ON THE INVARIANCE OF THE STRING TOPOLOGY COPRODUCT

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Abstract. We show that the Goresky-Hingston coproduct in string topology is invariant under homotopy equivalences that satisfy an additional assumption, either stated in terms the restriction to the "fake diagonal", or in terms of boundedness for the homotopy.

Introduction

For a closed oriented manifold M, the Chas-Sullivan product [4] is a product on the homology of its free loop space $\Lambda M = \mathrm{Maps}(S^1, M)$ of the form

$$\wedge : H_p(\Lambda M) \otimes H_q(\Lambda M) \longrightarrow H_{p+q-n}(\Lambda M),$$

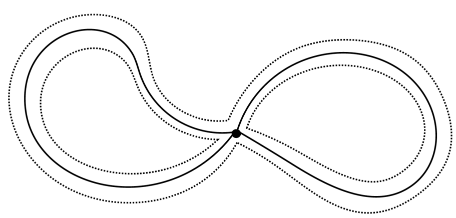
where $H_*(-) = H_*(-;\mathbb{Z})$ denotes singular homology with integral coefficients and $n = \dim M$. Goresky and Hingston defined in [10] a "dual" product in cohomology relative to the constant loops

$$\circledast : H^p(\Lambda M, M) \otimes H^q(\Lambda M, M) \to H^{p+q+n-1}(\Lambda M, M).$$

The associated homology coproduct

$$\vee : H_p(\Lambda M, M) \longrightarrow H_{p-n+1}(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M).^1$$

was defined by Sullivan [23]. The idea of the Chas-Sullivan product is to intersect the chains of basepoints of two chains of loops and concatenate the loops at the common basepoints. The homology coproduct looks instead for self-intersections at the basepoint within a single chain of loops and cuts. Both structures are represented geometrically by the same figure, either thought of as two loops intersecting at their basepoint,



or as a single loop having a self-intersection at its basepoint. For the coproduct, it is essential to look for self-intersections with the basepoint at all times t along the loop, including $t = 0$ and $t = 1$ (where all loops tautologically have a self-intersection!)—Tamanoi showed that the coproduct that only looks for self-intersections at $t = \frac{1}{2}$ is trivial [24]. In contrast, $H^*(\Lambda S^n, S^n)$ for an odd sphere S^n is generated by just 4 classes as an algebra with the cohomology product $\circledast$ [10, 15.3]. The domain $[0, 1]$ of the time variable t is the reason why the coproduct naturally is an operation in relative homology: it has non-trivial boundary terms coming from the boundary of the interval. This operation can be associated to a 1-parameter family of surfaces in the harmonic compactification of the moduli space of Riemann surfaces, making it a so-called *compactified operation*.

This coproduct detects *intersection multiplicity* of homology classes in the loop space, the largest number k such that any representing chain of a given homology class in $H_*(\Lambda M)$ necessarily has a loop with a k-fold self-intersection at its basepoint. (See [14, Thm 3.10].)

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[1] As we work with homology with integral coefficients, we do not in general have a Künneth isomorphism, which is why the coproduct in integral homology has target $H_*(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M)$ rather than $H_*(\Lambda M, M) \otimes H_*(\Lambda M, M)$.

While the above products and coproduct are all induced by chain maps (see eg., [14, Sec 1]), we will only consider in the present paper the induced maps in homology/cohomology.

It has been shown by several authors that the Chas-Sullivan product is homotopy invariant, in the sense that a homotopy equivalence $f : M_1 \to M_2$ induces an isomorphism $\Lambda f : H_*(\Lambda M_1) \to H_*(\Lambda M_2)$ of algebras with respect to the Chas-Sullivan product (see [7, Thm 1],[11, Prop 23],[8, Thm 3.7]). In contrast, a coproduct of the same flavour was used in [5, Sec 2.1] to define the differential in *string homology*, a knot invariant that detects the unknot. Also the paper [19] found indications of non-homotopy invariance in the S^1–equivariant version of the coproduct. And indeed, Naef showed recently through a computation with lens spaces that the coproduct is not homotopy invariant [18]. (See also Appendix B for an overview of Naef's example.) Note though that, over the rational and for simply-connected manifolds, Rivera-Wang showed that the Goresky-Hingston coproduct is homotopy invariant using a Hochschild homology model [22].

We give here two sufficient conditions for homotopy equivalences to respect the coproduct, one formulated in terms of the boundedness of the homotopy equivalence (condition B), and one formulated in terms of the "fake diagonal" (condition D):

A map $f : M_1 \to M_2$ is a homotopy equivalence if one can find a homotopy inverse g together with homotopies $g \circ f \simeq_{h_1} \mathrm{id}_{M_1}$ and $f \circ g \simeq_{h_2} \mathrm{id}_{M_2}$. We say that f is a *bounded homotopy equivalence* with respect to a metric on M_2 if one can find g, h_1, h_2 such that h_2 stays bounded and $h_2 \circ (f \times I)$ stays close to $f \circ h_1$ (see Definition 3.9). Our first condition on f is the following:

Condition B: The map f is a bounded homotopy equivalence with respect to a Riemannian metric on M_2.

For the other sufficient condition, let $\Delta_i \in M_i \times M_i$ denote the diagonal. We call

$$\Delta_1^f = \{(m, m') \in M_1 \times M_1 \mid f(m) = f(m')\}$$

the *fake diagonal*. Note that $\Delta_1 \subset \Delta_1^f = (f \times f)^{-1}(\Delta_2)$. Our alternative condition on f is:

Condition D: The inclusion $\Delta_1 \hookrightarrow \Delta_1^f$ is a weak homotopy equivalence.

Both conditions are trivially satisfied if f is a diffeomorphism. Condition D is also satisfied by elementary collapses, i.e. maps contracting a cell in M_1, see Example 2.2.

The two conditions will in addition require a "goodness assumption" on f, that we call Δ-*transversality* (see Definition 2.3). We expect that any f is homotopic to a map that is Δ-transverse (see Conjecture 2.4). For f not Δ-transverse, conditions D can instead be replaced by more technical versions of the conditions, involving neighborhoods of the fake diagonal, see condition D' in Section 2.1.

Conditions B and D both have the same purpose: ensuring that a certain map $j_0 : B^{\varepsilon_2}_{X_2} \to B^{\varepsilon_2}_{1,2}$ is a homology isomorphism, for certain spaces of "half-small loops" $B^{\varepsilon_2}_{X_2}$ and $B^{\varepsilon_2}_{1,2}$ defined in Section 2. We show in Proposition 2.5 that, under the Δ–transversality assumption, this map j_0 is a weak homotopy equivalence if and only if condition D holds. We expect that condition B is a strictly stronger condition.

Theorem A. *Let $f : M_1 \to M_2$ be a homotopy equivalence between closed oriented manifolds, homotopic to a Δ–transverse map satisfying either condition B or condition D, or to a map satisfying condition D'. Then the map induced by $\Lambda f : \Lambda M_1 \to \Lambda M_2$ in homology respects the coproduct, up to a sign given by the degree of f, that is the following diagram commute:*

$$\begin{array}{ccc} H_*(\Lambda M_1, M_1) & \xrightarrow{\quad\vee\quad} & H_{*+1-n}(\Lambda M_1 \times \Lambda M_1, M_1 \times \Lambda M_1 \cup \Lambda M_1 \times M_1) \\ \downarrow{\scriptstyle \Lambda f} & & \downarrow{\scriptstyle \Lambda f \times \Lambda f} \\ H_*(\Lambda M_2, M_2) & \xrightarrow{\deg(f)\ \vee} & H_{*+1-n}(\Lambda M_2 \times \Lambda M_2, M_2 \times \Lambda M_2 \cup \Lambda M_2 \times M_2). \end{array}$$

Likewise, the map induced by Λf in cohomology respects the corresponding cohomology products $\circledast$ and $\widehat{\circledast}$ (as defined in [10] and [14]) up to the same sign $\deg(f)$.

Remark 0.1. (i) Note that the question of whether Λf respects the coproduct only depends on the homotopy class of f. This is the reason it is enough to find a map $f' \simeq f$ satisfying the above properties. And in fact, Example 2.1 shows that condition D at least is not homotopy invariant.
(ii) Because the composition $M_1 \cong \Delta_1 \longrightarrow \Delta_1^f \longrightarrow \Delta_2 \cong M_2$ is the natural map induced by f, which is

assumed to be a homotopy equivalence, condition D that $\Delta_1 \to \Delta_1^f$ is a homotopy equivalence is equivalent to the condition that $\Delta_1^f \to \Delta_2$ is a homotopy equivalence, that is requiring that the product homotopy equivalence $f \times f : M_1 \times M_1 \to M_2 \times M_2$ restricts to a homotopy equivalence on the inverse image of Δ_2. As pointed out to us by Wolfgang Lück, this is a codimension n analogue of Cappell's splitting theorem, that gives conditions involving the Whitehead torsion for this to hold when considering a codimension 1 submanifold of the target [3].
(iii) A complete obstruction for a homotopy equivalence f to respect the coproduct can in principle be obtained by following through the diagram the potential failure of invertibility of the map labeled $[\![\bar{e}_I^* \overline{\tau}_e \cap]\!]$ in Diagram (2.1). In [18], a conjectural description of the obstruction is given in terms of the Whitehead torsion of the map.
(iv) The lens spaces $L(1,7)$ and $L(2,7)$ used in Naef's example in [18] also appeared in the work of Longoni-Salvatore, who showed that their 2-points configuration spaces $F_2(L(1,7))$ and $F_2(L(2,7))$ are not homotopy equivalent [16]. Note that these configuration spaces are exactly the complements of the diagonal in $M \times M$ for M the one or the other manifold. The complement of the fake diagonal Δ_1^f appears in [20] in the context of studying the question of homotopy invariance of configuration spaces. It is a natural question to ask whether a condition for a homotopy equivalence f to respect the coproduct can be formulated in terms of the configuration spaces of two points in the manifolds. (See also [19] for further relationship between the string topology product and coproduct, and the configuration space of two points in the manifold.)
(v) The coproduct is part of a larger (expected) family of string topology operations parametrized by the harmonic compactification of moduli space, see [27, Sec 6.6] for a rational version of this, or [9, 13] for work in this direction. Obstructions related to higher diagonals are to be expected for operations such as the families [26, Sec 4], and hence also related to configuration spaces of any number of points in the manifold.

We use in this paper the Cohen-Jones approach to string topology [6]: The products and coproducts described above are all lifts to the loop space ΛM of the intersection product on $H_*(M)$. While the intersection product is most easily defined as the Poincaré dual of the cup product on $H^*(M)$, the definition of the coproduct $\vee$ used here is, as in [6] for the product $\wedge$, a lift to the loop space of the alternative definition of the intersection product in terms of a Thom-Pontryagin construction.

Our proof of Theorem A is heavily inspired by the proof of homotopy invariance of the Chas-Sullivan product given by Gruher and Salvatore in [11]. For the coproduct, extra care is needed because of the boundary terms described above, and indeed the extra assumption in Theorem A comes from the relative terms (see Section 2.1). The proof can also be seen as a lift to the loop space of a proof of homotopy invariance of the intersection product using its Thom-Pontryagin definition. As the case of the intersection product is much simpler than that of the coproduct $\vee$, but yet already contains many of the crucial ingredients, we give it here in the first section of the paper.

Note that it is much easier to use Poincaré duality to show that the classical homology intersection product is homotopy invariant; it is in fact more generally invariant under degree d maps after an appropriate scaling, see e.g., [1, VI, Prop 14.2]. In [12], the first author proved that the same invariance under degree d maps holds for the restriction of the coproduct $\vee$ to the based loop space under the additional assumption that the target manifold M_2 is simply-connected. Given these two results, one can wonder whether this stronger invariance result holds more generally for the whole loop space, and also for the Chas-Sullivan product, under the simple connectivity assumption.

Organization of the paper. In Section 1, we give a proof of the homotopy invariance of the homology intersection product (Proposition 1.7) that does not use Poincaré duality. The intermediate results proved in that section will be used later in the paper. Section 2 gives a plan of the proof of homotopy invariance of the string coproduct, in particular setting up the key Diagram (2.1), whose commutativity is then proved in Section 3. The proof of Theorem A is given at the end of Section 3. Finally Appendix A details the relationship between the tubular neighborhoods of two composable embedding and the tubular neighborhood of their composition and Appendix B gives an overview of Naef's example.

Acknowledgements. The first version of this paper contained a "proof" of homotopy invariance of the coproduct that was contradicted by Naef's example [18]. The error in the original version of the paper was that the chosen relative term for the middle space in Diagram (2.1) made the maps labeled $[\![\bar{e}_I^*\overline{\eta}\cap]\!]$ in the diagram not an isomorphism. We have here corrected that relative term to make that map a standard suspension isomorphism, leading to the less standard map $[\![\bar{e}_I^*\overline{\tau}_e\cap]\!]$ not necessarily being an isomorphism, leading to the additional assumption in Theorem A. We thank Florian Naef for discussing his example with us early on. In the course of this work, the first author was supported by the Institute for Advanced Study and the second author by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) and the Copenhagen Centre for Geometry and Topology (DRNF151), the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 772960), the Heilbronn Institute for Mathematical Research, and would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Homotopy Harnessing Higher Structures (EPSRC grant numbers EP/K032208/1 and EP/R014604/1).

1. Homotopy invariance of the intersection product

The easiest way to prove the homotopy invariance of the intersection product on the homology of closed manifolds is to use its definition as Poincaré dual of the cup product, and use the homotopy invariance of the cup product. In fact, the compatibility is easily checked with any degree d map, up to appropriate scaling, see e.g., [1, VI,Prop 14.2]. We will present here a proof of the homotopy invariance using instead the definition of the intersection product via capping with a Thom class, as a warm-up, and partial preparation, for the invariance of the string topology coproduct.

Let M be a closed oriented Riemannian manifold of injectivity radius ρ, and let TM denote its ε–tangent bundle, where ε is small compared to ρ. (To be precise, the paper [14] whose constructions we use assumes $\varepsilon < \frac{\rho}{14}$.) We will also assume that $\varepsilon < \frac{1}{2}$.

The normal bundle of the diagonal embedding $M \hookrightarrow M \times M$ is isomorphic to TM, and we can choose as tubular embedding the map $\nu_M : TM \to M \times M$ given by $\nu_M(m, V) = (m, \exp_m V)$. (By an *embedding*, we will in this paper always mean a C^1–map that is an immersion and a homeomorphism onto its image.) Note that ν_M has image the ε–neighborhood of the diagonal

$$U = \{(m, m') \in M \times M \mid |m - m'| < \varepsilon\} \subset M \times M.$$

Let $\tau_M \in C^n(M \times M, U^c)$ be a chosen Thom class for this tubular neighborhood and $R\colon U \to M$ the associated retraction. After going through "small simplices" to get a chain inverse of excision, capping with the Thom class defines a map

$$[\tau_M\cap]\colon C_*(M \times M) \longrightarrow C_*(M \times M, U^c) \longrightarrow C_{*-n}(U).$$

(See eg., [14, A.2] for a detailed description.)

Up to sign, the intersection product on a closed manifold M can be defined as the composition

$$C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \longrightarrow C_{p+q}(M \times M, U^c) \xrightarrow{\tau_M\cap} C_{p+q-n}(U) \xrightarrow{R} C_{p+q-n}(M)$$

(See eg., [14, App B].)

Let $f : M_1 \to M_2$ be a homotopy equivalence between two closed oriented Riemannian manifolds M_1 and M_2. In particular f has degree ± 1, that is $f[M_1] = \pm[M_2]$. Let U_i be the ε_i–tubular neighborhoods of the diagonal in $M_i \times M_i$ as above, with Thom class τ_{M_i} and $R_i : U_i \to M_i$ the associated retraction. After changing f in its homotopy class, we may assume that

- f is smooth;
- ε_1 is small enough so that $f(U_1) \subset U_2$,

where the last property uses the compactness assumption. Leaving out the cross-product, which is well-known to be natural, we want to show that the diagram

$$\begin{array}{ccccccc} H_{p+q}(M_1\times M_1) & \longrightarrow & H_{p+q}(M_1\times M_1,U_1^c) & \xrightarrow{\tau_{M_1}\cap} & H_{p+q-n}(U_1) & \xrightarrow{R_1} & H_{p+q-n}(M_1) \\ \Big\downarrow{\scriptstyle f\times f} & & & & \Big\downarrow{\scriptstyle f\times f} & & \Big\downarrow{\scriptstyle f} \\ H_{p+q}(M_2\times M_2) & \longrightarrow & H_{p+q}(M_2\times M_2,U_2^c) & \xrightarrow{\tau_{M_2}\cap} & H_{p+q-n}(U_2) & \xrightarrow{R_2} & H_{p+q-n}(M_2) \end{array}$$

commutes. The difficulty is that we cannot assume that f takes U_1^c to U_2^c, so that we do not have a vertical map in the second column in the diagram. Likewise, we cannot use f directly to compare the Thom classes τ_{M_1} and τ_{M_2}. To make up for this, we will replace f by an embedding and use instead the larger diagram (1.4) below.

Pick an embedding $e : M_1 \hookrightarrow D^k$, for some k large. Then the map $(f,e)\colon M_1 \to M_2\times D^k$ is an embedding too and we can consider the composition of embeddings

(1.1)
$$\begin{array}{ccc} M_1 & \overset{\Delta_{M_1}}{\hookrightarrow} & M_1\times M_1 \\ & \underset{(\mathrm{id},f,e)}{\searrow} & \Big\downarrow{\scriptstyle id\times(f,e)} \\ & & M_1\times M_2\times D^k. \end{array}$$

To this situation, we can associate a diagram

$$\begin{array}{ccccc} TM_1 & & N_2 = M_1\times N(f,e) & & \\ \Big\downarrow{\scriptstyle p_1} & \overset{\nu_1=\nu_{M_1}}{\searrow} & \Big\downarrow{\scriptstyle p_2} & \overset{\nu_2=id\times\nu_e}{\searrow} & \\ M_1 & \overset{\Delta_1}{\hookrightarrow} & M_1\times M_1 & \overset{id\times(f,e)}{\hookrightarrow} & M_1\times M_2\times D^k, \end{array}$$

where $\nu_1 := \nu_{M_1} : TM_1 \to M_1\times M_1$ is the tubular neighborhood of ΔM_1 defined as above, and $N(f,e)$ is the normal bundle of (f,e) with $\nu_e : N(f,e) \to M_2\times D^k$ a chosen tubular embedding; we choose here, as in [2, Thm 12.11] to define ν_e using the restriction of the exponential map of $M_2\times D^k$ to the subbundle $N(f,e) \le TM_2\oplus\mathbb{R}^k$ of vectors normal to $d(f,e)TM_1$. Pick a Thom class $\tau_e \in C^k(M_2\times D^k,(\nu_e N(f,e))^c)$ for ν_e. Then

$$\overline{\tau}_e := 1\times\tau_e \in C^k(M_1\times M_2\times D^k,(\nu_2 N_2)^c) \tag{1.2}$$

is a Thom class of ν_2. Applying Proposition A.1, we now have the following:

Proposition 1.1. *Consider the sequence of embeddings* $M_1 \overset{\Delta_1}{\hookrightarrow} M_1\times M_1 \overset{id\times(f,e)}{\hookrightarrow} M_1\times M_2\times D^k$ *and let* ν_1,ν_2 *and* $\tau_1=\tau_{M_1},\tau_2=\overline{\tau}_e$ *be the embeddings and Thom classes chosen above. Then the following hold:*

(1) The bundle $N_3 := TM_1\oplus N(f,e)$ *is isomorphic to the normal bundle of the composed embedding* $(id,f,e) : M_1 \to M_1\times M_2\times D^k$.

