HOMOTOPY INVARIANCE OF THE STRING TOPOLOGY COPRODUCT

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Abstract. We show that the Goresky-Hingston coproduct in string topology, just like the Chas-Sullivan product, is homotopy invariant.

Introduction

For a closed oriented manifold $M$, the Chas-Sullivan product [4] is a product on the homology of its free loop space $\Lambda M = \text{Maps}(S^1, M)$ of the form

$$\wedge : H_p(\Lambda M) \otimes H_q(\Lambda M) \to H_{p+q-n}(\Lambda M),$$

where $H_*(-) = H_*(-; \mathbb{Z})$ denotes singular homology with integral coefficients and $n = \dim M$. Goresky-Hingston defined in [11] a “dual” product in cohomology relative to the constant loops

$$\otimes : H^p(\Lambda M, M) \otimes H^q(\Lambda M, M) \to H^{p+q+n-1}(\Lambda M, M).$$

The associated homology coproduct

$$\vee : H_p(\Lambda M, M) \to H_{p-n+1}(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M).$$

was defined by Sullivan [20]. The idea of the Chas-Sullivan product is to intersect the chains of basepoints of two chains of loops and concatenate the loops at the common basepoints. The homology coproduct looks instead for self-intersections at the basepoint within a single chain of loops and cuts. Both structures are represented geometrically by the same figure, either thought of as two loops intersecting at their basepoint,

or as a single loop having a self-intersection at its basepoint. For the coproduct, it is essential to look for self-intersections with the basepoint at all times $t$ along the loop, including $t = 0$ and $t = 1$ (where all loops tautologically have a self-intersection!)—Tamanoi showed that the coproduct that only looks for self-intersections at $t = \frac{1}{2}$ is trivial [21]. (In contrast, $H^*(\Lambda S^n, S^n)$ for an odd sphere $S^n$ is generated by just 4 classes as an algebra with the cohomology product $\otimes$ [11, 15.3].) The domain $[0,1]$ of the time variable $t$ is the reason why the coproduct naturally is an operation in relative homology: it has non-trivial boundary terms coming from the boundary of the interval. These boundary terms can be interpreted as coming from a certain compactification of the moduli space of Riemann surfaces, making it a so-called compactified operation.

As in [15], we will denote by

$$\tilde{\vee} : H_p(\Lambda M) \to H_{p-n+1}(\Lambda M \times \Lambda M)$$

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1As we work with homology with integral coefficients, we do not in general have a Kunneth isomorphism, which is why the coproduct in integral homology has target $H_*(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M)$ rather than $H_*(\Lambda M, M) \otimes H_*(\Lambda M, M)$. 
the extension by 0 of the coproduct induced by the natural splitting \( H_*(\Lambda M) \cong H_*(\Lambda M, M) \oplus H_*(M) \) (declaring the coproduct to be zero on the summand where it was not defined). The coproduct \( \vee \) shares the nice properties of the original coproduct \( \forall \), and detects intersection multiplicity of homology classes in the loop space, the largest number \( k \) such that any representing chain of a given homology class in \( H_*(\Lambda M) \) necessarily has a loop with a \( k \)-fold self-intersection at its basepoint. (See [15, Thm B, Thm D].)

While the above products and coproducts are all induced by chain maps, we will only consider in the present paper the induced maps in homology/cohomology.

It has been shown by several authors that the Chas-Sullivan product is homotopy invariant, in the sense that a homotopy equivalence \( f : M_1 \to M_2 \) induces an isomorphism \( \Lambda f : H_*(\Lambda M_1) \to H_*(\Lambda M_2) \) of algebras with respect to the Chas-Sullivan product (see [7, Thm 1],[12, Prop 23],[8, Thm 3.7]). This result left open the question whether a more complex operation like the Goresky-Hingston coproduct would, or would not, likewise be homotopy invariant. Note that a coproduct of the same flavour was used in [5, Sec 2.1] to define the differential in string homology, a knot invariant that detects the unknot. We show here that, as operations in homology, the coproducts \( \vee \) and \( \forall \), as well as the associated cohomology products \( \circ \) and \( \ast \), are in fact homotopy invariant:

**Theorem A.** Let \( f : M_1 \to M_2 \) be a homotopy equivalence between closed oriented manifolds. Then the following two diagrams commute:

\[
\begin{align*}
H_*(\Lambda M_1, M_1) & \xrightarrow{\vee} H_{n+1}(\Lambda M_1 \times \Lambda M_1, M_1 \times \Lambda M_1 \cup \Lambda M_1 \times M_1) \\
H_*(\Lambda M_2, M_2) & \xrightarrow{\text{deg}(f) \vee} H_{n+1}(\Lambda M_2 \times \Lambda M_2, M_2 \times \Lambda M_2 \cup \Lambda M_2 \times M_2).
\end{align*}
\]

\[
\begin{align*}
H_*(\Lambda M_1) & \xrightarrow{\forall} H_{n+1}(\Lambda M_1 \times \Lambda M_1) \\
H_*(\Lambda M_2) & \xrightarrow{\text{deg}(f) \forall} H_{n+1}(\Lambda M_2 \times \Lambda M_2).
\end{align*}
\]

Likewise, the map induced by \( \Lambda f \) in cohomology respects the corresponding cohomology products \( \circ \) and \( \ast \) (as defined in [11] and [15]) up to the same sign \( \text{deg}(f) \).

In particular, any quantity measured by the above operations only depends on the homotopy type of the manifold.

The above result is, as far as we know, the first homotopy invariance for a compactified string operation. One expects that, more generally, the harmonic compactification of moduli space defines a prop of operations in string topology, see [23, Sec 6.6] for a rational version of this, or [9, 14] for work in this direction. We do not know whether the arguments presented here would extend to such more general compactified operations.

We use in this paper the Cohen-Jones approach to string topology [6]: The products and coproducts described above are all lifts to the loop space \( \Lambda M \) of the intersection product on \( H_*(M) \). While the intersection product is easiest defined as the Poincaré dual of the cup product on \( H^*(M) \), the definition of the coproduct \( \vee \) used here is, as in [6] for the product \( \wedge \), a lift to the loop space of the alternative definition of the intersection product in terms of a Thom-Pontryagin construction.

Our proof of Theorem A is heavily inspired by the proof of homotopy invariance of the Chas-Sullivan product given by Gruher-Salvatore in [12]. For the coproduct, extra care is needed because of the boundary terms described above. The proof can also be seen as a lift to the loop space of a proof of homotopy invariance of the intersection product using its Thom-Pontryagin definition. As the case of the intersection product is much simpler than that of the coproduct \( \vee \), but yet already contains many of the crucial ingredients, we give it here in the first section of the paper.

Note that it is much easier to use Poincaré duality to show that the classical homology intersection product is homotopy invariant; it is in fact more generally invariant under degree \( d \) maps after an appropriate scaling, see e.g., [3, VI, Prop 14.2]. In [13], the first author proved that the same invariance
under degree $d$ maps holds for the restriction of the coproduct $\vee$ to the based loop space under the additional assumption that the target manifold $M_2$ is simply-connected. Given these two results, one could expect that this stronger invariance result holds more generally for the whole loop space, and also for the Chas-Sullivan product, but the proof of homotopy invariance given here does not directly generalize to prove such a result. Already for the intersection product, the proof given here assumes that the map $f$ is a homology isomorphism (see Proposition 1.7). For Theorem A, we use that $f$ is an actual homotopy equivalence in Section 3.7. In both cases, the assumption is used to show that certain maps are invertible.

**Remark 1** (Relation to Hochschild homology). When $M$ is simply connected, there are graded vector space isomorphisms $H^*(\Lambda M; k) \cong HH_*(C^*(M; k), C^*(M; k))$ and $H_*(\Lambda M; k) \cong HH_*(C^*(M; k), C^*(M; k))$ for $k$ any field (see [17], [10, App]). In [10], it is shown that, in characteristic zero, the Chas-Sullivan product becomes the classical cup product under that last isomorphism. The cohomology product has a description in terms of the first isomorphism, again assuming characteristic zero for the field, see [2, 19] that confirm the conjectural description of [1, Sec 6],[18, Conj 2.14]. A homotopy equivalence $f : M_1 \to M_2$ induces a quasi-isomorphism of cochain algebras, and thus a $k$-module isomorphism $HH_*(C^*(M_2; k), C^*(M_2; k)) \cong HH_*(C^*(M_1; k), C^*(M_1; k))$. However it is not clear that this isomorphism preserves the algebraic model of the Goresky-Hingston product since the algebraic product uses a (co)chain level model of the intersection product.

