\lambda, t))$ going around the top half, while it has the form $\beta(\gamma^{K,K}_t, h_2(f \circ \lambda, t))$ going around the lower half, and similarly for $\bar{\beta}^K_t$. Now such loops are canonically homotopic by gradually applying the homotopy K to λ. □

4.5. **Proof of Theorems A and C.** We give here a proof of the two formulas stated in the introduction, modulo characterizing the obstruction class that will arise in the proof, which we postpone to Section 4.6. We start with one last preparatory step.

To compute the obstruction to invariance, following (4.2) (see also Theorem 3.14), after applying the homotopy L_{N_2} we have to apply the intersection product pulled back from M_1 along the maps $\Lambda_{N_2} \to N_2 \to M_1 \times M_1$. This operation commutes, up to a sign, with the product operation, as shown by the following result.

Proposition 4.5. *For any cycle $A \in C_*(\Lambda_1)$ and $B \in C_*(\Lambda^0_{N_2})$, we have that*

$$int_{M_1}\big(\hat{L}_{N_2}(B\times I)\wedge^1_{\partial I^r} A\big) = (-1)^{rn} int_{M_1}\big(\hat{L}_{N_2}(B\times I)\big)\wedge^1_{\partial I^r} A \quad \in H_*(\Lambda_{N_2}|_{M_1}, \mathcal{R}^\varepsilon_{N_2}),$$

Proof. There are two evaluation maps

$$\mathbf{ev}_1, \mathbf{ev}_2 : (M_1\times I)\times_{h_1}\Lambda_{N_2}\times\Lambda_1 \longrightarrow M_1\times M_1$$

defined by $\mathbf{ev}_1((m,t),(m',\gamma,x),\lambda) = (m,\lambda(0))$ and $\mathbf{ev}_2((m,t),(m',\gamma,x),\lambda) = (m',m'')$ for m'' such that $(\gamma(\frac{1}{2}),x) = m''+W$ in $N(f,e)$. The left side of the equation is capping first with $\mathbf{ev}_1^*\tau_{M_1}$, crossing with $[b]$, and then with $\mathbf{ev}_2^*\tau_{M_1}$, while on the right side of the equation one first caps with $\mathbf{ev}_2^*\tau_{M_1}$. Commuting these cap and cross products gives a sign $(-1)^{n^2+(r-1)n} = (-1)^{rn}$. □

We are now ready to prove the main theorem.

Proof of Theorem A. Combining Lemma 4.1 and Theorem 3.14, we have that the failure of invariance is computed by

$$\mathrm{reread}\circ\mathrm{forget}\circ\hat{h}\circ int_{M_1}\circ L_{N_2}(\alpha([M_1]\wedge^1_{\partial I^r} A))\times I)\big)$$

which can be rewritten as

$$(-1)^{rn}\,\mathrm{reread}\circ\mathrm{forget}\circ\hat{h}\circ int_{M_1}(\hat{L}_{N_2}(\alpha[M_1]\times I)\wedge^1_{\partial I^r} A)$$

by Propositions 4.3, 4.4, 4.5. Let

$$T_{N_3} := (-1)^{rn} int_{M_1}(\hat{L}_{N_2}(\alpha[M_1]\times I)) \in H_1(\Lambda_{N_3}, \Lambda^0_{N_3})$$

with $T_{\bar{N}_3}\in H_1(\Lambda_{\bar{N}_3},\mathcal{R}_{\bar{N}_3})$ its image under $\hat{h}$. Given that $\hat{h}$ fixes Λ_1, we thus get that the failure can be written as

$$\mathrm{reread}\circ\mathrm{forget}(T_{\bar{N}_3}\wedge^1_{\partial T^r} A)$$

where we recall that, as indicated by the notation, the product $\wedge^1_{\partial I^r}$ is an intersection product pulled-back from M_1. The forgetfull map is the map from $\Lambda_{1,2}\times D^k$ to Λ_2 that only remembers the loop component. Now note that the product $\wedge^1_{\partial I^r}$ is here a product on $\Lambda_{\bar{N}_3}\times\Lambda_1$ relative to $\mathcal{R}_{\bar{N}_3}\times\Lambda_1$. Hence it satisfies Assumption (A) and by Theorem 3.9 it is equivalent to compute this product in Λ_2 after the forgetful map instead. It follows that we can write the obstruction as