(2) There is an isomorphism $\hat{\nu}_1 : N_3 = TM_1\oplus N(f,e) \xrightarrow{\cong} N_2|_{U_1} = (M_1\times N(f,e))|_{U_1}$, *as bundles over* $TM_1\cong U_1$, *with the property that*

$$\nu_3 := \nu_2\circ\hat{\nu}_1 : N_3 \longrightarrow M_1\times M_2\times D^k$$

is a tubular embedding for (id,f,e). *Explicitly,*

$$\nu_3(m,V,W) = \nu_2(\nu_1(m,V),W') = (m,\nu_e(\exp_m V,W'))$$

for $W' \in (N(f,e))_{\nu_1(m,V)}$ *the parallel transport of* W *along the path* $\nu_1(m,tV)$.

(3) The class

$$\tau_3 := p_2^*\tau_{M_1}\cup\overline{\tau}_e \in C^{n+k}(M_1\times M_2\times D^k,N_3^c)$$

is a Thom class for the tubular embedding ν_3 and the diagram

$$\begin{array}{ccc} H_{*+k}(M_1 \times M_2 \times D^k) & \xrightarrow{[\![\tau_e \cap]\!]} & H_*(M_1 \times M_1) \\ & \searrow^{[\![\tau_3 \cap]\!]} & \downarrow{\scriptstyle [\![\tau_{M_1} \cap]\!]} \\ & & H_{*-n}(M_1) \end{array}$$

commutes, where we use the notation $[\![\tau\cap]\!]\colon C_*(X) \to C_*(Y)$ for maps that are compositions of the form

$$[\![\tau\cap]\!] : C_*(X) \to C_*(X, U^c) \xrightarrow{[\tau\cap]} C_*(U) \xrightarrow{R} C_*(Y)$$

for U a tubular neighborhood of X inside Y with associated Thom class τ and R the associated retraction.

The embedding $(\mathrm{id}, f, e) : M_1 \to M_1 \times M_2 \times D^k$ is also equal to the composition of embeddings

(1.3)
$$\begin{array}{ccc} M_1 & \xrightarrow{(\mathrm{id},f)} & M_1 \times M_2 \\ & \searrow^{(\mathrm{id},f,e)} & \downarrow{\scriptstyle (\mathrm{id},e\circ\pi)} \\ & & M_1 \times M_2 \times D^k. \end{array}$$

for $(\mathrm{id}, e \circ \pi) : M_1 \times M_2 \to M_1 \times M_2 \times D^k$ taking (m, n) to $(m, n, e(m))$. This gives another associated diagram of tubular embeddings, namely

$$\begin{array}{ccccc} f^*TM_2 & & \bar{N}_2 = M_1 \times M_2 \times \mathbb{R}^k & & \\ \downarrow{\scriptstyle \bar{p}_1} & \searrow^{\bar{\nu}_1} & \downarrow{\scriptstyle \bar{p}_2} & \searrow^{\bar{\nu}_2} & \\ M_1 & \xrightarrow{(\mathrm{id},f)} & M_1 \times M_2 & \xrightarrow{(\mathrm{id},e\circ\pi)} & M_1 \times M_2 \times D^k, \end{array}$$

where we choose the explicit tubular embedding

$$\bar{\nu}_1(m, V) = (m, \exp_{f(m)} V) \quad \text{and} \quad \bar{\nu}_2(m, n, x) = (m, n, e(m) + \ell(x))$$

with ℓ a scaling map so that $\bar{\nu}_2$ is well-defined, recalling also that TM_2 denotes the ε_2–tangent bundle of M_2 so that $\bar{n}u_1$ is well-defined. Note that $\bar{\nu}_1$ has image

$$U_{1,2} := \{(m, n) \in M_1 \times M_2 \mid |f(m) - n| < \varepsilon_2\} = (f \times 1)^{-1}(U_2)$$

for $U_2 \subset M_2 \times M_2$ the ε–neighborhood of the diagonal as above.

Lemma 1.2. *The map $f \times 1$ induces a map of pairs $f \times 1 : (M_1 \times M_2, U_{1,2}^c) \to (M_2 \times M_2, U_2^c)$ and, if f is a homotopy equivalence, the class $(f \times 1)^*\tau_{M_2} \in C^n(M_1 \times M_2, U_{1,2}^c)$ is a Thom class for the tubular neighborhood $(f^*TM_2, \bar{\nu}_1)$.*

Proof. The Thom class $\tau \in C^n(M_1 \times M_2, U_{1,2}^c)$ is characterised by the property $\tau \cap [M_1 \times M_2] = [(1 \times f)M_1]$. We are interested in the class $\tau = (f \times 1)^*\tau_{M_2}$. By the naturality of the cap product,

$$\begin{aligned} (f \times 1)_*((f \times 1)^*\tau_{M_2} \cap [M_1 \times M_2]) &= \tau_{M_2} \cap (f \times 1)_*[M_1 \times M_2] \\ &= \deg(f)\tau_{M_2} \cap [M_2 \times M_2] = \deg(f)[\Delta M_2]. \end{aligned}$$

Now $(f \times 1)_*[(1 \times f)M_1] = \Delta f[\Delta M_1] = \deg(f)[\Delta M_2]$ and the result follows from the fact that $(f \times 1)_*$ is invertible. □

For the tubular neighborhood $(M_1 \times M_2 \times \mathbb{R}^k, \bar{\nu}_2)$, the inclusion $M_1 \times M_2 \times \partial D^k \subset \bar{\nu}_2 \bar{N}_2^c$ induces a quasi-isomorphism

$$j^* : C^*(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \xrightarrow{\simeq} C^*(M_1 \times M_2 \times D^k, (\bar{\nu}_2 \bar{N}_2)^c)$$

Let

$$\overline{\eta} := j^*(1 \times \eta)$$

for $\eta \in C^k(D^k, \partial D^k)$ a chosen representative of the dual of the fundamental class. Then $\overline{\eta}$ is a Thom class for $\bar{N}_2$ as $j_*(\overline{\eta} \cap [M_1 \times M_2 \times D^k]) = (1 \times \eta) \cap j_*[M_1 \times M_2 \times D^k] = [M_1 \times M_2 \times \{0\}]$ is homologous to the image of the zero-section of $\bar{N}_2$ under j_*. Indeed, the embedding $(\mathrm{id}, e \circ \pi)$ is isotopic to the embedding $(\mathrm{id}, 0)$, replacing e by the constant map at $0 \in D^k$, through an isotopy only affecting the disc.

Applying Proposition A.1 and Lemma 1.2 to this new factorisation of the embedding (id, f, e), we get

Proposition 1.3. *Consider the sequence of embeddings* $M_1 \overset{(\mathrm{id},f)}{\hookrightarrow} M_1 \times M_2 \overset{(\mathrm{id},e\circ\pi)}{\hookrightarrow} M_1 \times M_2 \times D^k$ *and let* $\bar{\nu}_1, \bar{\nu}_2$ *and* $\bar{\tau}_1 = (f \times 1)^*\tau_{M_2}, \bar{\tau}_2 = \overline{\eta}$ *be the embeddings and associated Thom classes chosen above. Assume that* f *is a homotopy equivalence. Then the following hold:*

(1) *The bundle* $\bar{N}_3 := f^*TM_2 \oplus \mathbb{R}^k$ *is isomorphic to the normal bundle of the composed embedding* $(id, f, e) : M_1 \longrightarrow M_1 \times M_2 \times D^k$.

(2) *There is an isomorphism* $\tilde{\nu}_1 : \bar{N}_3 = f^*TM_2 \oplus \mathbb{R}^k \xrightarrow{\cong} \bar{N}_2|_{U_{1,2}} = M_1 \times M_2 \times \mathbb{R}^k|_{U_{1,2}}$ *as bundles over* $f^*TM_2 \cong U_{1,2}$, *with the property that*

$$\bar{\nu}_3 := \bar{\nu}_2 \circ \tilde{\nu}_1 \colon \bar{N}_3 \longrightarrow M_1 \times M_2 \times D^k$$

is a tubular embedding for (id, f, e). *Explicitly,*

$$\bar{\nu}_3(m, V, W) = \bar{\nu}_2(\bar{\nu}_1(m, V), W) = (m, \exp_{f(m)} V, e(m) + W)$$

for $W \in \mathbb{R}^k_{(m,f(m))} \equiv \mathbb{R}^k_{(m,\exp_{f(m)} V)}$.

(3) *The class*

$$\bar{\tau}_3 := \bar{p}_2^*(f \times 1)^*\tau_{M_2} \cup \overline{\eta} \in C^{n+k}(M_1 \times M_2 \times D^k, \bar{\nu}_3\bar{N}_3^c)$$

is a Thom class for the tubular embedding $\bar{\nu}_3$ *and the diagram*

$$\begin{array}{ccc} H_{*+k}(M_1 \times M_2 \times D^k) & \xrightarrow{\llbracket \overline{\eta} \cap \rrbracket} & H_*(M_1 \times M_2) \\ & \searrow^{\llbracket \bar{\tau}_3 \cap \rrbracket} & \downarrow{\scriptstyle \llbracket (f\times 1)^*\tau_{M_2} \cap \rrbracket} \\ & & H_{*-n}(M_1) \end{array}$$

commutes, where the notation $\llbracket \tau \cap \rrbracket$ *is as in Proposition 1.1.*

We have given two explicit tubular neighborhoods (N_3, ν_3) and $(\bar{N}_3, \bar{\nu}_3)$ of the embedding (id, f, e), coming from two different ways of considering it as a composition of embeddings. The bundles N_3 and $\bar{N}_3$ are isomorphic, as they are both isomorphic to the normal bundle of the embedding. More than that, the uniqueness theorem for tubular neighborhoods implies that the embeddings are necessarily isotopic, and the following result holds:

Proposition 1.4. *Let* $(N_3, \nu_3), \tau_3$ *and* $(\bar{N}_3, \bar{\nu}_3), \bar{\tau}_3$ *be the tubular neighborhoods and Thom classes for the embedding* (id, f, e) *obtained in Propositions 1.1 and 1.3. There exists a bundle isomorphism* $\phi : N_3 \to \bar{N}_3$ *and a diffeomorphism* $h : M_1 \times M_2 \times D^k \longrightarrow M_1 \times M_2 \times D^k$ *fibered over the projection to* M_1 *such that*

(1) h *restricts to the identity on* $M_1 \overset{(\mathrm{id},f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$ *and outside* $\exp_{(m,f(m),e(m))} TM_2 \oplus \mathbb{R}^k \subset \{m\} \times M_2 \times D^k$ *for each* m. *In particular, it also fixes* $M_1 \times M_2 \times \partial D^k$;

(2) h *is isotopic over* M_1 *to the identity relative to the same subspaces;*

(3) $h \circ \nu_3 = \bar{\nu}_3 \circ \phi : \delta N_3 \longrightarrow M_1 \times M_2 \times D^k$, *for* δN_3 *(resp.* $\delta \bar{N}_3 = \phi(\delta N_3)$*) an appropriately chosen isomorphic subbundle.*

(4) *For all* $(m, n, x) \in M_1 \times M_2 \times D^k$, *we have* $h(m, n, x) = (m, n', x')$ *with* $|n - n'| < 2\varepsilon_2$.

(5) $[h^*\bar{\tau}_3] = [\tau_3] \in H^{n+k}(M_1 \times M_2 \times D^k, \nu_3(N_3)^c)$.

Note that (3) implies that $h(\nu_3\delta N_3) = \bar{\nu}_3\delta\bar{N}_3$ and $h((\nu_3\delta N_3)^c) = (\bar{\nu}_3\delta\bar{N}_3)^c$.

Proof. Uniqueness of tubular neighborhoods (see e.g. [15, Chap 4, Thm 5.3]) gives that the tubular neighborhoods (N_3, ν_3) and $(\bar{N}_3, \bar{\nu}_3)$ are isotopic, and this is what we will use to produce the diffeomorphism h. To obtain detailed properties of the resulting h in our particular case, we follow the actual construction of the isotopy in the proof of this theorem in [15]. The first step in the proof is to replace N_3 by an isomorphic subbundle δN_3 such that $\nu_3(\delta N_3) \subset \bar{\nu}_3(\bar{N}_3)$. Then one considers the map $g = \bar{\nu}_3^{-1} \circ \nu_3 : \delta N_3 \to \bar{N}_3$. In our case, because ν_3 and $\bar{\nu}_3$ both restrict to the identity on M_1 in the first factor, having the form $\nu_3(m,V,W) = (m, \nu_e(\exp_m V, W'))$ and $\bar{\nu}_3(m,X,Y) = (m, \exp_{f(m)} X, e(m)+Y)$, we have that g has the form $g(m,V,W) = (m,X,Y)$ sends fibers to fibers. Now the map g is fiberwise homotopic to its fiber derivative $T_{M_1}g : \delta N_3 \to \bar{N}_3$ through the homotopy

$$H(m,V,W,t) = \begin{cases} t^{-1}g(m,tV,tW) & t \in (0,1] \\ T_{M_1}g(m,V,W) & t = 0 \end{cases}$$

We set $\phi = T_{M_1}g$ and obtained the desired isotopy $F : \delta N_3 \times I \to M_1 \times M_2 \times D^k$ from $\nu_3 = \bar{\nu}_3 \circ g$ to $\bar{\nu}_3 \circ \phi$ by setting

$$F(m,V,W,1-t) = \bar{\nu}_3(H(m,V,W,t)).$$

By the isotopy extension theorem (see eg. [15, Chap 8, Thm 1.3]), there exists an isotopy of diffeomorphisms

$$\hat{F} : (M_1 \times M_2 \times D^k) \times I \longrightarrow M_1 \times M_2 \times D^k$$

such that $\hat{F}(-,0)$ is the identity on $M_1 \times M_2 \times D^k$ and such that $\hat{F}$ restricts to F on $\nu_3(\delta N_3) \times I$. As $\hat{F}$ is only non-trivial in a neighborhood of the image of F, can also be chosen to be fiberwise, and F has support inside $\bar{N}_3 \cong TM_2 \oplus \mathbb{R}^k$, $\hat{F}$ can be constructed so that it has support within the slices $\exp_{(m,f(m),e(m))} TM_2 \oplus \mathbb{R}^k \subset \{m\} \times M_2 \times D^k$. It also fixes $(\mathrm{id}, f, e)M_1$ because F fixes it. Hence by construction, $h := \hat{F}(-,1)$ is a diffeomorphism of $M_1 \times M_2 \times D^k$ satisfying (1)–(3) in the statement.

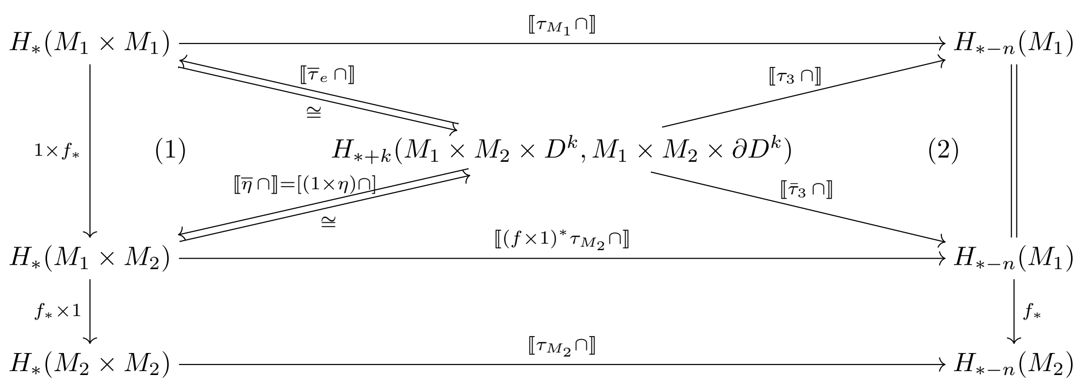
We check that (4) also holds. Possibly after replacing δN_3 by an even smaller subbundle, we can assume that $\hat{F}$ is the identity outside $\bar{N}_3$. So we only have to worry about points $(m,n,x) \in \bar{N}_3$. Now $\hat{F}$ is defined extending the time dependent vector field dF to a neighborhood of the image by scaling to the trivial vector field, and F being fiberwise, we have chosen this vector field to also be fiberwise, and hence the resulting homotopy will, like F, be fiberwise in $\bar{N}_3$. But now points (m,n,x) of $\bar{N}_3$ have the property that $|n - f(m)| < \varepsilon_2$, from which the result follows.

Because h is a diffeomorphism compatible with the tubular embeddings, it pulls back the Thom class to a Thom class. Unicity of the Thom class in cohomology gives the last part of the statement. □

Assumption 1. Recall that $N_3 \cong N_2|_{U_1}$ and $\bar{N}_3 \cong \bar{N}_2|_{U_{1,2}}$. We replace N_3 by the smaller subbundle δN_3, and $\bar{N}_3$ by $\delta\bar{N}_3 = \phi(\delta N_3)$, scaling U_1, N_2, $U_{1,2}$, U_2 and $\bar{N}_2$ accordingly.

Consider now the diagram

(1.4)



Invariance of the intersection product will follow from the commutativity of this diagram once we have also shown that the maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are isomorphisms. This last fact is a consequence of the homology suspension theorem for $[\![\overline{\eta} \cap]\!]$ as it identifies with the map $[(1 \times \eta)\cap]$, and, for $[\![\overline{\tau}_e \cap]\!]$, will follow from the commutativity of (1) in the diagram and the fact that f is a homology isomorphism. [2]

We analyse the diagram step by step. The top triangle commutes by Proposition 1.1(3), and the middle triangle by Proposition 1.3(3). The bottom square commutes by the naturality of the cap product. So what we need to show is that the triangles labeled (1) and (2) commute. To prove commutativity of (1), we use the following lemma:

[2]Note that the map $[\![\tau_e \cap]\!]$ in the diagram is not directly a case of a Thom isomorphism because the source of the map is not correct, and thus we cannot deduce that it is an isomorphism from the Thom isomorphism theorem.

Lemma 1.5. *Let $i : (M_2 \times D^k, M_2 \times \partial D^k) \longrightarrow (M_2 \times D^k, \nu_e(N(f,e))^c)$ be the inclusion of pairs. If f is a degree d map, then*

$$i^*[\tau_e] = d[1 \times \eta] \in H^k(M_2 \times D^k, M_2 \times \partial D^k)$$

for τ_e the Thom class of $(N(f,e), \nu_e)$ and $\eta \in C^k(D^k, \partial D^k)$ a generator as above.

Proof. Consider the diagram

$$\begin{array}{ccccc} H^*(N(f,e),\partial N(f,e)) & \xleftarrow[\cong]{\nu_e^*} & H^*(M_2\times D^k,\nu_e(N(f,e))^c) & \xrightarrow{i^*} & H^*(M_2\times D^k, M_2\times\partial D^k)\\ \downarrow{\scriptstyle \cap[N(f,e)]} && \downarrow{\scriptstyle\cap[M_2\times D^k]} && \downarrow{\scriptstyle\cap[M_2\times D^k]}\\ H_{n+k-*}(N(f,e)) & \xrightarrow{\nu_{e*}} & H_{n+k-*}(M_2\times D^k) & \xleftarrow{i_*=id} & H_{n+k-*}(M_2\times D^k)\end{array}$$

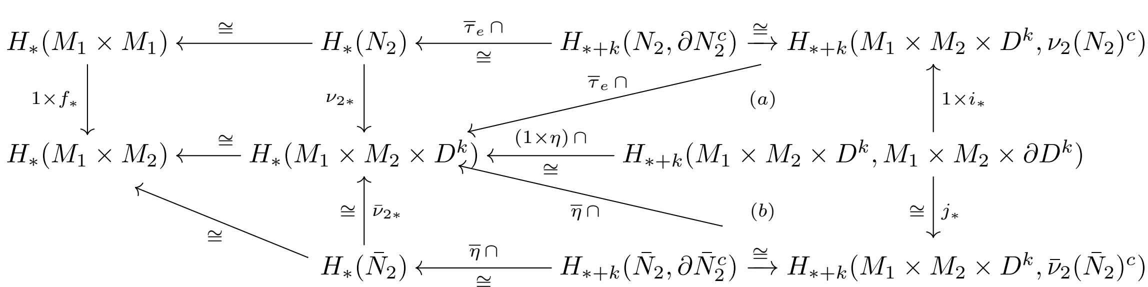
where the vertical maps cap with the fundamental class in each case and ν_e^* is an isomorphism by excision. The two squares commute by naturality of the cap product, given that $[M_2 \times D^k] = i_*[M_2 \times D^k] = \nu_{e*}[N(f,e)] \in H^*(M_2 \times D^k, N(f,e)^c)$. Hence in homology we have a commuting diagram

$$\begin{array}{ccccc} [\tau_e]\in & H^k(M_2\times D^k,\nu_e(N(f,e))^c) & \xrightarrow{i^*} & H^k(M_2\times D^k, M_2\times\partial D^k) & \ni \deg(f)[1\times\eta]\\ & \downarrow{\scriptstyle (\cap[N(e,f)])\circ\nu_e^*} && \downarrow{\scriptstyle\cap[M_2\times D^k]} & \\ & H_n(N(f,e)) & \xrightarrow{\nu_{e*}} & H_n(M_2\times D^k) & \\ & \uparrow{\scriptstyle\cong} && \downarrow{\scriptstyle\cong} & \\ [M_1]\in & H_n(M_1) & \xrightarrow{f} & H_n(M_2) & \ni \deg(f)[M_2]\end{array}$$

where the first square commutes by the commutativity of the previous diagram, and the bottom square commutes as it comes from a commuting square in the category of spaces. Now by definition of the Thom classes, $[\tau_e]$ is taken to $[M_1]$ along the left vertical maps, and $[1 \times \eta]$ to $[M_2]$ along the right vertical maps as $[1 \times \eta]$ is a Thom class for the trivial D^k–bundle on M_2. On the other hand f takes $[M_1]$ to $d[M_2]$ along the bottom horizontal map, from which the result follows. □

Proposition 1.6. *Assume that f is a degree 1 map. The subdiagram labeled (1) in diagram (1.4) commutes. Moreover, the maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are both isomorphisms.*

Proof. Recall that $N_2 = M_1 \times N(f,e)$. Spelling out the maps, the diagram becomes the following:



The subdiagram labeled (b) commutes by naturality of the cap product since $i^*[\overline{\tau}_e] = [1 \times \eta]$ by Lemma 1.5 as f has degree 1. Likewise subdiagram (b) commutes as $\overline{\eta} = j^*(1 \times \eta)$ by definition. The rest commutes on the chain or space level by definition of the maps.

We have marked in the diagrams the maps that are isomorphisms because of the Thom isomorphism, excision, or homotopy equivalences coming either from contracting a disc or a bundle, or from the equivalence $\bar{N}_2 \xrightarrow{\simeq} M_1 \times \mathcal{M}_2 \times D^k$ and likewise for the complements. From this we see that the map $[\![\overline{\eta} \cap]\!]$ is a composition of isomorphisms, and hence is itself an isomorphism. And because $1 \times f_*$ is an isomorphism, the commutativity of the diagram gives that also $[\![\overline{\tau}_e \cap]\!]$ is an isomorphism. □

Proposition 1.7. *Let $f : M_1 \to M_2$ be a homotopy equivalence between closed manifolds. Then f induces a ring map $H_*(M_1) \to H_*(M_2)$, with ring structure given by the intersection product, up to the sign* $\deg(f)$*, that is the diagram*

$$\begin{array}{ccc}
H_*(M_1 \times M_1) & \xrightarrow{[\![\tau_{M_1} \cap]\!]} & H_{*-n}(M_1) \\
{\scriptstyle f_* \times f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
H_*(M_2 \times M_2) & \xrightarrow{[\![\tau_{M_2} \cap]\!]} & H_{*-n}(M_2)
\end{array}$$

commutes up to the sign $\deg(f) = \pm 1$.

We repeat here that this is *not* the optimal result for invariance of the intersection product, and absolutely not the optimal proof! See eg., [1, VI, Prop 14.2] for the stronger statement for any degree d map with easier proof using Poincaré duality.

Proof. Given that f is a homotopy equivalence, it has degree ± 1. If $\deg(f) = -1$, changing the orientation of M_2, so that the degree of f becomes 1, changes the intersection product of M_2 by a sign (-1). Hence it is enough to consider the case where $\deg(f) = 1$.

We need to show that Diagram (1.4) commutes, and that the two maps $[\![\overline{\tau}_e \cap]\!]$ and $[\![\overline{\eta} \cap]\!]$ are isomorphisms. We have already seen that the top and middle triangles commute by Propositions 1.1(3) and 1.3(3), and the bottom square by the naturality of the cap product. The commutativity of the triangle labeled (1) and the two required isomorphisms are given by Proposition 1.6 (where the commutation only holds up to the sign (-1) if f has degree (-1)). For the triangle labeled (2), commutativity follows from Proposition 1.4. Indeed, the map h in the proposition gives a commutative diagram

$$\begin{array}{ccccccc}
H_{*+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) & \longrightarrow & H_{*+k}(M_1 \times M_2 \times D^k, N_3^c) & \xrightarrow{[\tau_3 \cap]} & H_{*-n}(N_3) & \longrightarrow & H_{*-n}(M_1) \\
& \searrow & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \| \\
& & H_{*+k}(M_1 \times M_2 \times D^k, \bar{N}_3^c) & \xrightarrow{[\bar{\tau}_3 \cap]} & H_{*-n}(\bar{N}_3) & \longrightarrow & H_{*-n}(M_1)
\end{array}$$

because h is isotopic to the identity relative to $M_1 \times M_2 \times \partial D^k$ (for the first triangle), pulls back $\bar{\tau}_3$ to τ_3 (for the middle square), and fixes M_1 (for the last square). Hence Diagram (1.4) commutes, which finishes the proof. ☐

2. Invariance of the string coproduct: plan of proof

We start by recalling the definition of the string coproduct, where we use a construction of the coproduct given in [14]. Let $\Lambda_i := \Lambda M_i$ denote the free loop space of M_i, for $i = 1, 2$. There is an evaluation map

$$e_I : \Lambda_i \times I \to M_i \times M_i$$

defined by $e_I(\gamma, s) = (\gamma(0), \gamma(s))$. Recall from the previous section the neighborhoods $U_1 \subset M_1 \times M_1$ and $U_2 \subset M_2 \times M_2$ of the diagonals. For $i = 1, 2$, let

$$U^i_{GH} = e_I^{-1}(U_i) = \big\{(\gamma, s) \in \Lambda_i \times I) \mid |\gamma(0) - \gamma(s)| < \varepsilon_i\big\} \ \subset \Lambda_i \times I.$$

Then $U_{GH,i}$ is a neighborhood of the self-intersecting loops

$$\mathcal{F}_i = \big\{(\gamma, s) \in \Lambda_i \times I \mid \gamma(0) = \gamma(s)\big\} = e_I^{-1}(\Delta_i)$$

for $\Delta_i = \Delta(M_i)$ the diagonal. Pulling back along the evaluation map e_I, we get classes

$$e_I^* \tau_{M_i} \in C^n(\Lambda_i \times I, (U^i_{GH})^c) \xleftarrow{e_I^*} C^n(M_i \times M_i, U_i^c) \ni \tau_{M_i}.$$

Let

$$B_i^{\varepsilon_i} = \{(\gamma, s) \in \Lambda_i \times I \mid |\gamma(t) - \gamma(0)| < \varepsilon_i \ \ \forall t \in [0, s] \text{ or } \forall t \in [s, 1]\}$$
$$\bar{B}_i = \{(\gamma, s) \in \Lambda_i \times I \mid \gamma(t) = \gamma(0) \ \ \forall t \in [0, s] \text{ or } \forall t \in [s, 1]\}$$

The *relative coproduct* for Λ_i can be defined as the composition

$$\begin{aligned}\vee\colon H_*(\Lambda_i, M_i) &\xrightarrow{\times I} H_{*+1}(\Lambda_i \times I, B_i^{\varepsilon_i}) \longrightarrow H_{*+1}(\Lambda_i \times I, B_i^{\varepsilon_i} \cup (U^i_{GH})^c) \cong H_{*+1}(U^i_{GH}, B_i^{\varepsilon_i} \cup \partial U^i_{GH}) \\ &\xrightarrow{e_I^*\tau_{M_i}\cap} H_{*+1-n}(U^i_{GH}, B_i^{\varepsilon_i}) \xrightarrow{R_i} H_{*+1-n}(\mathcal{F}_i, \bar{B}_i) \xrightarrow{cut} H_{*+1-n}(\Lambda_i \times \Lambda_i, M_i \times \Lambda_i \cup \Lambda_i \times M_i)\end{aligned}$$

where the map R_i is a retraction map and the cut map cuts at time s, see [14, Prop 3.6].

Following ideas of Gruher-Salvatore [11] who gave a proof of homotopy invariance for the Chas-Sullivan product, we will use intermediate spaces:

$$\begin{aligned}W_{11,2} &= \{(m, m', \gamma, s) \in M_1^2 \times \Lambda_2 \times I \mid f(m) = \gamma(0), f(m') = \gamma(s)\} \\ \Lambda_{1,2} &= \{(m, \gamma) \in M_1 \times \Lambda_2 \mid f(m) = \gamma(0)\}\end{aligned}$$

together with the following "evaluation" maps

$$\begin{aligned}e_I &\colon W_{11,2} \longrightarrow M_1 \times M_1 \\ e_I &\colon \Lambda_{1,2} \times I \longrightarrow M_1 \times M_2 \\ \bar{e}_I &\colon \Lambda_{1,2} \times I \times D^k \longrightarrow M_1 \times M_2 \times D^k\end{aligned}$$

defined by setting $e_I(m, m', \gamma, s) = (m, m')$ and $e_I(m, \gamma, s) = (m, \gamma(s))$ respectively, with $\bar{e}_I = e_I \times D^k$. The spaces $W_{11,2}$ and $\Lambda_{1,2}$ will play the roles of $M_1 \times M_1$ and $M_1 \times M_2$ in the proof of homotopy invariance of the intersection product given in the previous section.

The map Λf factors as a composition

$$\Lambda_1 \xrightarrow{f_{1,2}} \Lambda_{1,2} \xrightarrow{\pi} \Lambda_2,$$

with $f_{1,2}$ defined by $f_{1,2}(\gamma) = (\gamma(0), f \circ \gamma)$ and π the projection $\pi(m, \gamma) = \gamma$; we will more generally use the notation π or π_j for projection maps. These maps induce corresponding maps on the subspaces of self-intersecting loops: for

$$\mathcal{F}_{1,2} := \{(m, \gamma, s) \in M_1 \times \Lambda_2 \times I \mid f(m) = \gamma(0) = \gamma(s)\} \subset \Lambda_{1,2} \times I$$

the maps $f_{1,2} \times I$ and $\pi \equiv \pi \times I$ restrict to maps

$$\mathcal{F}_1 \xrightarrow{f_{11,2} \times I} \mathcal{F}_{1,2} \xrightarrow{\ \pi\ } \mathcal{F}_2\ .$$

We will in addition use that $f_{1,2} \times I$ factors as

$$\begin{array}{ccccc}\Lambda_1 \times I & \xrightarrow{f_{11,2}} & W_{11,2} & \xrightarrow{\qquad\pi_1\qquad} & \Lambda_{1,2} \times I, \\ & & {\scriptstyle e}\searrow & & \nearrow{\scriptstyle \pi_2} \\ & & & \Lambda_{1,2} \times I \times D^k & \end{array}$$

with $f_{11,2}(\gamma, s) = (\gamma(0), \gamma(s), f \circ \gamma, s)$ and $\pi_1(m, m', \gamma, s) = (m, \gamma, s)$, $e(m, m', \gamma, s) = (m, \gamma, s, e(m'))$, and π_2 denotes the projection away from the disc coordinate.

Denote by

$$\begin{aligned}B^{\varepsilon_2}_{11,2} &= \big\{(m, m', \gamma, s) \in W_{11,2} \mid |\gamma(t) - \gamma(0)| < \varepsilon_2 \ \ \forall t \in [0, s] \text{ or } \forall t \in [s, 1]\big\} \\ B^{\varepsilon_2}_{1,2} &= \big\{(m, \gamma, s) \in \Lambda_{1,2} \times I \mid |\gamma(t) - \gamma(0)| < \varepsilon_2 \ \ \forall t \in [0, s] \text{ or } \forall t \in [s, 1]\big\} \\ \overline{B^{\varepsilon_2}_{1,2}} &= B^{\varepsilon_2}_{1,2} \times D^k\end{aligned}$$

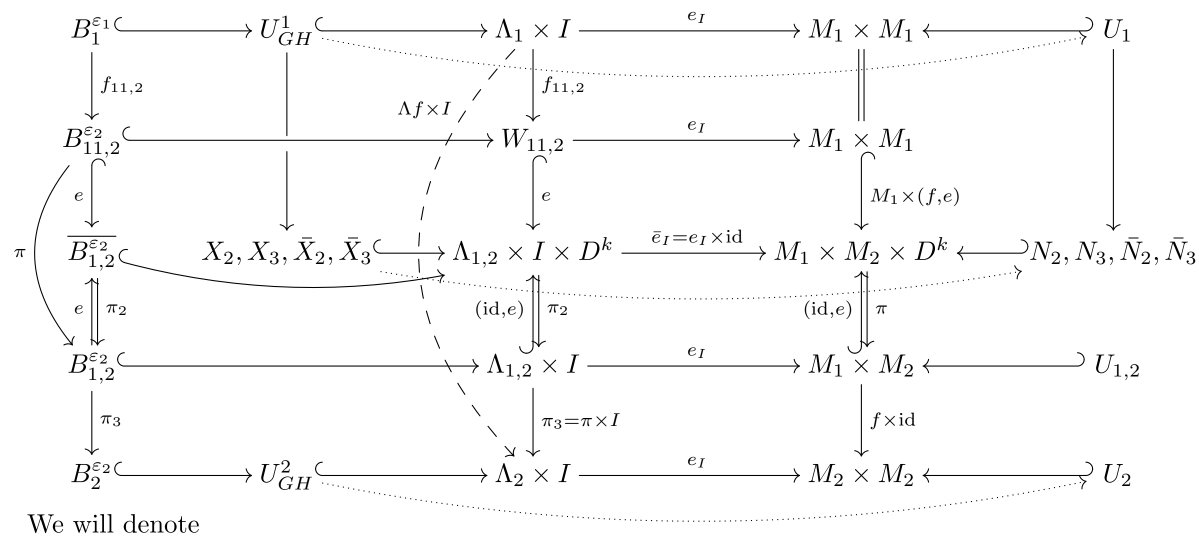
the subspaces of $W_{11,2}$, $\Lambda_{1,2} \times I$ and $\Lambda_{1,2} \times I \times D^k$ of "half small" loops.

Recall that $N_2 = M_1 \times N(f, e)$, $N_3 = TM_1 \oplus N(f, e)$, $\bar{N}_2 = M_1 \times M_2 \times \mathbb{R}^k$ and $\bar{N}_3 = f^*TM_2 \oplus \mathbb{R}^k$ are the normal bundles of Propositions 1.1 and 1.3, that embed inside $M_1 \times M_2 \times D^k$ via the maps ν_2, ν_3, $\bar{\nu}_2$ and $\bar{\nu}_3$ respectively. We write

$$X_2 := \bar{e}_I^{-1}(\nu_2 N_2), \quad X_3 := \bar{e}_I^{-1}(\nu_3 N_3), \quad \bar{X}_2 := \bar{e}_I^{-1}(\bar{\nu}_2 \bar{N}_2), \quad \bar{X}_3 := \bar{e}_I^{-1}(\bar{\nu}_3 \bar{N}_3) \ \text{ in } \Lambda_{1,2} \times I \times D^k$$

for their inverse images along the map $\bar{e}_I$.

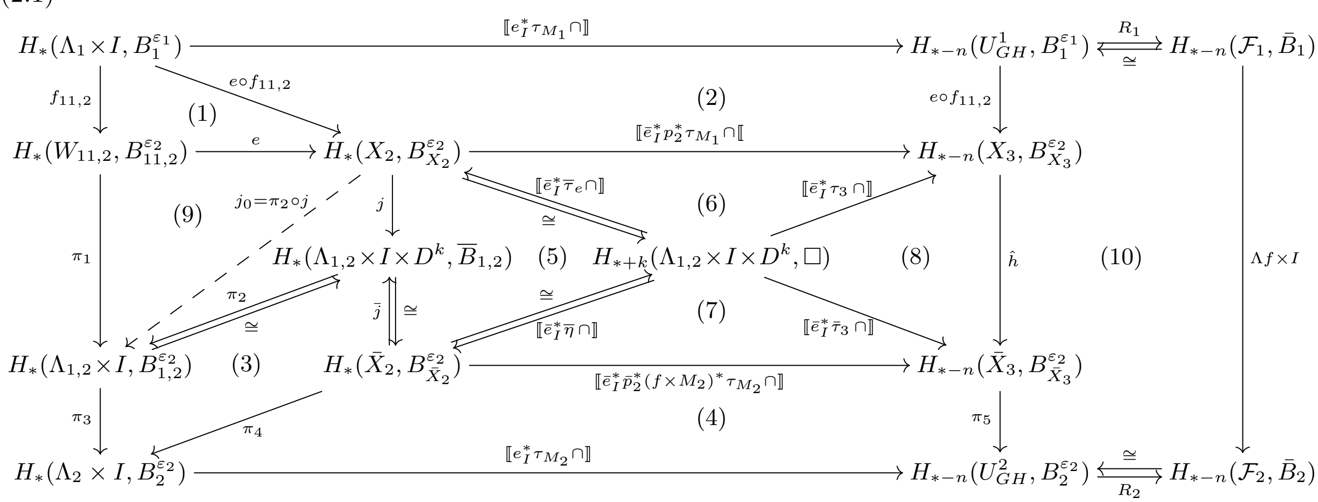
The spaces and maps introduced fit into a commutative diagram of maps and embeddings



We will denote

$$B^{\varepsilon_2}_{X_2} = \overline{B^{\varepsilon_2}_{1,2}} \cap X_2, \quad B^{\varepsilon_2}_{\bar{X}_2} = \overline{B^{\varepsilon_2}_{1,2}} \cap \bar{X}_2, \quad B^{\varepsilon_2}_{X_3} = \overline{B^{\varepsilon_2}_{1,2}} \cap X_3, \quad B^{\varepsilon_2}_{\bar{X}_3} = \overline{B^{\varepsilon_2}_{1,2}} \cap \bar{X}_3.$$

Our lift of Diagram (1.4), which we will analyse piece by piece below, is the following:
(2.1)



where $\square = \Lambda_{1,2} \times I \times \partial D^k \cup \overline{B}^{\varepsilon_2}_{1,2}$. The maps π_j are all projection maps, and the maps e, $f_{11,2}$, R_i were defined in the previous section. The maps $j : X_2 \to \Lambda_{1,2} \times I \times D^k$ and $\bar{j} : \bar{X}_2 \to \Lambda_{1,2} \times I \times D^k$ are the inclusions. Finally the map $\hat{h} : X_3 \to \bar{X}_3$ is induced by the map h of Proposition 1.4, and will be defined in Section 3.7.

To prove Theorem A, we will need to check that

(1) The maps in Diagram (2.1) are well-defined.
(2) Each subdiagram in (2.1) commutes.
(3) The maps π_2, $\bar{j}$, $[\![\bar{e}_I^* \overline{\tau}_e \cap]\!]$ and $[\![\bar{e}_I^* \overline{\eta} \cap]\!]$ in (2.1) are isomorphisms.

We will also use that the maps R_1 and R_2 are homotopy inverses of the inclusions, see [14, Lem 3.5], to simplify the proof of the commutativity of subdiabram (10).