$$\mathrm{reread}(T_{\Lambda_2}\wedge^2_{\partial I^r} f_*A)$$

for T_{Λ_2} the image of $T_{\bar{N}_3}$ in $H_1(\Lambda_2,\mathcal{R}_2)$. Finally, our choice of Browder operation (see Figure 4) is so that

$$\mathrm{reread}(B\wedge^2_{\partial I^r} f_*A) = \mathrm{reread}(B)\wedge^2_{\partial I^r} f_*A$$

where the product on the right hand side is the intersection product lifted from M_2 followed by the map

$$H_*(\mathrm{Maps}(S^1\times S^{r-1},M_2)\times_{M_2}\Lambda_2)\xrightarrow{[-,-]} H_{*+r-1}(\mathrm{Maps}(S^1\times S^{r-1},\mathcal{R}_2))$$

coming from a generalized little disc action of the class $\tilde{b}\in C_{r-1}(\mathrm{Emb}(D^2,S^1\times S^{r-1}))$ corresponding to the classe b of Figure 4 under the reread map $S^1\times S^{r-1}\to (I^r/\partial I^r)/_{\mathbf{0}\sim\mathbf{c}}$. Let

$$T_r := \mathrm{reread}(T_{\Lambda_2}) = (-1)^{rn}\,\mathrm{reread}\circ\mathrm{forget}\circ\hat{h}\circ int_{M_1}\circ L_{N_2}(\alpha[M_1]\times I) \quad \in H_1(\mathrm{Maps}(S^1\times S^{r-1},M_2),\mathcal{R}).$$

Putting all of this together, we have shown that the failure of invariance is computed by

$$T_r\wedge^2_{\partial I^r} f_*(A).$$

Setting $T_f := T_1$, this proves Theorem A in the case $r=1$ (up to the characterization of T_f). To prove Theorem C, we are left to identify the class T_r with the suspension $s^{r-1}T_1$ of the class in the case $s=1$. This follows from Proposition 3.17(iii). Indeed the proposition gives that $L_{N_2}(\alpha[M_1]\times I)\simeq \underline{K_{\mathcal{E}}}(\mathbf{ev}\alpha[M_1]\times I)$ is a class of thin loops of the form given by Lemma 3.15. After applying the composition $\mathrm{reread}\circ\mathrm{forget}\circ\hat{h}\circ int_{M_1}$, we see that it does not dependent on the S^{r-1}-factor, and can be given as a composition

$$S^1\times S^{r-1}\to S^1\xrightarrow{T_f} M_2$$

for T_f the obstruction class in the case $r=1$. □

4.6. **The obstruction class and the fake diagonal.** The obstruction class T_f has two main ingredients: the class $\alpha[M_1]$, and then the homotopy L_{N_2}. From Proposition 3.17, we know that $L_{N_2}(\alpha[M_1])$ can be constructed out of the homotopy data (f, g, h_1, h_2, K), and we will explain here in what sense $[\alpha[M_1]$ is a transverse version of the diagonal, hence explaining the end of the statement of Theorem A. We will then deduce two possible reasons for the obstruction to vanish: one from $\alpha[M_1]$ actually coming from $\Lambda^{\varepsilon}_{N_2}|_{M_1}$, and one from the higher homotopy K is "small".

Recall that $\Delta_1^f = \{(m, m') \in M_1 \times M_1 \mid f(m) = f(m')\}$. This subspace is essentially never a manifold, and in particular does not immediately define a homology class. The issue comes from the fact that the map $(f, f) : M_1 \times M_1 \to M_2 \times M_2$ is not transverse to the diagonal of M_2 (unless $\dim(M_2) = 0$). But we can always deform f by a small homotopy to a map f' transverse to f (see e.g.; [17, Cor IV.2.5]), i.e. such that the pair (f, f') is transverse to the diagonal in $M_2 \times M_2$.

So fix $f' \simeq_{h_0} f$ a small deformation of f transverse to it and let

$$\underline{\Delta}_1^f := \{(m, m') \in M_1 \times M_1 \mid f'(m) = f(m')\} \subset M_1 \times M_1.$$

This is now an n-dimensional submanifold of $M_1 \times M_1$, whose homology class is independent of the choice of f'. (See Figure 5 below for an example.) Define also the corresponding class of almost constant loops

$$\widehat{\underline{\Delta}}_1^f := \{(m, \gamma_{m,m'}, e(m')) \in \Lambda^{\varepsilon}_{N_2} \mid (m, m') \in \underline{\Delta}_1^f\}$$

where the loop $\gamma_{m,m'}$ is defined as

$$\gamma_{m,m'} : I^r \xrightarrow{1-d_{\mathbf{c}}(-)} I \xrightarrow{h_0(m,-)} M_2$$

for $d_{\mathbf{c}} : I^r \to I$ the normalised distance to the center of the cube. Note that, while the loops $\gamma_{m,m'}$ are all canonically contractible in M_2, they are not contractible anymore in general when considered as the tuple $(m, \gamma_{m,m'}, e(m')) \in \Lambda_{N_2}$, since there is no garantie that we can retract $f(m')$ to $f(m)$ along h_0 while staying inside N_2.

Proposition 4.6. *We have that $\alpha[M_1] = [\widehat{\underline{\Delta}}_1^f] \in H_n(\mathcal{R}^{\varepsilon}_{N_2})$, where $[M_1]$ is as before identified with the class of constant loops in Λ_1, and α is as in the previous sections.*

Proof. Let $D : M_1 \times D^k \to \Lambda_{N_2}$ be defined by $D(m, x) = (m, [f(m)], x)$, where $[f(m)]$ denotes the constant loop at $f(m)$. Then

$$\alpha[M_1] = \bar{p}^*\bar{\tau}_e \cap D[M_1 \times D^k] \in H_n(\mathcal{R}^{\varepsilon}_{N_2}).$$

The map D is homotopic to $D' : M_1 \times D^k \to \Lambda_{N_2}$ defined by $D(m, x) = (m, \Gamma_{h_0}(m, -), x)$. Hence $\bar{\alpha}[M_1]$ can also be computed as $\bar{p}^*\bar{\tau}_e \cap D'[M_1 \times D^k]$. We have

$$\begin{tikzcd}
M_1 \times D^k \arrow[rr, "D'"] & & \Lambda_{N_2} \arrow[r, "\mathbf{ev}"] & N_2 \subset M_1 \times M_2 \times D^k \\
\Sigma = (\mathbf{ev} \circ D')^{-1}(M_1 \times M_1) \arrow[u, hook] \arrow[rrr] & & & M_1 \times M_1. \arrow[u, hook, "\mathrm{id} \times (f,e)"']
\end{tikzcd}$$

By transversality, the normal bundle N_2 of $M_1 \times M_1$ in $M_1 \times M_2 \times D^k$ pulls back to a normal bundle of Σ in $M_1 \times D^k$, and thus $(D')^*\bar{p}^*\bar{\tau}_e$ is a Thom class for that embedding. Hence capping with that class intersects with the 0-section Σ, and $\bar{\alpha}[M_1]$ is the restriction of D' to Σ, which is precisely the class $[\widehat{\underline{\Delta}}_1^f]$ in the statement. □

Corollary 4.7. *If the class $\alpha[M_1] = [\widehat{\underline{\Delta}}_1^f] \in H_n(\Lambda^{\varepsilon}_{N_2})$ can be represented by a class with support in $\Lambda^{\varepsilon}_{N_2}|_{M_1}$, then T_f vanishes.*

Proof. The homotopy L_{N_2} is a homotopy relative to $\mathcal{R}_{N_2}|_{M_1}$, and in fact also preserves, up to homotopy $\Lambda^{\varepsilon}_{N_2}|_{M_1}$. It follows that T_f can be represented in constant loops and hence is 0 in $H_*(\Lambda_2, M_2)$. □

The fake diagonal Δ_1^f is typically not a submanifold of $M_1 \times M_1$, because of the lack of transversality encountered above. The following example shows that this can fail in a "tranverse way", with Δ_1^f is the union of Δ_1 and a transverse submanifold. We will also see in the example that $\underline{\Delta}_1^f$ is a better behaved object.