2.1. **Role of the assumptions on f in Theorem A.** To deduce commutativity of the outside square in Diagram 2.1 from the commutativity of the diagram, it is vital that the maps "going the wrong way", namely the maps π_2, $\bar{j}$, $[\![\bar{e}_I^* \overline{\tau}_e \cap]\!]$ and $[\![\bar{e}_I^* \overline{\eta} \cap]\!]$ are isomorphisms. For the map $[\![\bar{e}_I^* \overline{\eta} \cap]\!]$ to induce an isomorphism in relative homology (without any assumption on f), we need to use $\overline{B}^{\varepsilon_2}_{1,2} = B^{\varepsilon_2}_{1,2} \times D^k$ in the relative term of the source of that map (see Lemma 3.4), and that essentially determines the relative terms elsewhere in the diagram. The maps π_2 and $\bar{j}$ are relative homotopy equivalences, also without any assumption on f, so left is the question of whether the map $[\![\bar{e}_I^*\overline{\tau}_e \cap]\!]$ induces an isomorphism in relative homology, or equivalently the map j, or in fact j_0, in the diagram.

We will show in Proposition 3.11 that, for f any homotopy equivalence, the map $j_0 : X_2 \to \Lambda_{1,2} \times I$ is likewise a homotopy equivalence, but a stronger assumption is needed to show that also its restriction $j_0 \colon B^{\varepsilon_2}_{X_2} \to B^{\varepsilon_2}_{1,2}$, is a (weak) homotopy equivalence. And indeed, if this map was also a homotopy equivalence without any futher assumption on f, the above diagram would prove homotopy invariance of the coproduct, contradicting Naef's example [18] (see also Appendix B). We will show here how one set of assumptions in Theorem A, namely Condition D that $\Delta_1 \xrightarrow{\simeq} \Delta_1^f$, together with Δ–transitivity, suffices for j_0 to be a weak homotopy equivalence on the relative term. We will also give a version D' of condition D that does not require Δ-transitivity. The proof that also Condition B (boundedness assumption on the homotopy equivalence) together with Δ–transitivity suffices, will be given in Section 3.8. We expect that this latter condition is stronger.

Recall that $B^{\varepsilon_2}_{X_2} = (B^{\varepsilon_2}_{1,2} \times D^k) \cap \bar{e}_I^{-1}(N_2)$. So the evaluation map $\bar{e}_I$ takes $B^{\varepsilon_2}_{X_2}$ to $(U_{1,2} \times D^k) \cap N_2$, where we recall that $N_2 = M_1 \times N(f, e)$. Condition D originates in the following diagram

$$\begin{array}{ccccc} B^{\varepsilon_2}_{X_2} & \xrightarrow{\bar{e}_I} & (U_{1,2} \times D^k) \cap N_2 & \xrightarrow{p_2} & V_e(\Delta_1) \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle p_0} & & {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle \Delta_1} \\ B^{\varepsilon_2}_{1,2} & \xrightarrow{e_I} & U_{1,2} & \xrightarrow{\simeq} & M_1 \end{array} \tag{2.2}$$

where

$$V_e(\Delta_1) = \{(m, m') \in M_1 \times M_1 \mid \exists W \in N(f, e)_{m'} \text{ with } \nu_e(m', W) = (n, x) \text{ and } |f(m) - n| < \varepsilon_2\}.$$

The map p_0 is the projection away from the disc, p_1 is the projection to the first factor, with its right inverse Δ_1, and $p_2 : N_2 \to M_1 \times M_1$ is the projection map of the bundle N_2, here restricted to $(U_{1,2} \times D^k) \cap N_2$.

We start by relating $V_e(\Delta_1)$ to the fake diagonal $\Delta_1^f = \{(m, m') \in M_1 \times M_1 \mid f(m) = f(m')\}$. Define

$$U_\varepsilon(\Delta_1^f) = \{(m, m') \in M_1 \times M_1 \mid |f(m) - f(m')| < \varepsilon\}.$$

We have

$$\Delta_1^f \subset U_{\varepsilon_2}(\Delta_1^f) \subset V_e(\Delta_1) \subset U_{2\varepsilon_2}(\Delta_1^f). \tag{2.3}$$

where the second inclusion arises from taking $W = 0$ in the definition of $V_e(\Delta_1)$, and the third from noting that any pair $(m, m') \in V_e(\Delta_1)$ does satisfy that $|f(m) - f(m')| < 2\varepsilon_2$.

The fake diagonal Δ_1^f is typically not a submanifold of $M_1 \times M_1$, as illustrated by the following examples, though it seems likely that f can be perturbed so that Δ_1^f is the union of Δ_1 and a transverse submanifold, as is the case in the first example.

Example 2.1. Let $f : I \to I$ be a smooth map that folds part of the interval, as illustrated in Figure 1. Then f is homotopic to the identity, but the map $\Delta_1 \hookrightarrow \Delta_1^f \cong \Delta_1 \cup_{S^0} S^1$ is not a homotopy equivalence. In particular, we see that the question of whether $\Delta_1 \hookrightarrow \Delta_1^f$ is a homotopy equivalence is itself not a property invariant under replacing f by a homotopic map.

Example 2.2. Suppose that $M_2 = M_1/B$ for $B \subset M_1$ a contractible subspace, and suppose that $f : M_1 \to M_2$ is the collapse map. Then $\Delta_1^f = \Delta_1 \cup_{\Delta B} (B \times B)$ is homotopic to Δ_1. Indeed, the map $p_B : \Delta_1^f \to \Delta_1$ defined on $B \times B$ by $p_B(b, b') = (b, b)$ and on Δ_1 as the identity, is a homotopy inverse to the inclusion.

The following question was raised by Goodwillie:

Question 1. Can one always modify f within its homotopy class so that $\Delta_1^f = \Delta_1 \cup D^f$ with D^f a submanifold of $M_1 \times M_1$ that is transverse to Δ_1?

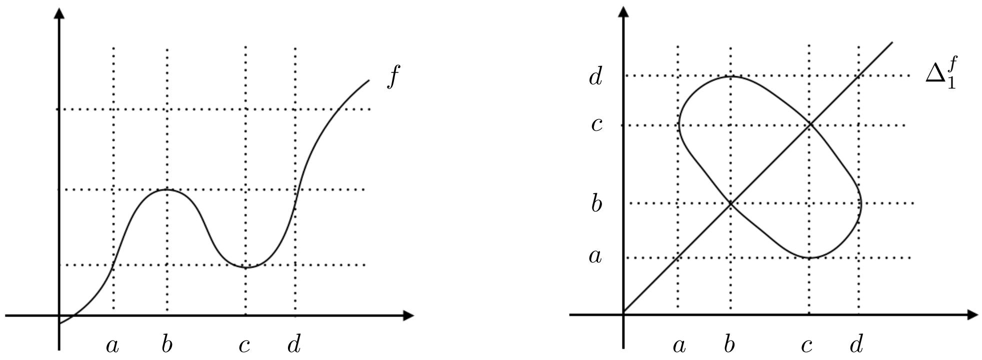


FIGURE 1. Graph of the map $f : I \to I$ of Example 2.1 and associated fake diagonal Δ_1^f

A positive answer to this question would allow to show that all the neighborhoods of Δ_1^f appearing in (2.3) are homotopic "tubular neighborhoods" of Δ_1^f. This leads to the following definition:

Definition 2.3. We say that f is Δ–*transverse* if there is an $\varepsilon > 0$ such that the inclusions in (2.3) are all homotopy equivalences for every $\varepsilon_2 \le \varepsilon$.

Note that for f a Δ–transverse homotopy equivalence, the map $\Delta_1 \hookrightarrow \Delta_1^f$ is a weak homotopy equivalence (i.e. condition D holds) if and only if the right vertical map $p_1 : V_e(\Delta_1) \to M_1$ in Diagram 2.2 is a weak equivalence. If f is not Δ–transverse, Proposition 2.5 below shows that condition D in Theorem A can be replaced by that condition:

Condition D': The projection map $p_1 : V_e(\Delta_1) \to M_1$ is a weak homotopy equivalence.

We expect however that Δ–transversality is not an actual restriction:

Conjecture 2.4. Any smooth map $f : M_1 \to M_2$ between compact Riemannian manifolds is homotopic a map that is Δ–transverse.

For Theorem A, we need to know that the left vertical arrow in Diagram 2.2 is a homology isomorphism. The next result will use the diagram to show that this holds if the rightmost vertical arrow in the diagram is a weak equivalence, which in turns holds if f is a Δ–transverse homotopy equivalence satisfying condition D by the above remarks:

Proposition 2.5. *The map $j_0 : B^{\varepsilon_2}_{X_2} \to B^{\varepsilon_2}_{1,2}$ is a weak homotopy equivalence if and only if f satisfies condition D'. Moreover, condition D' holds if f is a Δ–transverse homotopy equivalence satisfying condition D.*

To prove Proposition 2.5, we start with the following lemma.

Lemma 2.6. *$V_e(\Delta_1^f)$ is an open neighborhood of Δ_1^f and the projection map $p_2 : (U_{1,2} \times D^k) \cap N_2 \to V_e(\Delta_1^f)$ is a deformation retraction.*

Proof. Recall from Section 1 that the tubular embedding $\nu_e : N(f,e) \to M_2 \times D^k$ was defined using the exponential map of $M_2 \times D^k$ on the subbundle $N(f,e) \le TM_2 \oplus \mathbb{R}^k$ of vectors normal to $d(f,e)TM_1$. Recall also that TM_2 denotes the ε_2–tangent bundle of M_2, with $\varepsilon_2 < \rho_2/14$, for ρ_2 the injectivity radius of M_2. For $m \in M_1$, let

$$V_m = \{m' \in M_1 \mid \exists W \in N(f,e)_{m'} \text{ with } \nu_e(m', W) = (n, x) \text{ and } |f(m) - n| < \varepsilon_2\}.$$

We claim that V_m is open. Indeed, V_m is the image under the open map $p : N(f,e) \to M_1$ of the open set $\nu_e^{-1}(pr^{-1}_{M_2}(B_{\varepsilon_2}(f(m)) \cap \nu_e(N(f,e)))$. Hence $V_e(\Delta_1^f) = \bigcup_{m \in M_1} V_m$ is also open.

For $(m, m') \in V_e(\Delta_1^f)$, consider the fiber

$$F_{m,m'} = \{W \in N_{m'}(f, e) \mid \nu_e(m', W) = (n, x) \text{ and } |f(m) - n| < \varepsilon_2\}$$

of the projection $p_2 : (U_{1,2} \times D^k) \cap N_2 \to V_e(\Delta_1^f)$. This is the inverse image under the projection $\pi : N_{m'}(f, e) \leq T_{f(m')}M_2 \times \mathbb{R}^k \twoheadrightarrow T_{f(m')}M_2$ of the ball $B_{\varepsilon_2}(f(m))$ identified as a ball in $T_{f(m')}M_2$ by the exponential map, where we use that $|f(m) - f(m')| < 2\varepsilon_2 << \rho_2$ for ρ_2 the injectivity radius of M_2. The space $F_{m,m'}$ is in particular convex, and we can use the smallest W to define the claimed deformation retraction. □

The other main ingredient for the proof of Proposition 2.5 is the following result:

Proposition 2.7. *The projection $p_0 : (U_{1,2} \times D^k) \cap N_2 \to U_{1,2}$ is a weak homotopy equivalence if and only if the map $j_0 : B_{X_2}^{\varepsilon_2} \to B_{1,2}^{\varepsilon_2}$ is a weak homotopy equivalence.*

Proof. The diagram

$$\begin{array}{ccc} B_{X_2}^{\varepsilon_2} & \xrightarrow{j_0} & B_{1,2}^{\varepsilon_2} \\ \downarrow{\scriptstyle \bar{e}_I} & & \downarrow{\scriptstyle e_I} \\ (U_{1,2} \times D^k) \cap N_2 & \xrightarrow{p_0} & U_{1,2}, \end{array}$$

is a pull-back diagram. Hence it is enough to show that $e_I : B_{1,2}^{\varepsilon_2} \to U_{1,2}$ is a Serre fibration. So consider a lifting diagram

$$\begin{array}{ccc} D^r \times \{0\} & \xrightarrow{\hat{g}^0} & B_{1,2}^{\varepsilon_2} \\ \downarrow & \overset{\hat{g}}{\nearrow} & \downarrow{\scriptstyle e_I} \\ D^r \times I & \xrightarrow[g=(g_1,g_2)]{} & U_{1,2} \subset M_1 \times M_2. \end{array}$$

For $t \in I$, let $\delta_t : I \to \mathbb{R}$ be defined by

$$\delta_t(s) = \max\{|f \circ g_1(x, s) - f \circ g_1(x, t)| \mid x \in D^r\},$$

and define

$$U_t := \delta_t^{-1}(-\infty, \rho/28) \subset I.$$

The sets U_t define an open covering of I, and we let $\frac{1}{L}$ be a Lebesgues number for that cover, so that each interval $[\frac{i}{L}, \frac{i+1}{L}]$ lies within one U_{t_i}.

The map $\hat{g} = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$ has three components as $B_{1,2}^{\varepsilon_2} \subset M_1 \times \Lambda_2 \times I$. The first component $\hat{g}_1 = g_1$ is determined by g. We define the time component $\hat{g}_3(x, t) = (1 - t)\hat{g}_3(x, 0) + t\frac{1}{2}$ so that $\hat{g}_3(x, t)$ has the property that it is equal to $\hat{g}_3(x, 0)$ when $t = 0$, and it is otherwise always strictly between 0 and 1.

We construct the remaining loop space component $\hat{g}_2$ inductively on the intervals $[\frac{i}{L}, \frac{i+1}{L}]$, with induction start the second component of $\hat{g}^0$. So suppose we have already defined $\hat{g}_2$ on $[0, \frac{i}{L}]$. We want to extend the definition of $\hat{g}_2$ to $[0, \frac{i+1}{L}]$. Let t_i be such that $[\frac{i}{L}, \frac{i+1}{L}] \subset U_{t_i}$, and let $(m_x, \gamma_x, s_x) = \hat{g}(x, \frac{i}{L}) \in B_{1,2}^{\varepsilon_2}$. By our choices, we have that

$$|f \circ g_1(x, t') - f \circ g_1(x, t)| < \rho/14 \quad \text{for every} \quad t, t' \in [\frac{i}{L}, \frac{i+1}{L}].$$

Take $t' = \frac{i}{L}$. We have $f \circ g_1(x, \frac{i}{L}) = f(m_x) = \gamma_x(0)$ as $g_1 = \hat{g}_1$. Hence for each $t \in [\frac{i}{L}, \frac{i+1}{L}]$, the push map $h : U_{M,\rho} \to \mathrm{Diff}(M)$ of [14, Lem 2.1] can be applied to the pair $(f(m_x), f \circ g_1(x, t))$ yielding, continuously in $M \times M$, a diffeomorphism of M that takes $f(m_x)$ to $f \circ g_1(x, t)$, and restricts to the identity away from the ball of radius $\rho/2$ around $f(m_x)$. The map h is defined so that it takes the ball of radius $\rho/14$ around $f(m_x)$ to the the ball of radius $\rho/14$ around $f \circ g_1(x, t)$. It follows that $\gamma_{x,t} := h(f(m_x), f \circ g_1(x, t))(\gamma_x)$ is a loop based at $f \circ g_1(x, t))$ and satisfying that $|\gamma_{x,t}(t') - \gamma_{x,t}(0)| < \varepsilon < \rho/14$ for all $t' \in [0, s_x]$ or for all $t' \in [s_x, 1]$ as the same was true for γ_x. Hence $(g_1(x, t), \gamma_{x,t}, s_x) \in B_{1,2}^{\varepsilon_2}$. We still have to modify $\gamma_{x,t}$ to be compatible with g_2, and reparametrize it so that it is compatible with $\hat{g}_3$: we need to ensure that $\gamma_{x,t}(s_{x,t}) = g_2(x,t)$ for $s_{x,t} = \hat{g}_3(x,t)$. But note that $g_2(x,t)$ is also in the ball of radius ε around $f \circ g_1(x,t)$, so we can modify $\gamma_{x,t}$ by adding the geodesic arc $\overline{\gamma_{x,t}(s_x)g_2(x,t)}$ We set

$$\hat{g}_2(x,t) = \gamma_{x,t}[0,s_x] \star \overline{\gamma_{x,t}(s_x)g_2(x,t)} *_{s_{x,t}} \overline{g_2(x,t)\gamma_{x,t}(s_x)} \star \gamma_{x,t}[s_x,1].$$

Note that the concatenation at time $s_{x,t}$ is possible because we have defined $s_{x,t} = \hat{g}_3(x,t)$ so that it lies in $(0,1)$. And the triple $\hat{g}(x,t) = (g_1(x,t), \hat{g}_2(x,t), s_{x,t})$ still lies in $B_{1,2}^{\varepsilon_2}$ by construction. □

Proof of Proposition 2.5. The first part of the statement says that the left vertical map in Diagram (2.2) is a weak equivalence if and only if the right vertical map is one, which we can prove using the above preliminary results: By Proposition 2.7, the map right hand map is a weak equivalence if and only if the middle map is one, and by Lemma 2.6, the top map in the right hand square is a homotopy equivalence. The result then follows from the fact that the lower map in that square is also a homotopy equivalence, noting that the projection $p_1 : V_e(\Delta_1^f) \to M_1$ is a right inverse to the inclusion $M_1 \cong \Delta_1 \hookrightarrow V_e(\Delta_1^f)$.

The second part of the statement follows directly from the definitions. □

2.2. **The lifted coproduct.** Consider the maps $M_i \xrightarrow{i} \Lambda_i \xrightarrow{e} M_i$ including M_i as the constant loops and evaluating at 0. They define a canonical splitting $H_*(\Lambda_i) \cong H_*(M_i) \oplus H_*(\Lambda_i, M_i)$, and similarly for $H_*(\Lambda \times \Lambda)$. In [14], a lifted coproduct

$$\widehat{\vee} \colon H_*(\Lambda_i) \longrightarrow H_{*+1-n}(\Lambda_i \times \Lambda_i)$$

is defined as the extension by 0 on $H_*(M)$ of the relative coproduct $\vee$ under this splitting; see [14, Thm 4.2]. The following result shows that a map $f : M_1 \to M_2$ preserves the lifted coproduct if and only if it preserves the unlifted one:

Lemma 2.8. *Let $f : M_1 \to M_2$ be a continuous map. Then the map $\Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2)$ respects the lifted coproduct $\widehat{\vee}$ in the sense of Theorem A, i.e.*

$$\widehat{\vee} \circ \Lambda f = (\Lambda f \times \Lambda f) \circ \widehat{\vee} : H_*(\Lambda_1) \longrightarrow H_*(\Lambda_2 \times \Lambda_2)$$

if and only if the relative map $\Lambda f_* : H_*(\Lambda_1, M_1) \to H_*(\Lambda_2, M_2)$ respects the unlifted coproduct $\vee$, i.e.

$$\vee \circ \Lambda f = (\Lambda f \times \Lambda f) \circ \vee : H_*(\Lambda_1, M_1) \longrightarrow H_*(\Lambda_2 \times \Lambda_2, M_2 \times \Lambda_2 \cup \Lambda_2 \times M_2).$$

Proof. The result follows from the fact that a continuous map $f : M_1 \to M_2$ induces a map $\Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2)$ that respects the splitting $H_*(\Lambda_i) \cong H_*(M_i) \oplus H_*(\Lambda_i, M_i)$ and the corresponding splitting of $H_*(\Lambda_i \times \Lambda_i)$ as it commutes with both the inclusion of the constant loops i and the evaluation map e. □

3. Analysis of the subdiagrams in Diagram 2.1

In this section, we analyse each of the 10 subdiagrams in Diagram 2.1, showing that the maps are well-defined, that they commute, and that appropriate maps are invertible.