Example 4.8. Let $f : I \to I$ be a smooth map that folds part of the interval, as illustrated in Figure 5. Then f is homotopic to the identity, but the map $\Delta_1 \hookrightarrow \Delta_1^f \cong \Delta_1 \cup_{S^0} S^1$ is not a homotopy equivalence. In particular, we see that the question of whether $\Delta_1 \hookrightarrow \Delta_1^f$ is a homotopy equivalence is itself not a property invariant under replacing f by a homotopic map.

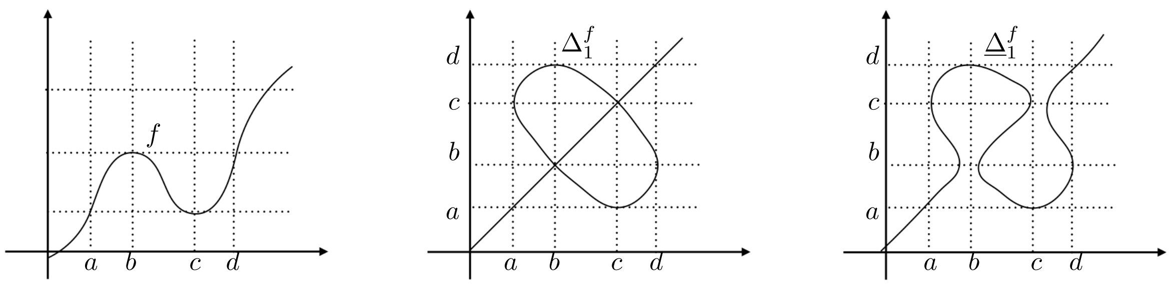

FIGURE 5. Graph of the map $f : I \to I$ of Example 4.8 and associated fake diagonal Δ_1^f

Example 4.9. Suppose that $M_2 = M_1/B$ for $B \subset M_1$ a contractible subspace, and suppose that $f : M_1 \to M_2$ is the collapse map. Then $\Delta_1^f = \Delta_1 \cup_{\Delta B} (B \times B)$ is homotopic to Δ_1. Indeed, the map $p_B : \Delta_1^f \to \Delta_1$ defined on $B \times B$ by $p_B(b, b') = (b, b)$ and on Δ_1 as the identity, is a homotopy inverse to the inclusion. In particular, Corollary 4.7 implies that T_f vanishes on elementary collapses.

The following question was in part raised by Tom Goodwillie[3]:

Question 1. Can one always modify f within its homotopy class so that $\Delta_1^f = \Delta_1 \cup D^f$ with D^f a submanifold of $M_1 \times M_1$ that is transverse to Δ_1? and if yes, are the classes $[D^f]$ and $[\underline{\Delta}_1^f]$ equal in $H_n(\Lambda^\varepsilon_{N_2}, \Lambda^\varepsilon_{N_2}|_{U_{M_1}})$?

Remark 4.10 (Simple homotopy equivalences). The previous example shows that T_f vanishes on elementary collapses. To show that T_f vanishes on simple homotopy equivalences more generally, as expected give the result of [22], one needs to understand how the obstruction T_f behaves under composition.

For our second vanishing condition, we start by defining a "smallness" condition for homotopies.

Definition 4.11. We call $f : M_1 \to M_2$ an ε*–bounded homotopy equivalence* if we can choose g, h_1, h_2 as above such that

(i) $|n - n'| < \varepsilon$ implies $|h_2(n, t) - h_2(n', t)| < \varepsilon$ for all $n, n' \in M_2$ and all $t \in I$;
(ii) $|f \circ h_1(m, t) - h_2(f(m), t)| < \varepsilon$ for every $m \in M_1$ and all $t \in I$.

Note that when f is ε–bounded with ε smaller than the injectivity radius of M_2, then we can choose $K(m, t, -)$ to be the geodesic path from $f \circ h_1(m, t)$ to $h_2(f(m), t)$, that is we can choose K to consist of small paths.