3.1. **Diagram (1).** The maps in Diagram (1) are

$$f_{11,2} \colon \Lambda_1 \times I \longrightarrow W_{11,2}, \quad f_{11,2}(\gamma, s) = (\gamma(0), \gamma(s), f \circ \gamma, s).$$
$$e \colon W_{11,2} \longrightarrow X_2 \subset \Lambda_{1,2} \times I \times D^k, \quad e(m, m', \gamma, s) = (m, \gamma, s, e(m')).$$

Here we need to check that e indeed has image inside $X_2 = \bar{e}_I^{-1}(\nu_2 N_2)$. This follows from the fact that

$$\bar{e}_I \circ e(m, m', \gamma, s) = (m, \gamma(s), e(m')) = (m, f(m'), e(m')) \in (\mathrm{id} \times (f,e))M_1 \times M_1 \subset M_1 \times M_2 \times D^k$$

in fact lies in the image of the zero section of $N_2 = M_1 \times N(e,f)$.

We need to further check that the maps are maps of pairs

$$f_{11,2} \colon (\Lambda_1 \times I, B_1^{\varepsilon_1}) \longrightarrow (W_{11,2}, B_{11,2}^{\varepsilon_2})$$
$$e \colon (W_{11,2}, B_{11,2}^{\varepsilon_2}) \longrightarrow (X_2, B_{X_2}^{\varepsilon_2}).$$

This is immediate as almost constant loops are mapped to almost constant loops in both cases, and the time parameter s is unaffected by the maps. Hence the maps are well-defined.

Diagram (1) commutes by definition.

3.2. **Diagram (2).** Recall from Proposition 1.1 the maps

$$\begin{array}{ccccc} & & N_3 = TM_1 \oplus N(f,e) & & \\ & & \downarrow \hat{\nu}_1 & \searrow \nu_3 & \\ M_1 \times M_1 & \xleftarrow{p_2} & N_2 = M_1 \times N(f,e) & \xhookrightarrow{\nu_2} & M_1 \times M_2 \times D^k \\ \| & & \| & & \| \\ U_1 \sqcup U_1^c & \longleftarrow & N_2|_{U_1} \sqcup N_2|_{U_1^c} & \longrightarrow & \nu_3 N_3 \sqcup \nu_3 N_3^c. \end{array} \tag{3.1}$$

Proposition 1.1(2) gives that $\hat{\nu}_1 : N_3 \xrightarrow{\cong} N_2|_{U_1}$ as bundles over $TM_1 \cong U_1$, and thus also $(\hat{\nu}_1 N_3)^c = \nu_2 N_2|_{U_1^c}$.

Note that $B^{\varepsilon_2}_{X_3} := (B^{\varepsilon_2}_{1,2} \times D^k) \cap X_3 = B^{\varepsilon_2}_{X_2} \cap X_3$. We will use the above to prove the following:

Lemma 3.1. *Subdiagram (2) in Diagram (2.1)*

$$\begin{array}{ccccc} H_*(\Lambda_1 \times I, B_1^{\varepsilon_1}) & \longrightarrow & H_*(\Lambda_1 \times I, (U_{GH,1})^c \cup B_1^{\varepsilon_1}) & \xrightarrow{[e_I^*\tau_{M_1}\cap]} & H_{*-n}(U_{GH,1}, B_1^{\varepsilon_1}) \\ \downarrow e\circ f_{11,2} & & \downarrow e\circ f_{11,2} & & \downarrow e\circ f_{11,2} \\ H_*(X_2, B^{\varepsilon_2}_{X_2}) & \longrightarrow & H_*(X_2, X_3^c \cup B^{\varepsilon_2}_{X_2}) & \xrightarrow{[\bar{e}_I^* p_2^* \tau_{M_1}\cap]} & H_{*-n}(X_3, B^{\varepsilon_2}_{X_3}) \end{array} \tag{3.2}$$

is well-defined and commutes.

Note that the composition $\Lambda_1 \times I \xrightarrow{f_{11,2}} W_{11,2} \xrightarrow{e} \Lambda_{1,2} \times I \times D^k$ takes (γ, s) to $(\gamma(0), f \circ \gamma, s, e(\gamma(s)))$.

Proof. We have already seen above that the map $e \circ f_{11,2}$ takes $(\Lambda_1 \times I, B_1^{\varepsilon_1})$ to $(X_2, B^{\varepsilon_2}_{X_2})$. We need to check that it takes $U_{GH,1}$ to X_3 and also $(U_{GH,1})^c$ to $X_2 \cap X_3^c$. This follows from taking the inverse image under evaluation maps of Diagram 3.1:

$$\begin{array}{ccccccc} & & \Lambda_1 \times I & \xrightarrow{e\circ f_{11,2}} & X_2 & \hookrightarrow & \Lambda_{1,2} \times I \times D^k \\ & & \| & & \| & & \| \\ & & U_{GH,1} \sqcup U^c_{GH,1} & \longrightarrow & X_3 \sqcup (X_3^c \cap X_2) & \hookrightarrow & X_3 \sqcup X_3^c \\ & & \downarrow e_I & & \downarrow \bar{e}_I & & \downarrow \bar{e}_I \\ M_1 \times M_1 & = & U_1 \sqcup U_1^c & \xrightarrow{\mathrm{id}\times(f,e)} & \nu_2(N_2|_{U_1}) \sqcup \nu_2(N_2|_{U_1^c}) & \hookrightarrow & M_1 \times M_2 \times D^k \end{array}$$

Now the left square in Diagram (3.2) clearly commutes. For the right square, the above gives also a commuting diagram of pairs

$$\begin{array}{ccccc} (\Lambda_1 \times I, U^c_{GH,1}) & \xrightarrow{e\circ f_{11,2}} & (X_2, X_2 \cap X_3^c) & \subset & \Lambda_{1,2} \times I \times D^k \\ \downarrow e_I & & \downarrow \bar{e}_I & & \downarrow \bar{e}_I \\ (M_1 \times M_1, U_1^c) & \xleftarrow{p_2} & (\nu_2 N_2, \nu_2 N_2|_{U_1^c}) & \subset & M_1 \times M_2 \times D^k. \end{array}$$

Hence the bottom right horizontal map in Diagram (3.2), capping with $\bar{e}_I^* p_2^* \tau_{M_1}$, is well-defined. Moreover, the commutativity of the diagram gives that

$$(e \circ f_{11,2})^* \big(\bar{e}_I^* p_2^* \tau_{M_1}\big) = e_I^* \tau_{M_1} \in C^n(\Lambda_1 \times I, U^c_{GH,1}),$$

which, together with the naturality of the cap product, gives the commutativity of the right square in (3.2). □

3.3. **Diagrams (3).** Recall that the map $\pi_3 : \Lambda_{1,2} \times I \to \Lambda_2 \times I$ is the forgetful map, taking (m, γ, s) to (γ, s). Recall also that $\bar{N}_2 \xrightarrow{\simeq} M_1 \times M_2 \times D^k$ is a homotopy equivalence via a homotopy that rescales the disc coordinate, giving also that its lift to the loop space

$$\bar{j} \colon \bar{X}_2 \overset{\simeq}{\hookrightarrow} \Lambda_{1,2} \times I \times D^k$$

using the same homotopy. Likewise the projection $\pi_2 : \Lambda_{1,2} \times I \times D^k \to \Lambda_{1,2} \times I$ is a homotopy equivalence. All maps and homotopies are relative with respect to the appropriate subspaces $B_2^{\varepsilon_2}, B_{1,2}^{\varepsilon_2}, \overline{B}_{1,2}^{\varepsilon_2}$ and $B_{\bar{X}_2}^{\varepsilon_2}$ as they do not affect the loop or time coordinates. Diagram (3)

$$\begin{array}{ccc}
H_*(\Lambda_{1,2} \times I, B_{1,2}^{\varepsilon_2}) & \xleftarrow[\cong]{\pi_2} & H_*(\Lambda_{1,2} \times I \times D^k, \overline{B}_{1,2}^{\varepsilon_2}) \\
\downarrow{\scriptstyle \pi_3} & & \uparrow{\scriptstyle \cong\, \bar{j}} \\
H_*(\Lambda_2 \times I, B_2^{\varepsilon_2}) & \xleftarrow[\pi_4]{} & H_*(\bar{X}_2, B_{\bar{X}_2}^{\varepsilon_2})
\end{array}$$

commutes by definition of the maps given that the last projection map $\pi_4 \colon \bar{X}_2 \to \Lambda_2 \times I$ is the restriction of $\pi_3 \circ \pi_2$ to $\bar{X}_2$. Also, the maps π_2 and $\bar{j}$ being relative homotopy equivalences, induce isomorphisms in relative homology.

3.4. **Diagrams (4).** Diagram (4) is very similar to Diagram (2). We need to check that the map

$$\pi_4 : \bar{X}_2 \hookrightarrow \Lambda_{1,2} \times I \times D^k \longrightarrow \Lambda_2 \times I$$

takes $\bar{X}_3$ to U_{GH}^2, and likewise for their complements. We do this as part of the following lemma, noting first that $B_{\bar{X}_3}^{\varepsilon_2} := (B_{1,2}^{\varepsilon_2} \times D^k) \cap \bar{X}_3 = B_{\bar{X}_2}^{\varepsilon_2} \cap \bar{X}_3$ as $\bar{X}_3 \subset \bar{X}_2$.

Lemma 3.2. *Subdiagram (4) in (2.1)*

$$\begin{array}{ccccc}
H_*(\bar{X}_2, B_{\bar{X}_2}^{\varepsilon_2}) & \longrightarrow & H_*(\bar{X}_2, \bar{X}_3^c \cup B_{\bar{X}_2}^{\varepsilon_2}) & \xrightarrow{[\bar{e}_I^* \bar{p}_2^* (f \times \mathrm{id})^* \tau_{M_2} \cap]} & H_{*-n}(\bar{X}_3, B_{\bar{X}_3}^{\varepsilon_2}) \\
\downarrow{\scriptstyle \pi_4} & & \downarrow{\scriptstyle \pi_4} & & \downarrow{\scriptstyle \pi_5} \\
H_*(\Lambda_2 \times I, B_2^{\varepsilon_2}) & \longrightarrow & H_*(\Lambda_2 \times I, (U_{GH}^2)^c \cup B_2^{\varepsilon_2}) & \xrightarrow{[e_I^* \tau_{M_2} \cap]} & H_{*-n}(U_{GH}^2, B_2^{\varepsilon_2})
\end{array}$$

is well-defined and commutes.

Proof. Recall from Proposition 1.3 that $\tilde{\nu}_1 \bar{N}_3 \cong \bar{N}_2|_{U_{1,2}}$. Also $f \times \mathrm{id}$ takes $U_{1,2}$ to U_2 and its complement to U_2^c. Hence we have a commuting the diagram

$$\begin{array}{ccccc}
\Lambda_{1,2} \times I \times D^k & \hookleftarrow & \bar{X}_2 = \bar{X}_3 \sqcup \bar{X}_3^c & \xrightarrow{\pi_4} & \Lambda_2 \times I = U_{GH,2} \sqcup U_{GH,2}^c \\
\downarrow{\scriptstyle \bar{e}_I} & & \downarrow{\scriptstyle \bar{e}_I} & & \downarrow{\scriptstyle e_I} \\
M_1 \times M_2 \times D^k & \hookleftarrow & \bar{N}_2 = \bar{N}_2|U_{1,2} \sqcup \bar{N}_2|_{U_{1,2}^c} & \xrightarrow{(f \times \mathrm{id}) \circ \bar{p}_2} & M_2 \times M_2 = U_2 \sqcup U_2^c
\end{array}$$

where the maps respects the unions into disjoint subsets. Thus π_3 takes $\bar{X}_3$ to U_{GH}^2, and likewise for their complements. Moreover we have a commuting diagram of pairs

$$\begin{array}{ccc}
(\bar{X}_2, \bar{X}_3^c) & \xrightarrow{\pi_4} & (\Lambda_2 \times I, U_{GH,2}^c) \\
\downarrow{\scriptstyle \bar{p}_2 \circ \bar{e}_I = e_I} & & \downarrow{\scriptstyle e_I} \\
(M_1 \times M_2, U_{1,2}^c) & \xrightarrow{f \times \mathrm{id}} & (M_2 \times M_2, U_2^c).
\end{array}$$

Hence $\bar{e}_I^* \bar{p}_2^* (f \times \mathrm{id})^* \tau_{M_2} = \pi_4^* \circ e_I^* \tau_{M_2}$ and the right square in the statement commutes by naturality of the cap product. $\square$

3.5. **Diagram (5) and the maps $\bar{j}$ and $[\![\bar{e}_I^*\overline{\eta}\cap]\!]$.** Let

$$\iota\colon (\Lambda_{1,2}\times I\times D^k, (\Lambda_{1,2}\times I\times\partial D^k)\cup \overline{B}_{1,2}^{\varepsilon_2}) \longrightarrow (\Lambda_{1,2}\times I\times D^k, X_2^c\cup\overline{B}_{1,2}^{\varepsilon_2})$$
$$j\colon (X_2, B_{X_2}^{\varepsilon_2}) \hookrightarrow (\Lambda_{1,2}\times I\times D^k, \overline{B}_{1,2}^{\varepsilon_2})$$

denote the inclusions of pairs, and similarly for $\bar{\iota}$ and $\bar{j}$. A little more explicitly, Diagram (5) is, after mirroring it, the following:

$$\begin{array}{ccc} H_{*+k}(\Lambda_{1,2}\times I\times D^k, X_2^c\cup\overline{B}_{1,2}^{\varepsilon_2}) & \overset{[\bar{e}_I^*\overline{\tau}_e\cap]}{\dashrightarrow} & H_*(X_2, B_{X_2}^{\varepsilon_2}) \\ \uparrow{\scriptstyle \iota} & \nearrow{\scriptstyle [\![\bar{e}_I^*\overline{\tau}_e\cap]\!]} & \downarrow{\scriptstyle j} \\ H_{*+k}(\Lambda_{1,2}\times I\times D^k, (\Lambda_{1,2}\times I\times\partial D^k)\cup\overline{B}_{1,2}^{\varepsilon_2}) & \overset{\bar{e}_I^*(1\times\eta)\cap}{\dashrightarrow} & H_*(\Lambda_{1,2}\times I\times D^k, \overline{B}_{1,2}^{\varepsilon_2}) \\ \downarrow{\scriptstyle \bar{\iota}} & \searrow{\scriptstyle [\![\bar{e}_I^*\overline{\eta}\cap]\!]} & \cong\uparrow{\scriptstyle \bar{j}} \\ H_{*+k}(\Lambda_{1,2}\times I\times D^k, X_2^c\cup\overline{B}_{1,2}^{\varepsilon_2}) & \underset{[\bar{e}_I^*\overline{\eta}\cap]}{\dashrightarrow} & H_*(\bar{X}_2, B_{\bar{X}_2}^{\varepsilon_2}) \end{array} \tag{3.3}$$

where the dashed arrows are added for clarity; by definition the two triangles on the left side of the diagram commute.

Lemma 3.3. *Assume f is a degree 1 map. Then subdiagram (5) in Diagram (2.1) is well-defined and commutes.*

Proof. Lemma 1.5 together with the commutativity of

$$\begin{array}{ccc} (\Lambda_{1,2}\times I\times D^k, \Lambda_{1,2}\times I\times\partial D^k) & \xrightarrow{\ \iota\ } & (\Lambda_{1,2}\times I\times D^k, X_2^c) \\ {\scriptstyle \bar{e}_I}\downarrow & & \downarrow{\scriptstyle \bar{e}_I} \\ (M_1\times M_2\times D^k, M_1\times M_2\times\partial D^k) & \xrightarrow{\mathrm{id}\times i} & (M_1\times M_2\times D^k, \nu_2(N_2)^c) \end{array}$$

and naturality of the cap product gives that the top square in Diagram (3.3) commutes. The commutativity of the bottom square follows similarly by definition of $\overline{\eta}$. □

We would also like to know that the maps in Diagram (5) are homology isomorphisms. We start by showing that the map labelled $\bar{e}_I^*\overline{\eta}\cap$ is one, by a double application of the suspension isomorphism in homology:

Lemma 3.4. *The maps* $[\![\bar{e}_I^*\overline{\eta}\cap]\!] : H_{*+k}(\Lambda_{1,2}\times I\times D^k, \Lambda_{1,2}\times I\times\partial D^k\cup\overline{B}_{1,2}^{\varepsilon_2})) \longrightarrow H_*(\bar{X}_2, B_{\bar{X}_2}^{\varepsilon_2})$ *and* $\bar{j}_*\colon H_*(\bar{X}_2, B_{\bar{X}_2}^{\varepsilon_2}) \longrightarrow H_*(\Lambda_{1,2}\times I\times D^k, \overline{B}_{1,2}^{\varepsilon_2})$ *are isomorphisms.*

Proof. For the second map, recall that $\bar{\nu}_2 : \bar{N}_2 = M_1\times M_2\times\mathbb{R}^k \xrightarrow{\simeq} M_1\times M_2\times D^k$ is defined by $\bar{\nu}_2(m,n,x) = (m,n,e(m)+\ell(x))$, where ℓ is a scaling map. In particular, $\bar{\nu}_2$ is homotopic to the homotopy equivalence $\mathrm{id}_{M_1\times M_2}\times\ell$. This homotopy lifts to show that also $\bar{j} : \bar{X}_2 \xrightarrow{\simeq} \Lambda_{1,2}\times I\times D^k$ is a homotopy equivalence, that restricts to a homotopy equivalence $B_{\bar{X}_2}^{\varepsilon_2} \xrightarrow{\simeq} \overline{B}_{1,2}^{\varepsilon_2} = B_{1,2}^{\varepsilon_2}\times D^k$ as both maps are likewise homotopic to maps that only affect the disc coordinates.

Given that $\bar{j}$ induces an isomorphism in relative homology, the first part of the statment follows from the commutativity of Diagram (3.3), if we can show that the maps

$$\bar{e}_I^*\overline{\eta}\cap\colon H_{*+k}(\Lambda_{1,2}\times I\times D^k, \Lambda_{1,2}\times I\times\partial D^k\cup\overline{B}_{1,2}^{\varepsilon_2})) \longrightarrow H_*(\Lambda_{1,2}\times I\times D^k, \overline{B}_{1,2}^{\varepsilon_2}) \cong H_*(\Lambda_{1,2}\times I, B_{1,2}^{\varepsilon_2})$$

is an isomorphism. For this, consider the triple

$$\Lambda_{1,2}\times I\times\partial D^k \ \subset\ \Lambda_{1,2}\times I\times\partial D^k\cup\overline{B}_{1,2}^{\varepsilon_2} \ \subset\ \Lambda_{1,2}\times I\times D^k$$

The associated long exact sequence in homology is

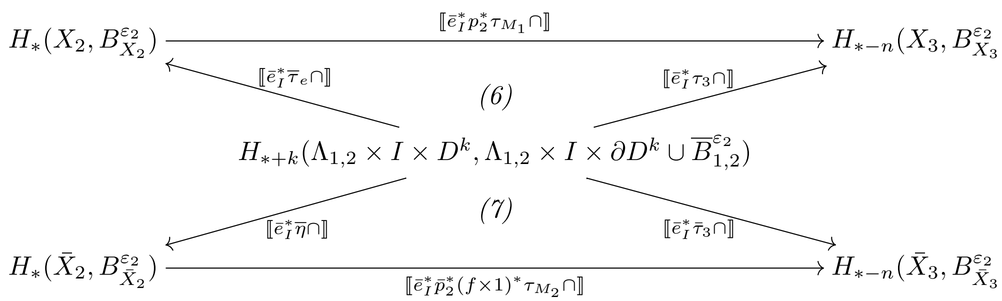
$$\dots\longrightarrow H_{*+k}(\overline{B}_{1,2}^{\varepsilon_2}, B_{1,2}^{\varepsilon_2}\times\partial D^k) \longrightarrow H_{*+k}(\Lambda_{1,2}\times I\times D^k\ ,\ \Lambda_{1,2}\times I\times\partial D^k)$$
$$\longrightarrow H_{*+k}(\Lambda_{1,2}\times I\times D^k\ ,\ \Lambda_{1,2}\times I\times\partial D^k\ \cup\ \overline{B}_{1,2}^{\varepsilon_2}) \longrightarrow\dots$$

where we use that $H_*(\overline{B}^{\varepsilon_2}_{1,2}, B^{\varepsilon_2}_{1,2} \times \partial D^k) \cong H_*(\Lambda_{1,2} \times I \times \partial D^k \cup B^{\varepsilon_2}_{1,2} \times D^k, \Lambda_{1,2} \times I \times \partial D^k)$ by excision. Now crossing with a disc gives a degree k map from the long exact sequence of the pair $(\Lambda_{1,2} \times I \times D^k, B^{\varepsilon_2}_{1,2} \times D^k)$ to the above sequence. The suspension isomorphism combined with the 5-lemma then gives the desired isomorphism, whose inverse can be describe by capping with the corresponding Thom class , which is the map we were interested in. $\square$

By the commutativity of Diagram (5), invertibility of the map $[\![\bar{e}_I^*\overline{\tau}_e\cap]\!]$ is thus equivalent to the invertibility of the map j. This will be shown below, when analysing diagram (9).

3.6. **Diagrams (6) and (7).**

Lemma 3.5. *Subdiagrams (6) and (7) in Diagram (2.1)*



are well-defined and commute.

Proof. Recall from Propositions 1.1 and 1.3 that the classes $\tau_3 \in C^{n+k}(M_1 \times M_2 \times D^k, \nu_3(N_3)^c)$ and $\bar{\tau}_3 \in C^{n+k}(M_1 \times M_2 \times D^k, \nu_3(\bar{N}_3)^c)$ are Thom classes for the bundles N_3 and $\bar{N}_3$. The maps $[\![\bar{e}_I^*\tau_3\cap]\!]$ and $[\![\bar{e}_I^*\bar{\tau}_3\cap]\!]$ are well-defined as $X_3, \bar{X}_3$ are by definition the inverse images of $\nu_3 N_3, \bar{\nu}_3\bar{N}_3$ under the map e_I, and likewise for their complements.

The triangles are the lift to $\Lambda_{1,2} \times I \times D^k$, along the map $\bar{e}_I$, and relatively to $\overline{B}^{\varepsilon_2}_{1,2}$, of the triangles occurring in Propositions 1.1(3) and 1.3(3). Made completely explicit, these triangle take the form of Diagram (A.3), used to prove that they commute. The input in that proof is the relations $\tau_3 := p_2^*\tau_{M_1}\cup\overline{\tau}_e \in C^{n+k}(M_1 \times M_2 \times D^k, N_3^c)$ and $\bar{\tau}_3 := \bar{p}_2^*(f \times 1)^*\tau_{M_2} \cup \overline{\eta} \in C^{n+k}(M_1 \times M_2 \times D^k, \bar{N}_3^c)$. Pulling back the classes along e_I, we get the analogous relations in the case considered here:

$$\bar{e}_I^*\tau_3 := \bar{e}_I^* p_2^*\tau_{M_1} \cup \bar{e}_I^*\overline{\tau}_e \in C^{n+k}(\Lambda_{1,2} \times I \times D^k, \bar{e}_I^{-1}(N_3^c))$$

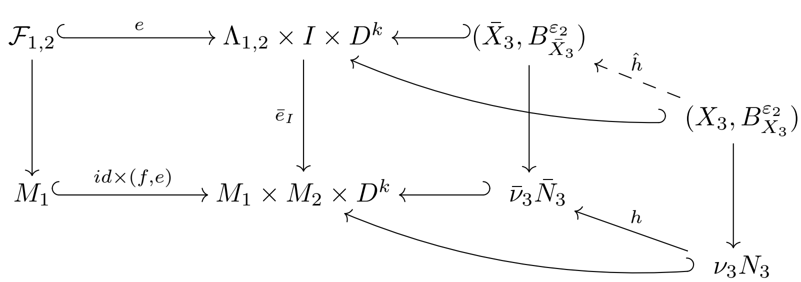
and

$$\bar{e}_I^*\bar{\tau}_3 := \bar{e}_I^*\bar{p}_2^*(f \times 1)^*\tau_{M_2} \cup \bar{e}_I^*\overline{\eta} \in C^{n+k}(\Lambda_{1,2} \times I \times D^k, \bar{e}_I^{-1}(\bar{N}_3^c))$$

which gives the commutativity of the diagrams by the same proof. $\square$

3.7. **The map $\hat{h}$ and Diagram (8).** We start by defining the map $\hat{h}$ of Diagram (8).

Recall from Proposition 1.4 the diffeomorphism $h : M_1 \times M_2 \times D^k \longrightarrow M_1 \times M_2 \times D^k$, isotopic to the identity, that identifies the tubular neighborhoods $\nu_3 N_3$ and $\bar{\nu}_3\bar{N}_3$ fixing their 0-section M_1. We want to lift h to a map $\hat{h} : \Lambda_{1,2} \times I \times D^k \to \Lambda_{1,2} \times I \times D^k$, likewise homotopic to the identity, and now taking the pair $(X_3 = \bar{e}_I^{-1}(\nu_3 N_3), B^{\varepsilon_2}_{X_3})$ to the pair $(\bar{X}_3 = \bar{e}_I^{-1}(\bar{\nu}_3\bar{N}_3), B^{\varepsilon_2}_{\bar{X}_3})$ and fixing the embedding of $\mathcal{F}_{1,2}$ in $X_3 \cap \bar{X}_3$:



for $e : \mathcal{F}_{1,2} \to \Lambda_{1,2} \times I \times D^k$ defined by $e(m, \gamma, s) = (m, \gamma, s, e(m))$.

Proposition 3.6. *There is a continuous map*

$$\hat{h}\colon \Lambda_{1,2} \times I \times D^k \longrightarrow \Lambda_{1,2} \times I \times D^k$$

such that

1. *The evaluation map $\bar{e}_I : \Lambda_{1,2} \times I \times D^k \to M_1 \times M_2 \times D^k$ intertwines the maps $\hat{h}$ and h, for h the diffeomorphism of Proposition 1.4(2). In particular, $\hat{h}(X_3) \subset \bar{X}_3$ and $\hat{h}(X_3^c) \subset \bar{X}_3^c$;*
2. *$\hat{h}$ fixes $e(\mathcal{F}_{1,2})$ and preserves $\Lambda_{1,2} \times I \times \partial D^k$ and $\overline{B}^{\varepsilon_2}_{1,2}$;*
3. *$\hat{h}$ is homotopic to the identity relative to $e(\mathcal{F}_{1,2})$, by a homotopy that preserves $\Lambda_{1,2} \times I \times \partial D^k$ and $\overline{B}^{\varepsilon_2}_{1,2}$.*

Proof. From Proposition 1.4 we have that $h : M_1 \times M_2 \times D^k \to M_1 \times M_2 \times D^k$ has the form $h(m,n,x) = (m,n',x')$ with $|n-n'| < 2\varepsilon_2$. If $h(m,\gamma(s),x) = (m,n',x')$, we define

$$\hat{h}(m,\gamma,s,x) = (m,\gamma',s',x')$$

with

$$\gamma' = \gamma[0,s] \star \overline{\gamma(s)n'} *_{s'} \overline{n'\gamma(s)} \star \gamma[s,1]$$

where the geodesic $\overline{\gamma(s)n'}$ is well-defined by the property of h just mentioned, and where $\star$ denotes the optimal concatenation of paths and $*_{s'}$ the concatenation at time s', see [14, Sec 1.2]. Here s' is defined by

$$s' = \begin{cases} \max(|x-e(m)|, s) & s \leq \frac{1}{2} \\ s & s = \frac{1}{2} \\ \min(1-|x-e(m)|, s) & s \geq \frac{1}{2}, \end{cases}$$

recalling that D^k is a small disc of radius ε, so that $|x-e(m)| << \frac{1}{2}$.

We first check that $\hat{h}$ is well-defined. The time s' is defined so that $s'=0$ is only possible if $s=0$ and $x=e(m)$ and $s'=1$ is only possible if $s=1$ and $x=e(m)$. In both cases, $(m,\gamma(s),x) = (m,f(m),e(m))$ is in the image of the 0-section M_1, which is fixed by h. Hence $n'=\gamma(s)$ in that case and the concatenation at time s' is always well defined.

For the same reason, we see that $\hat{h}$ fixes $e(\mathcal{F}_{1,2})$. And because h fixes $M_1 \times M_2 \times \partial D^k$, we have that $\hat{h}$ preserves $\Lambda_{1,2} \times I \times \partial D^k$ as in that case we also have $n'=\gamma(s)$ and $x'=x$, and only the time coordinate will be affected by $\hat{h}$.

Finally, $\hat{h}$ also preserves $\overline{B}^{\varepsilon_2}_{1,2}$ as the geodesic path $\overline{\gamma(s)n'}$ stays within distance ε_2 of $f(m) = \gamma(0)$, as both $\gamma(s)$ and n' are in the ball of radius ε_2 around $f(m)$, proving (2).

By construction, the diagram

$$\begin{array}{ccc} \Lambda_{1,2} \times I \times D^k & \xrightarrow{\hat{h}} & \Lambda_{1,2} \times I \times D^k \\ \downarrow{\scriptstyle \bar{e}_I} & & \downarrow{\scriptstyle \bar{e}_I} \\ M_1 \times M_2 \times D^k & \xrightarrow{h} & M_1 \times M_2 \times D^k \end{array}$$

commutes. As h takes $\nu_3 N_3$ to $\bar{\nu}_3 \bar{N}_3$ (Proposition 1.4(1)), it follows that $\hat{h}$ takes X_3 to $\bar{X}_3$, and likewise for the complements, which proves (1).

Let $\hat{F}$ be the isotopy $\hat{h} \simeq_{\hat{F}} \mathrm{id}$ given in Proposition 1.4, and write $\hat{F}(m,\gamma(s),x,t) = (m,n'_t,x'_t)$, We define a homotopy H between $\hat{h}$ and the identity by directly generalising the definition of $\hat{h}$, setting

$$H(m,\gamma,s,x,t) = (m,\gamma'_t,s'_t,x'_t)$$

with

$$\gamma' = \gamma[0,s] \star \overline{\gamma(s),n'_t} *_{s'_t} \overline{n'_t,\gamma(s)} \star \gamma[s,1]$$

with $s'_t = (1-t)s' + ts$. This homotopy fixes $e(\mathcal{F}_{1,2})$ and preserves both $\Lambda_{1,2} \times I \times \partial D^k$ and $\overline{B}^{\varepsilon_2}_{1,2}$ by the same argument as above, because $\hat{F}$ has the same properties as h, see Proposition 1.4(2). □

Lemma 3.7. *Subdiagram (8) in Diagram (2.1)*

$$\begin{array}{ccc} H_{*+k}(\Lambda_{1,2}\times I\times D^k,\Lambda_{1,2}\times I\times\partial D^k\cup\overline{B}^{\varepsilon_2}_{1,2}) & \xrightarrow{[\![\bar e_I^*\tau_3\cap]\!]} & H_{*-n}(X_3,B^{\varepsilon_2}_{X_3}) \\ & \searrow^{[\![\bar e_I^*\bar\tau_3\cap]\!]} & \downarrow \hat h \\ & & H_{*-n}(\bar X_3,B^{\varepsilon_2}_{\bar X_3}) \end{array}$$

is well-defined and commutes.

Proof. More precisely, the diagram is the following

$$\begin{array}{ccc} H_{*+k}(\Lambda_{1,2}\times I\times D^k,\Lambda_{1,2}\times I\times\partial D^k\cup\overline{B}^{\varepsilon_2}_{1,2}) & & \\ \downarrow & \searrow & \\ H_{*+k}(\Lambda_{1,2}\times I\times D^k,X_3^c\cup\overline{B}^{\varepsilon_2}_{1,2}) & \xrightarrow{\hat h} & H_{*+k}(\Lambda_{1,2}\times I\times D^k,\bar X_3^c\cup\overline{B}^{\varepsilon_2}_{1,2}) \\ \cong\downarrow & & \downarrow\cong \\ H_{*+k}(X_3,\partial X_3\cup B^{\varepsilon_2}_{X_3}) & \xrightarrow{\hat h} & H_{*+k}(\bar X_3,\partial\bar X_3\cup B^{\varepsilon_2}_{\bar X_3}) \\ \bar e_I^*\tau_3\cap\downarrow & & \downarrow\bar e_I^*\bar\tau_3\cap \\ H_{*-n}(X_3,B^{\varepsilon_2}_{X_3}) & \xrightarrow{\hat h} & H_{*-n}(\bar X_3,B^{\varepsilon_2}_{\bar X_3}) \end{array}$$

The maps are well-defined by (1),(2) in Proposition 3.6. The triangle commutes by (3) in the same proposition, and the bottom square by (5) in Proposition 1.4 using the naturality of the cap product. The middle square commutes as the inverse of excision is the inclusion. $\square$

3.8. **Diagram (9) and invertibility of j.** Diagram (9)

$$\begin{array}{ccc} H_*(W_{11,2},B^{\varepsilon_2}_{11,2}) & \xrightarrow{e} & H_*(X_2,B^{\varepsilon_2}_{X_2}) \\ \pi_1\downarrow & \swarrow^{j_0} & \downarrow j \\ H_*(\Lambda_{1,2}\times I,B^{\varepsilon_2}_{1,2}) & \xleftarrow[\cong]{\pi_2} & H_*(\Lambda_{1,2}\times I\times D^k,\overline{B}^{\varepsilon_2}_{1,2}) \end{array}$$

commutes on the space level as $j_0=\pi_2\circ j$ by definition, and the composition $j_0\circ e$ is precisely the map π_1.

We want to show that the map j, or equivalently j_0, induces an isomorphism in these relative homology groups, as it will give the invertibility of $[\bar e_I^*\overline{\tau}_e\cap]$ via Diagram (5). We will do this by showing here that the maps e and π_1 in Diagram (9) do induce isomorphisms in homology. Lemma 3.8 gives that e is a *non-relative* homotopy equivalence for any f and Proposition 3.10 that the same holds for the map π_1, and hence also for j_0, when f is a homotopy equivalence. As already discussed in Section 2.1, an additional assumption on f is necessary for this to hold also on the relative terms. We showed in that section that j_0 does also induce a homotopy equivalence on the relative terms if f satisfies condition D', or condition D together with Δ-transversality. In the present section, we will show that also condition B together with Δ–transitivity is sufficient by showing in Lemma 3.8 that e restricts to a homotopy equivalence on the relative terms if f is Δ–transverse, and in Proposition 3.10 that the same holds for the map π_1 if f satisfies condition B.

We start by considering the map $e:(W_{11,2},B^{\varepsilon_2}_{11,2})\to(X_2,B^{\varepsilon_2}_{X_2})$.

Lemma 3.8. *The map $e:W_{11,2}\to X_2$ is always a homotopy equivalence. If f is moreover Δ–transverse, it restricts to a homotopy equivalence $e:B^{\varepsilon_2}_{11,2}\to B^{\varepsilon_2}_{X_2}$. In particular, it then induces a relative homology isomorphism $e:H_*(W_{11,2},B^{\varepsilon_2}_{11,2})\to H_*(X_2,B^{\varepsilon_2}_{X_2})$.*

Proof. We prove the statement by constructing a homotopy inverse $r : X_2 \to W_{11,2}$ as follows. Given a point $(m, \gamma, s, x) \in X_2 = \bar{e}_I^{-1}(N_2)$, we have that $(m, \gamma(s), x) \in N_2 = M_1 \times N(f, e)$, that is $(\gamma(s), x)$ is the image of an element $(m', W) \in N(f, e)$ under the embedding $\nu_e : N(f, e) \to M_2 \times D^k$. Define a path $\delta_W : [0, 1] \to M_2$ by setting

$$\delta_W(t) = \nu_e(m', tW)_{M_2},$$

to be the first component of $\nu_e(m', tW) \in M_2 \times D^k$. Then $\delta_W(0) = f(m')$ and $\delta_W(1) = \gamma(s)$. Recall that D^k is a disc of radius ε. In particular, $|x - e(m)| << \frac{1}{2}$ is small. Define

$$s' = \begin{cases} \max(|x - e(m)|, s) & s \leq \frac{1}{2} \\ s & s = \frac{1}{2} \\ \min(1 - |x - e(m)|, s) & s \geq \frac{1}{2} \end{cases}$$

and

$$\gamma' = \gamma[0, s] \star \delta_W^{-1} *_{s'} \delta_W \star \gamma[s, 1]$$

Finally, set $r(m, \gamma, s, x) = (m, m', \gamma', s')$.

The path γ' is well-defined because $s' = 0$ or $s' = 1$ is only possible if $x = e(m')$ and $s = 0$ or $s = 1$ accordingly, which itself implies that $\gamma(s) = \gamma(0) = f(m)$, so that $(\gamma(s), x)$ lies on the 0-section of $N(f, e)$ and $W = 0$ is a trivial vector. It follows that δ_W is a constant path in that case, with $s' = s$ and $\gamma' = \gamma$. By construction, $(m, m', \gamma', s') \in W_{11,2}$.

The map r restricts to a map $r : B_2^{\varepsilon_2} \to B_{11,2}^{2\varepsilon_2}$ as $|\gamma(0) - \gamma(s)| < \varepsilon_2$ implies that the path δ_W stays within a radius $2\varepsilon_2$ of $f(m) = \gamma(0)$.

We have that $r \circ e \simeq \mathrm{id}_{W_{11,2}}$. Indeed, if $(m, m', \gamma, s) \in W_{11,2}$, then $e(m, m', \gamma, s) = (m, \gamma, s, e(m'))$ also has the property that $(\gamma(s), x) = (f(m'), e(m'))$ lies on the zero section, so δ_W is a trivial path. Now setting $s'_t = ts + (1 - t)s'$ and $\gamma'_t = \gamma[0, s] *_{s'_t} \gamma[s, 1]$ defines the required homotopy (by reparametrization) from (m, m', γ, s) to (m, m', γ', s'). This homotopy restricts on $B_{11,2}^{\varepsilon_2}$ to a homotopy between $r \circ e$ and the inclusion $B_{11,2}^{\varepsilon_2} \hookrightarrow B_{11,2}^{2\varepsilon_2}$.

Finally, we consider the composition $e \circ r : X_2 \to X_2$. It takes a tuple (m, γ, s, x) to $(m, \gamma', s', e(m'))$ for m', γ' and s' defined as above. We define a homotopy H to the identity by setting

$$H(m, \gamma, s, x, t) = (m, \gamma'_t, s'_t = ts + (1 - t)s', x_t = \nu_e(m', tW)_{D^k})$$

with

$$\gamma_t = \gamma[0, s] \star \delta_W[0, 1 - t]^{-1} *_{s'_t} \delta_W[0, 1 - t] \star \gamma[s, 1]$$

and s', W as above. This has image in X_2 at all times t as

$$(\gamma'_t(s'_t), x_t) = (\nu_e(m', (1 - t)W)_{M_2}, \nu_e(m', (1 - t)W)_{D^k}) \in N(f, e).$$

And again, this homotopy restricts on $B_{X_2}^{\varepsilon_2}$ to a homotopy between $e \circ r$ and the inclusion $B_{X_2}^{\varepsilon_2} \hookrightarrow B_{X_2}^{2\varepsilon_2}$.