Corollary 4.12. *Suppose that the homotopy equivalence $f : M_1 \to M_2$ is ε-bounded. Then T_f vanishes.*

Proof. By Proposition 3.17(iii), we have that $L_{N_2}(\alpha[M_1] \times I)$ can be computed as $\underline{K}_{\mathcal{E}}(\mathbf{ev}\alpha[M_1] \times I)$, for $\underline{K}_{\mathcal{E}}$ the map of Lemma 3.15. Now the boundedness assumption on f is exactly ensuring that the map $\underline{K}_{\mathcal{E}}$ has image in small loops. Hence T_f is 0 in $H_*(\Lambda_2, M_2)$. □

Remark 4.13. (i) The lens spaces $L(1, 7)$ and $L(2, 7)$ used in Naef's example in [20] also appeared in the work of Longoni-Salvatore, who showed that their 2-points configuration spaces $F_2(L(1, 7))$ and $F_2(L(2, 7))$ are not homotopy equivalent [18]. Note that these configuration spaces are exactly the complements of the diagonal in $M \times M$ for M the one or the other manifold. The complement of the fake diagonal Δ_1^f appears in [24] in the context of studying the question of homotopy invariance of configuration spaces. It

[3]In a question session during the Andrew Ranicki memorial conference.

is a natural question to ask whether a condition for a homotopy equivalence f to respect the coproduct can be formulated in terms of the configuration spaces of two points in the manifolds. See also [23] for further relationship between the string topology product and coproduct, and the configuration space of two points in the manifold.

(ii) A naive version of the assumption of Corollary 4.7 would be that the inclusion $\Delta_1 \to \Delta_1^f$ is a homotopy equivalence, which is equivalent to the condition that $\Delta_1^f \to \Delta_2$ is a homotopy equivalence, that is requiring that the product homotopy equivalence $f \times f : M_1 \times M_1 \to M_2 \times M_2$ restricts to a homotopy equivalence on the inverse image of Δ_2. As pointed out to us by Wolfgang Lück, this is a codimension n analogue of Cappell's splitting theorem, that gives conditions involving the Whitehead torsion for this to hold when considering a codimension 1 submanifold of the target [3].

(iii) The coproduct is part of a larger (expected) family of string topology operations parametrized by the harmonic compactification of moduli space, see [31, Sec 6.6] for a rational version of this, or [9, 13] for work in this direction. Obstructions related to higher diagonals are to be expected for operations such as the families [30, Sec 4], and hence also related to configuration spaces of any number of points in the manifold.

Appendix A. Composing tubular embeddings

Given a composition of embeddings $M_1 \overset{f_1}{\hookrightarrow} M_2 \overset{f_2}{\hookrightarrow} M_3$ and choices of tubular neighborhoods for each embedding, we will construct here an associated tubular neighborhood for the composed map and study its properties.

Proposition A.1. *Consider*

(A.1)
$$\begin{tikzcd} N_1 \arrow[d, "p_1"'] \arrow[dr, hook, "\nu_1"] & N_2 \arrow[d, "p_2"'] \arrow[dr, hook, "\nu_2"] & \\ M_1 \arrow[r, hook, "f_1"] & M_2 \arrow[r, hook, "f_2"] & M_3. \end{tikzcd}$$

with f_1, f_2 codimension k and l embeddings respectively, N_1, N_2 the associated normal bundles and ν_1, ν_2 chosen tubular neighborhoods. Then

1. *The bundle $N_{12} := N_1 \oplus f_1^* N_2 \longrightarrow M_1$ is isomorphic to the normal bundle of $f_2 \circ f_1$.*
2. *There is an isomorphism*

$$\begin{tikzcd} N_{12} = N_1 \oplus f_1^*N_2 \arrow[rr, dashed, "\cong"] \arrow[d] \arrow[dr, "p"] & & \nu_1^*N_2 \cong N_2|_{\nu_1 N_1} \arrow[r, hook] \arrow[dl] & N_2 \arrow[d, "p_2"] \arrow[dr, "\nu_2"] & \\ M_1 \arrow[rrr, bend right, "f_1"'] & N_1 \arrow[l, "p_1"'] \arrow[rr, hook, "\nu_1"] & & M_2 \arrow[r, "f_2"'] & M_3 \end{tikzcd}$$

as bundles over N_1, where p is the natural projection, with the property that the resulting composition

$$\nu_{12} := \nu_2 \circ \hat{\nu}_1 : N_{12} \xrightarrow{\hat{\nu}_1} N_2 \xrightarrow{\nu_2} M_3$$

is a tubular neighborhood for $f_2 \circ f_1 : M_1 \hookrightarrow M_3$. Explicitly, $\nu_{12}(m, V, W) = \nu_2(\nu_1(m, V), W')$ for $W' \in (N_2)_{\nu_1(m,V)}$ the parallel transport of W along the path $\nu_1(m, tV)$.