Hence the maps r and e are homotopy inverse of each other as non-relative maps for any f. For the relative terms, we have maps

$$B_{11,2}^{\varepsilon_2} \xrightarrow{e} B_{X_2}^{\varepsilon_2} \xrightarrow{r} B_{11,2}^{2\varepsilon_2} \xrightarrow{e} B_{X_2}^{2\varepsilon_2}$$

with each double composition homotopic to the inclusion. So it is enough to show that both inclusions $B_{11,2}^{\varepsilon_2} \hookrightarrow B_{11,2}^{2\varepsilon_2}$ and $B_{X_2}^{\varepsilon_2} \hookrightarrow B_{X_2}^{\varepsilon_2}$ are homotopy equivalences. First observe that $B_{11,2}^{\varepsilon_2} \to U_{\varepsilon_2}(\Delta_1^f)$ is a fibration, as it is the pull-back along the map $U_{\varepsilon_2}(\Delta_1^f) \to U_{1,2}$ of the fibration $B_{1,2}^{\varepsilon_2} \to U_{1,2}$ of Proposition 2.7. Likewise $B_{X_2}^{\varepsilon_2} \to (U_{1,2} \times D^k) \cap N_2$ is a fibration. Hence it suffices to show that the maps $U_{\varepsilon_2}(\Delta_1^f) \to U_{2\varepsilon_2}(\Delta_1^f)$ and $(U_{1,2} \times D^k) \cap N_2 \to (U_{1,2}^{2\varepsilon_2} \times D^k) \cap N_2$ are weak homotopy equivalences, which they are under the Δ–transverse assumption (with $\varepsilon = 4\varepsilon_2$), using also Lemma 2.6 for the second map. $\square$

For the map $\pi_1 : W_{11,2} \to \Lambda_{1,2} \times I$, we will use the notion of a *bounded homotopy equivalence*: For $f : M_1 \to M_2$ a homotopy equivalence, and $g : M_2 \to M_1$ a homotopy inverse, with homotopies $h_1 : M_1 \times I \to M_1$ from $g \circ f$ to the identity and $h_2 : M_2 \times I \to M_2$ from $f \circ g$ to the identity, one can ask whether there also are homotopies

$$f \circ h_1 \simeq_{K_1} h_2 \circ (f \times 1) : M_1 \times I \longrightarrow M_2 \qquad \text{relative to } M_1 \times \partial I$$
$$g \circ h_2 \simeq_{K_2} h_1 \circ (g \times 1) : M_2 \times I \longrightarrow M_1 \qquad \text{relative to } M_2 \times \partial I$$

A homotopy equivalence f, such that K_1, K_2 exists for appropriate choices of g, h_1, h_2 was briefly called a *strong homotopy equivalence*, until Vogt showed that for any homotopy equivalence f and choices of g and h_1, there exists a homotopy h_2 such that K_1 and K_2 also exist [25].

Definition 3.9. Assume that M_2 has a metric. We call $f : M_1 \to M_2$ an *ε–bounded homotopy equivalence* if we can choose g, h_1, h_2 as above such that

(i) $|n - n'| < \varepsilon$ implies $|h_2(n,t) - h_2(n',t)| < \varepsilon$ for all $n, n' \in M_2$ and all $t \in I$;
(ii) $|f \circ h_1(m,t) - h_2(f(m),t)| < \varepsilon$ for every $m \in M_1$ and all $t \in I$.

Note that when $t = 0$, condition (i) says that $f \circ g$ stays bounded in the above sense. Also, if (f, g, h_1, h_2) is an ε–bounded strong homotopy equivalence, and M_2 is a Riemannian manifold with injectivity radius greater than ε, then we define K_1 as above using geodesics in M_2.

Recall from the introduction that condition B is the condition that f is a bounded homotopy equivalence in this sense.

Proposition 3.10. *Assume that f is a homotopy equivalence. Then the map $\pi_1 : W_{11,2} \to \Lambda_{1,2} \times I$ is a homotopy equivalence. If f is moreover ε_2–bounded (i.e. satisfies condition B), the map π_1 restricts to a homotopy equivalence $\pi_1 : B^{\varepsilon_2}_{11,2} \to B^{\varepsilon_2}_{1,2}$.*

Proof. Let g be a homotopy inverse for g. We choose h_1, h_2 and $K := K_1 : M_1 \times I \times I \to M_2$ as above, which is possible by Vogt's lemma [25]. (The homotopy K_2 will not be used.) If f is ε_2–bounded, we moreover choose K_1 to be such that each path $K_1(m, t, -)$ is a geodesic in M_2. This is possible by the above remarks. We define a map $G : \Lambda_{1,2} \times I \longrightarrow W_{11,2}$ by setting $G(m, \gamma, s) = (g \circ f(m), g \circ \gamma(s), f \circ g \circ \gamma, s)$. As $\gamma(0) = f(m)$, we have that $f \circ g \circ \gamma(0) = f \circ g \circ f(m)$ and G does have image in $W_{11,2}$.

If f is moreover bounded, we have that G restricts to a map $G : B^{\varepsilon_2}_{1,2} \to B^{\varepsilon_2}_{11,2}$: indeed, (m, γ, s) lies in the subspace $B^{\varepsilon_2}_{11,2}$ if γ satisfies that $|\gamma(t) - \gamma(0)| < \varepsilon_2$ for all $t \in [0, s]$ or for all $t \in [s, 1]$, and the bounded assumption ensures that if this holds for γ, then the same then holds for $f \circ g \circ \gamma$.

We have that $\pi_1 \circ G(m, \gamma, s) = (g \circ f(m), f \circ g \circ \gamma, s)$. We will use concatenation of the homotopy $h_2(\gamma, -)$ from $f \circ g \circ \gamma$ to γ and the homotopy K from $f \circ h_1(m, -)$ to $h_2(f(m), -)$ to define a homotopy L between $\pi_1 \circ G$ and the identity. Define $L : \Lambda_{1,2} \times I \times I \to \Lambda_{1,2} \times I$ by

$$L(m, \gamma, s, t) = (h_1(m,t), K(m, t, -) \star h_2(\gamma, t) \star K(m, t, -)^{-1}, s)$$

where $K(m, t, -) : I \to M_2$ is the path from $f(h_1(m,t))$ to $h_2(f(m), t)$ defined at time u by $K(m, t, u)$. (See Figure 2.) As $h_2(\gamma, t)$ is a loop based at $h_2(f(m), t)$, we see that the concatenation is well-defined. Also, the resulting loop is based at $f(h_1(m,t))$, as required. So L is well-defined. As K is a homotopy relative to $M_1 \times \partial I$, we have that $K(m, t, -)$ is a trivial path when $t = 0$ or 1, giving that L is a homotopy between $\pi_1 \circ G$ and the identity.

The boundedness assumption on f ensures that L restricts to a homotopy equivalence between $G \circ \pi_1$ and the inclusion $B^{\varepsilon_2}_{1,2} \hookrightarrow B^{2\varepsilon_2}_{1,2}$ as both h_2 and K stay bounded under this assumption, yielding together loops of length at most $2\varepsilon_2$.

The other way around, we have that $G \circ \pi_1(m, m', \gamma, s) = (g \circ f(m), g \circ f(m'), f \circ g \circ \gamma, s)$ as $\gamma(s) = f(m')$. Similarly to the previous case, concatenating $h_2(\gamma, .-)$ with K from $f \circ h_1(m, -)$ to $h_2(f(m), -)$ as well as K from $f \circ h_1(m', -)$ to $h_2(f(m'), -)$ gives a homotopy between $G \circ \pi_1$ and the identity. Explicitly, define $L' : W_{11,2} \times I \to W_{11,2}$ by

$$L'(m, m', \gamma, s, t) = (h_1(m,t), h_1(m',t), \gamma'_t, s'_t)$$

with

$$\gamma'_t = K(m, t, -) \star h_2(\gamma[0, s], t) \star K(m', t, -)^{-1} *_{s'_t} K(m', t, -) \star h_2(\gamma[s, 1], t) \star K(m, t, -)^{-1}$$

and

$$s'_t = s + (\frac{1}{2} - s)\min(t, 1 - t).$$

The time s'_t has the property that $s'_t = s$ if $t = 0$ or $t = 1$, allowing us to recover our original maps at time 0 and 1, while $0 < s'_t < 1$ when $0 < t < 1$ so that the concatenation is well-defined, as $K(m, t, -)$ and $K(m', t, -)$ will in general be non-trivial paths when $0 < t < 1$.

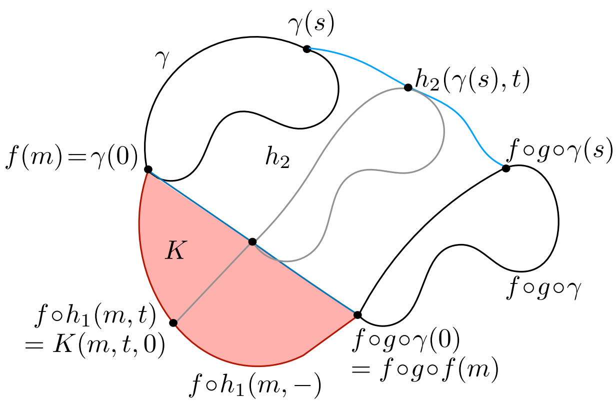


FIGURE 2. For $(m,\gamma,s)\in\Lambda_{1,2}\times I$, the figure illustrates the 2nd component of $L(m,\gamma,s,-)$, a homotopy between γ and $f\circ g\circ\gamma$ through loops originating in $\Lambda_{1,2}$, given at time t by $K(m,t,-)\star h_2(\gamma,t)\star K(m,t,-)^{-1}$.

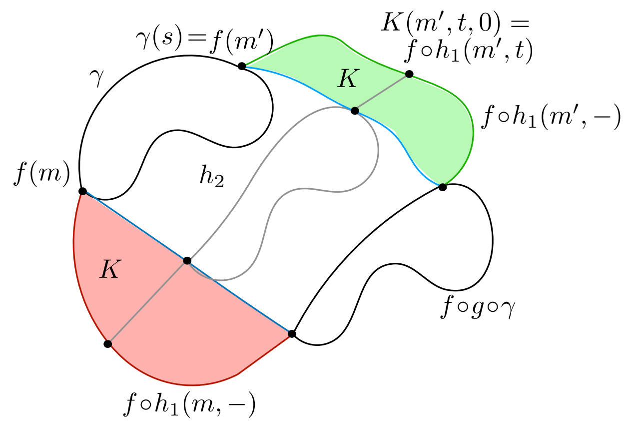


FIGURE 3. For $(m,m',\gamma,s)\in W_{11,2}$, the figure illustrates the 2nd component of $L'(m,m',\gamma,s,-)$, a homotopy between γ and $f\circ g\circ\gamma$ through loops originating in $W_{11,2}$, given at time t by $K(m,t,-)\star h_2(\gamma[0,s],t)\star K(m',t,-)^{-1} *_{s'} K(m',t,-)\star h_2(\gamma[s,1],t)\star K(m,t,-)^{-1}$.

As above, γ'_t is a loop based at $f(h_1(m,t))$, and γ'_t now passes through $f(h_1(m',t))$ at time s'_t, as required to have a well-defined element of $W_{11,2}$. As $\gamma(0)=f(m)$ and $\gamma(s)=f(m')$, we see that the concatenations defining γ'_t make sense: $K(m,t,-)$ is a path from $f(h_1(m,t))$ to $h_2(f(m),t))$ and $K(m',t,-)$ from $f(h_1(m',t))$ to $h_2(f(m'),t))$. Also, these two paths are trivial at time $t=0$ and $t=1$, giving that $\gamma'_0=f\circ g\circ\gamma$ while $\gamma'_1=\gamma$. Hence L' is a homotopy between $G\circ f'_2$ and the identity.

The boundedness assumption on f here gives that L restricts to a homotopy equivalence between $G\circ\pi_1$ and the inclusion $B^{\varepsilon_2}_{1,2}\hookrightarrow B^{3\varepsilon_2}_{1,2}$ as the boundedness of h_2 and K this time gives loops of length at most $3\varepsilon_2$.

In conclusion, G is a homotopy inverse for π_1 under the assumption that f is a homotopy equivalence. For the relative terms, note that $B^{\varepsilon_2}_{1,2}\hookrightarrow B^{2\varepsilon_2}_{1,2}\hookrightarrow B^{3\varepsilon_2}_{1,2}$ are homotopy equivalences as these spaces are fibered over the homotopy equivalences $U_{1,2}\hookrightarrow U^{2\varepsilon_2}_{1,2}\hookrightarrow U^{3\varepsilon_2}_{1,2}$. It follows that G is also a homotopy inverse for π_1 when restricted to the relative terms, under the stronger assumption that f is a bounded homotopy equivalence. This proves the result. □

Proposition 3.11. *If f is a homotopy equivalence, then $j_0:X_2\to\Lambda_{1,2}\times I$ is also a homotopy equivalence. If f is moreover satisfies condition D', or is Δ–transverse and satisfies either Condition B or Condition D, then j_0 restricts to a weak homotopy equivalence $j_0:B^{\varepsilon_2}_{X_2}\to B^{\varepsilon_2}_{1,2}$. In particular, the induced map in relative homology $j_0:H_*(X_2,B^{\varepsilon_2}_{X_2})\to H_*(\Lambda_{1,2}\times I,B^{\varepsilon_2}_{1,2})$ is an isomorphism.*

Proof. From Lemma 3.8 and Proposition 3.10, we have that $X_2\xleftarrow{e}W_{11,2}\xrightarrow{\pi_1}\Lambda_{1,2}\times I$ are homotopy equivalences if f is a homotopy equivalence. Hence the same holds for $j_0:X_2\to\Lambda_{1,2}\times I$ as the maps are related by $j_0\circ e=\pi_1$. Now the restriction $j_0:B^{\varepsilon_2}_{X_2}\to B^{\varepsilon_2}_{1,2}$ is also a homotopy by the same Lemma 3.8 and Proposition 3.10 if we assume Δ–transitivity and condition B, and a weak homotopy equivalence by Proposition 2.5 if we assume condition D' or Δ–transitivity together with condition D. The last part of the statement follows from the long exact sequence in homology. □

3.9. **Diagram (10).** Recall from [14, Lem 3.5] (taking $\varepsilon=\varepsilon_1=\varepsilon_i$ in the lemma) that, for $i=1,2$, the inclusion maps

$$(\mathcal{F}_i,\bar{B}_i)\hookrightarrow(U_{GH,i},B^{\varepsilon_i}_i)$$

are homotopy equivalences of pairs. We have denoted here

$$R_i\colon (U_{GH,i}, B_i^{\varepsilon_i}) \longrightarrow (\mathcal{F}_i, \bar{B}_i)$$

the homotopy inverses. To show that diagram (10) commutes, it is thus equivalent to show that the following diagram commutes in homology:

$$\begin{array}{ccc}
(U_{GH,1}, B_1^{\varepsilon_1}) & \hookleftarrow & (\mathcal{F}_1, \bar{B}_1) \\
\downarrow{\scriptstyle \pi_5\circ\hat{h}\circ e\circ f_{11,2}} & (10') & \downarrow{\scriptstyle \Lambda f\times I} \\
(U_{GH,2}, B_2^{\varepsilon_2}) & \hookleftarrow & (\mathcal{F}_2, \bar{B}_2)
\end{array}$$

Proposition 3.12. *Diagram (10') commutes on the space level.*

Proof. Let $(\gamma, s) \in \mathcal{F}_1 \subset U_{GH,1}$. We need to compute the composition of maps on the left vertical map in the diagram applied to (γ, s). We have

- $f_{11,2}(\gamma, s) = (m, m, f \circ \gamma, s) \in W_{11,2}$ where $f(m) = \gamma(0) = \gamma(s)$;
- $e(m, m, f \circ \gamma, s) = (m, f \circ \gamma, s, e(m)) \in e(\mathcal{F}_{1,2}) \subset X_3 \subset \Lambda_{1,2} \times I \times D^k$;
- $\hat{h}(m, f \circ \gamma, s, e(m)) = (m, f \circ \gamma, s, e(m))$ because $\hat{h}$ fixes $e(\mathcal{F}_{1,2})$ by Proposition 3.6(2);
- $\pi_5(m, f \circ \gamma, s, e(m)) = (f \circ \gamma, s)$;

And indeed, we see that the result equals $(\Lambda f \times I)(\gamma, s)$. □

3.10. **Proof of Theorem A.** Suppose $f : M_1 \to M_2$ is a homotopy equivalence between two closed Riemannian manifolds M_1 and M_2. Then f has degree ± 1. If f has degree -1, change the orientation of M_2 so that it becomes a degree 1 map. This change of orientation of M_2 has the effect of changing the sign of the Thom class τ_{M_2}, and hence multiplies the old coproduct by (-1). This reduces the question to the case of a degree 1 homotopy equivalence.

By Sections 3.1–3.9, we know that each subdiagram in Diagram (2.1) is well-defined and commutes. We also need that the maps π_2, $\bar{j}$, $[\![\bar{e}_I^*\bar{\tau}_e\cap]\!]$ and $[\![\bar{e}_I^*\bar{\eta}\cap]\!]$ are isomorphisms. The map π_2 is an isomorphism as it is a relative homotopy equivalence (see Section 3.3). Lemma 3.4 gives that $\bar{j}$ and $[\![\bar{e}_I^*\bar{\eta}\cap]\!]$ are isomorphisms. Proposition 3.11 gives that the map j_0 is an isomorphism under the assumption of Theorem A. As $j_0 = \pi_2 \circ j$, this implies that also j is an isomorphism which, by the commutativity of Subdiagram (5), gives that so is the map $[\![\bar{e}_I^*\bar{\tau}_e\cap]\!]$.

Hence the maps going the wrong way in the diagram (marked with double arrows in Diagram (2.1)) are all invertible. The commutativity of each of the subdiagrams thus implies the commutativity of the outside square. Now consider the diagram

$$\begin{array}{ccccccc}
H_*(\Lambda_1, M_1) & \xrightarrow{\times I} & H_*(\Lambda_1 \times I, B_1^{\varepsilon_1}) & \xrightarrow{R_1\circ[\![e_I^*\tau_{M_1}\cap]\!]} & H_*(\mathcal{F}_1, \bar{B}_1) & \xrightarrow{cut} & H_*(\Lambda_1 \times \Lambda_1, M_1 \times \Lambda_1 \cup \Lambda_1 \times M_1) \\
\downarrow & & \downarrow & (2.1) & \downarrow & & \downarrow \\
H_*(\Lambda_2, M_2) & \xrightarrow[\times I]{} & H_*(\Lambda_2 \times I, B_2^{\varepsilon_2}) & \xrightarrow[R_2\circ[\![e_I^*\tau_{M_2}\cap]\!]]{} & H_*(\mathcal{F}_2, \bar{B}_2) & \xrightarrow[cut]{} & H_*(\Lambda_2 \times \Lambda_2, M_2 \times \Lambda_2 \cup \Lambda_2 \times M_2)
\end{array}$$

The middle square commutes as it identifies with the outside of Diagram (2.1), and the left and right squares commute by the naturality of the cross and cut maps. Homotopy invariance of the coproduct follows since the top and bottom compositions are the definition of the coproduct of Λ_1 and Λ_2, where we used [14, Prop 3.6] to compute the coproducts.

The statement in cohomology follows from the universal coefficient theorem as the coproducts are maps induced in homology by chain maps, and the cohomology products are the maps in cohomology induced by these same chain maps (see [14, Def 1.6]).

Appendix A. Composing tubular embeddings

Given a composition of embeddings $M_1 \overset{f_1}{\hookrightarrow} M_2 \overset{f_2}{\hookrightarrow} M_3$ and choices of tubular neighborhoods for each embedding, we will construct here an associated tubular neighborhood for the composed map and study its properties.

Proposition A.1. *Consider*

(A.1)
$$\begin{array}{ccccc} N_1 & \hookrightarrow & N_2 & \hookrightarrow & \\ \downarrow p_1 & \searrow^{\nu_1} & \downarrow p_2 & \searrow^{\nu_2} & \\ M_1 & \xhookrightarrow{f_1} & M_2 & \xhookrightarrow{f_2} & M_3. \end{array}$$

with f_1, f_2 codimension k and l embeddings respectively, N_1, N_2 the associated normal bundles and ν_1, ν_2 chosen tubular neighborhoods. Then

1. *The bundle $N_{12} := N_1 \oplus f_1^* N_2 \longrightarrow M_1$ is isomorphic to the normal bundle of $f_2 \circ f_1$.*
2. *There is an isomorphism*

$$\begin{array}{ccccccc} N_{12} = N_1 \oplus f_1^* N_2 & \overset{\cong}{\dashrightarrow} & \nu_1^* N_2 \cong N_2|_{\nu_1 N_1} & \hookrightarrow & N_2 & & \\ \downarrow \quad \searrow^{p} & & \swarrow & & \downarrow p_2 & \searrow^{\nu_2} & \\ M_1 \xleftarrow{p_1} & N_1 & \xhookrightarrow{\nu_1} & & M_2 & \xrightarrow{f_2} & M_3 \end{array}$$
$$M_1 \xrightarrow{f_1} M_2$$

as bundles over N_1, where p is the natural projection, with the property that the resulting composition

$$\nu_{12} := \nu_2 \circ \hat{\nu}_1 : N_{12} \xrightarrow{\hat{\nu}_1} N_2 \xrightarrow{\nu_2} M_3$$

is a tubular neighborhood for $f_2 \circ f_1 : M_1 \hookrightarrow M_3$. Explicitly, $\nu_{12}(m, V, W) = \nu_2(\nu_1(m, V), W')$ for $W' \in (N_2)_{\nu_1(m,V)}$ the parallel transport of W along the path $\nu_1(m, tV)$.

3. *Let $\tau_1 \in C^k(M_2, \nu_1(N_1)^c)$ and $\tau_2 \in C^l(M_3, \nu_2(N_2)^c)$ be Thom classes for (N_1, ν_1) and (N_2, ν_2). Let $p_2^* \tau_1$ be the image of τ_1 along the maps*

$$C^k(M_2, \nu_1(N_1)^c) \xrightarrow{p_2^*} C^k(N_2, \hat{\nu}_1(N_{12})^c) \xrightarrow{\sim} C^k(M_3, \nu_{12}(N_{12})^c),$$

where the second map is the quasi-isomorphism induced by ν_2 and excision. Then

$$\tau_{12} := p_2^* \tau_1 \cup \tau_2 \in C^{k+l}(M_3, \nu_{12}(N_{12})^c)$$

is a Thom class for the tubular embedding ν_{12}. Moreover, the diagram

$$\begin{array}{ccc} H_*(M_3) & \xrightarrow{\llbracket \tau_2 \cap \rrbracket} & H_{*-l}(M_2) \\ & \searrow^{\llbracket \tau_{12} \cap \rrbracket} & \downarrow \llbracket \tau_1 \cap \rrbracket \\ & & H_{*-k-l}(M_1) \end{array}$$

commutes, where

$$\llbracket \tau_1 \cap \rrbracket : C_*(M_2) \to C_*(M_2, \nu_1(N_1)^c) \xrightarrow{[\tau_1 \cap]} C_{*-k}(N_1) \xrightarrow{\simeq} C_{*-k}(M_1)$$
$$\llbracket \tau_2 \cap \rrbracket : C_*(M_3) \to C_*(M_3, \nu_2(N_2)^c) \xrightarrow{[\tau_2 \cap]} C_{*-l}(N_2) \xrightarrow{\simeq} C_{*-k}(M_2)$$
$$\llbracket \tau_{12} \cap \rrbracket : C_*(M_3) \to C_*(M_3, \nu_{12}(N_{12})^c) \xrightarrow{[\tau_{12} \cap]} C_{*-k-l}(N_{12}) \xrightarrow{\simeq} C_{*-k-l}(M_1)$$

are the maps induced by capping with the respective Thom classes after taking small simplices (see [14, A.2] *for details about the map $[\tau \cap]$), and retracting to the zero section at the end.*

To prove the proposition, we will, as in the rest of the paper, pick Riemannian metrics. This allows us for example to consider the normal bundle of an embedding as a subbundle of the tangent bundle of the target, as we explain now: Let

$$f : M \hookrightarrow M'$$

be an embedding of manifolds, with M compact and M' a Riemannian manifold. Let $p : N \to M$ be the normal bundle of f. The metric of M' gives an identification of N with a subbundle of $TM'|_M$:

(A.2)
$$N \equiv TM^{\perp} =: \{(m, W) \mid m \in M,\, W \in T_{f(m)} M' \text{ and } W \perp f_* T_m M\}$$

and thus splits the exact sequence

$$0 \to TM \to TM' \to N \to 0$$

of bundles over M. We will identify N with this subbundle.

The following lemma is an elementary consequence of the inverse function theorem.

Lemma A.2. *Let $N \to M$ be the normal bundle of the embedding $f : M \hookrightarrow M'$ as above. Suppose that*

$$\nu : N \longrightarrow M'$$

has the following properties:

(i) $\nu(m, 0) = f(m)$ *for all* $m \in M$.
(ii) The map

$$d\nu : T_{(m,0)}N \to T_{f(m)}M'$$

is surjective.

Then there exists $\varepsilon > 0$ so that the restriction of ν to N_ε is a tubular embedding of M in M', for $N_\varepsilon = \{(m, W) \in N \mid |W| < \varepsilon\}$.

Proof of Proposition A.1. Choose a Riemannian metric on M_3. Let M_1 and M_2 carry the induced metrics. Statement (1) follows from (suppressing the inclusion maps f_1, f_2):

$$N \equiv TM_1^\perp \subset TM_3;\ \ N_1 \equiv TM_1^\perp \subset TM_2;\ N_2 \equiv TM_2^\perp \subset TM_3,$$

so that for each $m \in M_1$,

$$N_m \equiv (N_1)_m \oplus (N_2)_{f(m)}.$$

As a consequence we can and will identify

$$N \cong N_{12} \equiv \left\{ (m, V, W) \;\middle|\; \begin{array}{l} m \in M_1, \\ V \in T_{f_1(m)}M_2,\ V \perp df_1 T_m M_1, \text{ and} \\ W \in T_{f(m)}M_3,\ W \perp df_2 T_{f_1(m)} M_2 \end{array} \right\}.$$

To prove (2), define $\hat{\nu}_1 : N_{12} \to N_2$ by

$$\hat{\nu}_1(m, V, W) = (\nu_1(m, V), W')$$

where $W' \in (N_2)_{\nu_1(m,V)}$ is the vector obtained from W by parallel transport along the path $\nu_1(m, tV)$. The inverse is given by parallel transporting back, which gives the claimed isomorphism by uniqueness of such transports.

We then apply the lemma in the case $M = M_1$, $M' = M_3$, $f = f_2 \circ f_1$, and $\nu = \nu_{12}$. We need to check condition (ii) of the lemma. Fix $m \in M_1$. Because $\nu_1 : N_1 \to M_2$ is a tubular embedding, the map $d\nu_1 : T_{(m,0)}N_1 \to T_m M_2$ is surjective, and the tangent space to M_2 in M_3 is in the image of $d\nu_{12} =: d(\nu_2 \circ \hat{\nu}_1)$. Now let $W \in N_2$; that is, $W \in T_m M_3$, with $W \perp T_m M_2$. We find by definition $\hat{\nu}_1(m, 0, W) = (m, W) \in N_2$ and $\nu_{12}(m, 0, W) = \nu_2 \circ \hat{\nu}_1(m, 0, W) = (m, W)$. (When $V = 0$, the parallel transport is trivial.) Since $T_m M_3$ is spanned by $T_m M_2$ and $d\nu_2 N_2$, we conclude that

$$\nu_{12*} : T_{(m,0)}N_{12} \to T_m M_3$$

is surjective.

To show the commutativity of the diagram in statement (3), we decompose the diagram as

$$\begin{array}{ccccccc}
H_*(M_3) & \longrightarrow & H_*(M_3, N_2^c) & \xrightarrow{[\tau_2\cap]} & H_{*-k}(N_2) & \xrightarrow{p_2} & H_{*-k}(M_2) \\
& \searrow & & & \downarrow & & \downarrow \\
& & H_*(M_3, N_{12}^c) & & H_{*-k}(N_2, N_{12}^c) & \xrightarrow{p_2} & H_{*-k}(M_2, N_1^c) \\
& & & {\scriptstyle [\tau_{12}\cap]}\searrow & \downarrow{\scriptstyle [p_2^*\tau_1\cap]} & & \downarrow{\scriptstyle [\tau_1\cap]} \\
& & & & H_{*-k-l}(N_{12}) & \xrightarrow{p_2} & H_{*-k-l}(N_1) \\
& & & & & {\scriptstyle p_{12}}\searrow & \downarrow{\scriptstyle p_1} \\
& & & & & & H_{*-k-l}(M_1)
\end{array} \tag{A.3}$$

where we have suppressed the embeddings ν_i, $\hat{\nu}_i$ for readability. Commutativity follows from the naturality of the cap product $[\tau\cap]$ [14, Lem A.1] and the compatibility of the cup and cap products.

We are left to check that τ_{12} is the Thom class of the composition, which follows from its definition as

$$\tau_{12}\cap[M_3] = p_2^*\tau_1\cap(\tau_2\cap[M_3]) = p_2^*\tau_1\cap(f_2)_*[M_2] = (f_2)_*(f_2^*p_2^*\tau_1\cap[M_2]) = (f_2)_*(\tau_1\cap[M_2]) = (f_2)_*(f_1)_*[M_1],$$

where we used that $p_2 \circ f_2 : M_2 \to N_2 \to M_2$ is the identity. □

Appendix B. Naef's lens space example

Let S^3 be the 3-sphere, considered as the unit sphere in $\mathbb{C}^2$. We will write elements of S^3 in spherical coordinates as tuples $((r_1, \theta_1), (r_2, \theta_2))$ with $r_i \geq 0$ and $\theta_i \in \mathbb{R}/\mathbb{Z}$, with $r_1^2 + r_2^2 = 1$.

The lens space $L(k, 7)$ is the quotient of S^3 by the relation

$$((r_1, \theta_1), (r_2, \theta_2)) \ \sim \ ((r_1, \theta_1 + \frac{1}{7}), (r_2, \theta_2 + \frac{k}{7}))$$

The spaces $L(1, 7)$ and $L(2, 7)$ are known to be homotopy equivalent and not homeomorphic. (The classification of lens spaces goes back to Reidemeister [21]. See eg., [17, Sec 6.4] for a modern description.) A homotopy equivalence can be obtained from the degree 8 map $f_{2,4} : L(1, 7) \to L(2, 7)$ taking (z_1, z_2) to (z_1^2, z_2^4), by wedging $f_{2,4}$ with a degree -7 map via a composition

$$g : L(1, 7) \longrightarrow L(1, 7) \vee S^3 \xrightarrow{f_{2,4}\vee(-1)} L(2, 7) \vee S^3 \longrightarrow L(2, 7)$$

where "(-1)" denotes a degree (-1) self-map of S^3, that is then composed with the natural (degree 7) projection to $L(2, 7)$.

Naef's computation can be formulated as follows:

Proposition B.1. [18] *No homotopy equivalence $f : M_1 = L(1, 7) \longrightarrow L(2, 7) = M_2$ preserves the coproduct $\vee$ on the homology of the associated free loop spaces $\Lambda L(1, 7)$ and $\Lambda L(2, 7)$.*

We will give here for the reader's convenience a sketch of proof of this result, following the computations in [18].

We have that $\pi_1 L(k, 7) = \mathbb{Z}/7 = \pi_0 \Lambda L(k, 7)$. Let $\Lambda_\ell L(k, 7)$ denote the component of the loop

$$t \ \ \mapsto \ \ ((1, \ell t), (0, 0)).$$

Note that this naming of the components works for all k. Now any homotopy equivalence $f : L(1, 7) \to L(2, 7)$ has degree 1 and induces multiplication by 2 on the fundamental group [17, Lem 6.9]. In particular, Λf takes $\Lambda_1 L(1, 7)$ to $\Lambda_2 L(2, 7)$.

The coproduct is a map

$$H_*(\Lambda L(k, 7)) \longrightarrow H_{*-3+1}(\Lambda L(k, 7) \times \Lambda L(k, 7)).$$

We will partially compute the coproduct on the components $\Lambda_1 L(1, 7)$ and $\Lambda_2 L(2, 7)$, and in homological degree $* = 3$. Even more precisely, we will exhibit a class $\rho_{1,0} \in H_3(\Lambda_1 L(1, 7))$ whose coproduct is trivial, and show that its image in $H_3(\Lambda_2 L(2, 7))$ cannot have a trivial coproduct.

Sketch proof of Proposition B.1. Using the path-loop fibration, one finds that $H_3(\Lambda_\ell L(k,7)) = \mathbb{Z}\oplus\mathbb{Z}/7$ (see [18, Sec 2.1]), coming from a short exact sequence

$$H_1(L(k,7), H_2(\Omega_\ell L(k,7))) \longrightarrow H_3(\Lambda_\ell L(k,7)) \longrightarrow H_3(L(k,7)) \cong \mathbb{Z},$$

where Ω denotes the based loop space. In particular, there is a copy of $1+\mathbb{Z}/7$ inside $H_3(\Lambda_\ell L(k,7))$ of classes evaluating to the fundamental class of the lens space. We will work with such classes, all parametrized by $L(k,7)$ itself.

In $H_3(\Lambda_1 L(1,7))$ there is a class $\rho_{1,0} : L(1,7) \to \Lambda_1 L(1,7)$ of simple loops:

$$\rho_{1,0}((r_1,\theta_1),(r_2,\theta_2))\colon \left\{ \begin{array}{lcl} S^1 & \longrightarrow & L(1,7) \\ t & \mapsto & ((r_1,\theta_1+\frac{t}{7}),(r_2,\theta_2+\frac{t}{7})). \end{array}\right.$$

Note that $e\circ\rho_{1,0} = \mathrm{id}_{L(1,7)}$ so the corresponding class $\rho_{1,0}\in H_3(\Lambda L(1,7))$ evaluates to the fundamental class in $H_3(L(1,7))$. And because the loops in the image are all simple, it follows from [14, Thm 3.10] that $\vee\rho_{1,0} = 0$.

Now let $f : L(1,7) \to L(2,7)$ be a homotopy equivalence. By the discussion above, we know that $f_*(\rho_{1,0}) \in H_3(\Lambda_2 L(2,7))$ and it evaluates to the fundamental class in $H_3(L(2,7))$ as f is a homotopy equivalence.

Consider the classes $\rho_{2,m}$ in $H_3(\Lambda_2 L(2,7))$, parametrized by $L(2,7)$, defined as follows:

$$\rho_{2,m}((r_1,\theta_1),(r_2,\theta_2))\colon \left\{ \begin{array}{lcl} S^1 & \longrightarrow & L(2,7) \\ t & \mapsto & ((r_1,\theta_1+\frac{2t}{7}),(r_2,\theta_2+\frac{4t}{7}+mt)). \end{array}\right.$$

Note that these classes likewise have the property that they evaluate to the fundamental class in $H_3(L(2,7))$. We will only need the classes $\rho_{2,m}$ for $m=1,\ldots,7$.

To compute the coproduct of $\rho_{2,m}$, we can use [14, Prop 3.7], that says that, for such a "nice" representative of a homology class, the coproduct is computed by transerves self-intersections at the basepoint away from the constant loops. Because the constant loops live in a separate component and will not play a role here, we will do the whole computation away from them, only considering the part of the image of $\vee\rho_{2,m}$ in the summands $H_1(\Lambda_\ell L(2,7)\times\Lambda_{\ell'}L(2,7))$ with $\ell,\ell'\neq 0$. As we must have $\ell+\ell' = 2 \mod 7$, that gives the components $(\ell,\ell') = (1,1),(3,6),(4,5),(5,4),(6,3)$.

One checks that the image of $\rho_{2,m}$ only contains loops with non-trivial self-intersections at the basepoint when either $r_1 = 0$ or $r_2 = 0$. If $r_2 = 0$, we have $(r_1,\theta_1) = (1,\theta)$ and these are the loops

$$t\mapsto ((1,\theta+\frac{2t}{7}),(0,0)).$$

These loops have exactly one non-trivial self-intersection at time $t=\frac{1}{2}$ with left and right loop both in the component $\Lambda_1 L(2,7)$. When $r_1 = 0$, we have $(r_2,\theta_2) = (1,\theta)$ corresponding to the loops

$$t\mapsto ((0,0),(1,\theta+\frac{4t}{7}+mt)) = ((0,0),(1,\theta+\frac{(4+7m)}{7}t)).$$

These loops have a potentially non-trivial self-intersection at time $t=\frac{i}{4+7m}$ for $i\in\{1,\ldots,3+7m\}$. The left loop will however be in the component of the constant loops when $i = 0 \mod 7$ and the right loop when $i = 4 \mod 7$, and we discard those for simplicity. This leaves in total the following self-intersections:

(i) If $r_1 = 0$ and $t = \{\frac{1}{4+7m},\frac{2}{4+7m},\frac{3}{4+7m}\}\bigcup_{i=1}^m\{\frac{7i-2}{4+7m},\frac{7i-1}{4+7m},\frac{7i+1}{4+7m},\frac{7i+2}{4+7m},\frac{7i+3}{4+7m}\}$
(ii) If $r_2 = 0$ and $t=\frac{1}{2}$,

each coming in the 1-parameter families described above. One checks that these self-intersections are transverse, and that they all give the standard generator $\beta_{\ell,\ell'}\in H_1(\Lambda_\ell L(2,7)\times\Lambda_{\ell'}L(2,7))$, namely the image under the S^1–action of any basepoint in the given component. We are left to check which components the cutting map takes these self-intersections to. When $r_1 = 0$ and $t=\frac{i}{4+7m}$, the left loop will be in the component of the loop

$$t'\mapsto ((0,0)(1,\frac{it'}{7}))$$

which is the component $\Lambda_\ell L(2,7)$ for ℓ such that $2\ell = i$. Hence when $i = 1 \mod 7$, cutting will land in the component $(\ell,\ell') = (4,5)$, when $i = 2 \mod 7$, in the component $(\ell,\ell') = (1,1)$, when $i = 3 \mod 7$, in the component $(\ell,\ell')=(5,4)$, when $i=5 \mod 7$, in the component $(\ell,\ell')=(6,3)$, and when $i=6 \mod 7$, in the component $(\ell,\ell')=(3,6)$.

Together with the self-intersection coming from $r_2=0$, counting all the above classes gives the following:

$$\vee^0\rho_{2,m}=(2+m)\beta_{1,1}+(1+m)(\beta_{4,5}+\beta_{5,4})+m(\beta_{3,6}+\beta_{6,3})$$

in $H_1\big(\Lambda L(2,7)\times\Lambda L(2,7), L(2,7)\times\Lambda L(2,7)\cup\Lambda L(2,7)\times L(2,7)\big)$, where $\vee^0$ denotes the restriction of $\vee$ away from the components of trivial loops. Naef computes that $f_*\rho_{1,0}=\rho_{2,3}$ for the above map f [18, Lem 9], which concludes the proof at least for that map as $\rho_{2,3}$ has non-trivial coproduct by the above computation. Alternatively, one can use that the classes $\rho_{2,m}$ with $m=1,\dots,7$ have linearly independent coproduct, and hence must be linearly independent in $H_3(\Lambda L(2,7))$, and must generate $H_3(\Lambda_2 L(2,7))$ as they all evaluate to the fundamental class of $L(2,7)$. Now one can compute that no linear combination of $\rho_{2,m}$'s with coefficients s_m such that $\sum_{m=1}^7 s_m=1$ (to have the property that the class evaluates to the fundamental class) has a trivial coproduct, which gives an alternative way to finish the proof. □

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