3. *Let $\tau_1 \in C^k(M_2, \nu_1(N_1)^c)$ and $\tau_2 \in C^l(M_3, \nu_2(N_2)^c)$ be Thom classes for (N_1, ν_1) and (N_2, ν_2). Let $p_2^*\tau_1$ be the image of τ_1 along the maps*

$$C^k(M_2, \nu_1(N_1)^c) \xrightarrow{p_2^*} C^k(N_2, \hat{\nu}_1(N_{12})^c) \xrightarrow{\sim} C^k(M_3, \nu_{12}(N_{12})^c),$$

where the second map is the quasi-isomorphism induced by ν_2 and excision. Then

$$\tau_{12} := p_2^*\tau_1 \cup \tau_2 \in C^{k+l}(M_3, \nu_{12}(N_{12})^c)$$

is a Thom class for the tubular embedding ν_{12}. Moreover, the diagram

$$\begin{tikzcd} C_*(M_3) \arrow[r, "\llbracket\tau_2\cap\rrbracket"] \arrow[dr, "\llbracket\tau_{12}\cap\rrbracket"'] & C_{*-l}(M_2) \arrow[d, "\llbracket\tau_1\cap\rrbracket"] \\ & C_{*-k-l}(M_1) \end{tikzcd}$$

commutes up to homotopy, where

$$\llbracket\tau_1\cap\rrbracket\colon C_*(M_2) \to C_*(M_2, \nu_1(N_1)^c) \xrightarrow{[\tau_1\cap]} C_{*-k}(N_1) \xrightarrow{\simeq} C_{*-k}(M_1)$$

$$\llbracket\tau_2\cap\rrbracket\colon C_*(M_3) \to C_*(M_3, \nu_2(N_2)^c) \xrightarrow{[\tau_2\cap]} C_{*-l}(N_2) \xrightarrow{\simeq} C_{*-k}(M_2)$$

$$\llbracket\tau_{12}\cap\rrbracket\colon C_*(M_3) \to C_*(M_3, \nu_{12}(N_{12})^c) \xrightarrow{[\tau_{12}\cap]} C_{*-k-l}(N_{12}) \xrightarrow{\simeq} C_{*-k-l}(M_1)$$

are the maps induced by capping with the respective Thom classes after taking small simplices (see [14, A.2] *for details about the map $[\tau\cap]$), and retracting to the zero section at the end.*

To prove the proposition, we will, as in the rest of the paper, pick Riemannian metrics. This allows us for example to consider the normal bundle of an embedding as a subbundle of the tangent bundle of the target, as we explain now: Let

$$f : M \hookrightarrow M'$$

be an embedding of manifolds, with M compact and M' a Riemannian manifold. Let $p : N \to M$ be the normal bundle of f. The metric of M' gives an identification of N with a subbundle of $TM'|_M$:

$$N \equiv TM^\perp =: \{(m, W) \mid m \in M,\ W \in T_{f(m)}M' \text{ and } W \perp f_* T_m M\} \tag{A.2}$$

and thus splits the exact sequence

$$0 \to TM \to TM' \to N \to 0$$

of bundles over M. We will identify N with this subbundle.

The following lemma is an elementary consequence of the inverse function theorem.

Lemma A.2. *Let $N \to M$ be the normal bundle of the embedding $f : M \hookrightarrow M'$ as above. Suppose that*

$$\nu : N \longrightarrow M'$$

has the following properties:

(i) $\nu(m, 0) = f(m)$ *for all* $m \in M$.

(ii) The map

$$d\nu : T_{(m,0)}N \to T_{f(m)}M'$$

is surjective.

Then there exists $\varepsilon > 0$ so that the restriction of ν to N_ε is a tubular embedding of M in M', for $N_\varepsilon = \{(m, W) \in N \mid |W| < \varepsilon\}$.

Proof of Proposition A.1. Choose a Riemannian metric on M_3. Let M_1 and M_2 carry the induced metrics. Statement (1) follows from (suppressing the inclusion maps f_1, f_2):

$$N \equiv TM_1^\perp \subset TM_3;\ \ N_1 \equiv TM_1^\perp \subset TM_2;\ N_2 \equiv TM_2^\perp \subset TM_3,$$

so that for each $m \in M_1$,

$$N_m \equiv (N_1)_m \oplus (N_2)_{f(m)}.$$

As a consequence we can and will identify

$$N \cong N_{12} \equiv \left\{ (m, V, W) \;\middle|\; \begin{array}{l} m \in M_1, \\ V \in T_{f_1(m)}M_2,\ V \perp df_1 T_m M_1, \text{ and} \\ W \in T_{f(m)}M_3,\ W \perp df_2 T_{f_1(m)} M_2 \end{array} \right\}.$$

To prove (2), define $\hat{\nu}_1 : N_{12} \to N_2$ by

$$\hat{\nu}_1(m, V, W) = (\nu_1(m, V), W')$$

where $W' \in (N_2)_{\nu_1(m,V)}$ is the vector obtained from W by parallel transport along the path $\nu_1(m, tV)$. The inverse is given by parallel transporting back, which gives the claimed isomorphism by uniqueness of such transports.

We then apply the lemma in the case $M = M_1$, $M' = M_3$, $f = f_2 \circ f_1$, and $\nu = \nu_{12}$. We need to check condition (ii) of the lemma. Fix $m \in M_1$. Because $\nu_1 : N_1 \to M_2$ is a tubular embedding, the map $d\nu_1 : T_{(m,0)}N_1 \to T_m M_2$ is surjective, and the tangent space to M_2 in M_3 is in the image of $d\nu_{12} =: d(\nu_2 \circ \hat{\nu}_1)$. Now let $W \in N_2$; that is, $W \in T_m M_3$, with $W \perp T_m M_2$. We find by definition $\hat{\nu}_1(m, 0, W) = (m, W) \in N_2$ and $\nu_{12}(m, 0, W) = \nu_2 \circ \hat{\nu}_1(m, 0, W) = (m, W)$. (When $V = 0$, the parallel transport is trivial.) Since $T_m M_3$ is spanned by $T_m M_2$ and $d\nu_2 N_2$, we conclude that

$$\nu_{12*} : T_{(m,0)}N_{12} \to T_m M_3$$

is surjective.

To show the commutativity of the diagram in statement (3), we decompose the diagram as

$$\begin{tikzcd}
C_*(M_3) \arrow[r] \arrow[dr] & C_*(M_3, N_2^c) \arrow[r, "{[\tau_2\cap]}"] & C_{*-k}(N_2) \arrow[r, "p_2"] \arrow[d] & C_{*-k}(M_2) \arrow[d] \\
& C_*(M_3, N_{12}^c) \arrow[dr, "{[\tau_{12}\cap]}"'] & C_{*-k}(N_2, N_{12}^c) \arrow[r, "p_2"] \arrow[d, "{[p_2^*\tau_1\cap]}"] & C_{*-k}(M_2, N_1^c) \arrow[d, "{[\tau_1\cap]}"] \\
& & C_{*-k-l}(N_{12}) \arrow[r, "p_2"] \arrow[dr, "p_{12}"] & C_{*-k-l}(N_1) \arrow[d, "p_1"] \\
& & & C_{*-k-l}(M_1)
\end{tikzcd} \tag{A.3}$$

where we have suppressed the embeddings ν_i, $\hat{\nu}_i$ for readability. Commutativity follows from the naturality of the cap product $[\tau\cap]$ and the compatibility of the cup and cap products.

We are left to check that τ_{12} is the Thom class of the composition, which follows from its definition as

$$\tau_{12}\cap[M_3] = p_2^*\tau_1\cap(\tau_2\cap[M_3]) = p_2^*\tau_1\cap(f_2)_*[M_2] = (f_2)_*(f_2^*p_2^*\tau_1\cap[M_2]) = (f_2)_*(\tau_1\cap[M_2]) = (f_2)_*(f_1)_*[M_1],$$

in homology, where we used that $p_2 \circ f_2 : M_2 \to N_2 \to M_2$ is the identity. □

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