**Organization of the paper.** In Section 1, we give a proof of the invariance of the homology intersection product under homology isomorphisms (Proposition 1.7) that does not use Poincaré duality. The intermediate results proved in that section will be used later in the paper. Section 2 gives a plan of the proof of homotopy invariance of the coproducts $\vee$ and $\check{\vee}$, in particular setting up the key Diagram (2.1), whose commutativity is then proved in Section 3. The proof of Theorem A is given at the end of Section 3. Finally Appendix A at the end of the paper details the relationship between the tubular neighborhoods of two composable embedding and the tubular neighborhood of their composition.

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## 1. Homotopy invariance of the intersection product

The easiest way to prove the homotopy invariance of the intersection product on the homology of closed manifolds is to use its definition as Poincaré dual of the cup product, and use the homotopy invariance of the cup product. In fact, the compatibility is easily checked with any degree $d$ map, up to appropriate scaling, see e.g., [3, VI, Prop 14.2]. We will here present a proof of the homotopy invariance using instead the definition of the intersection product via capping with a Thom class, as a warm-up, and partial preparation, for the homotopy invariance of the string topology coproduct. The proof given here for the intersection product in fact only uses that $f$ induces an isomorphism in homology, so that is what we will assume for this section.

Let $M$ be a closed oriented Riemannian manifold of injectivity radius $\rho$, and let $TM$ denote its $\varepsilon$-tangent bundle, where $\varepsilon$ is small compared to $\rho$. We will also assume that $\varepsilon < \frac{\rho}{2}$.

The normal bundle of the diagonal embedding $M \hookrightarrow M \times M$ is isomorphic to $TM$, and we can choose as tubular embedding the map $\nu_M : TM \to M \times M$ given by $\nu_M(m, V) = (m, \exp_m V)$. (By an embedding, we will in this paper always mean a $C^1$–map that is an immersion and a homeomorphism onto its image.) Note that $\nu_M$ has image the $\varepsilon$–neighborhood

$$U = \{(m, m') \in M \times M \mid |m - m'| < \varepsilon\} \subset M \times M.$$  

Let $\tau_M \in C^n(M \times M, U^c)$ be a chosen Thom class for this tubular neighborhood.
Up to sign, the intersection product on a closed manifold $M$ can be defined as the composition

$$C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \rightarrow C_{p+q}(M \times M, U^c) \xrightarrow{\tau_{M_1}(\cdot)} C_{p+q-n}(U) \rightarrow C_{p+q-n}(M)$$

(See eg., [15, App B].)

Let $f : M_1 \rightarrow M_2$ be a homology isomorphism between two closed oriented Riemannian manifolds $M_1$ and $M_2$. We may assume that $f$ is smooth and, possibly after changing the orientation of one of the manifolds, that it preserves the orientation. In particular, $f$ has degree 1, that is $f[M_1] = [M_2]$. Leaving out the cross-product, which is well-known to be natural, we want to show that the diagram

$$\begin{array}{ccc}
H_{p+q}(M_1 \times M_1) & \xrightarrow{f \times f} & H_{p+q}(M_2 \times M_2) \\
\downarrow & & \downarrow \\
H_{p+q-n}(U_1) & \xrightarrow{\tau_{M_1}(\cdot)} & H_{p+q-n}(U_2)
\end{array}$$

commutes, for $U_1, U_2$ the $\varepsilon$–tubular neighborhoods of the diagonals in $M_1 \times M_1$ and $M_2 \times M_2$. We could approximate $f$ by a Lipschitz map, and assume that $f$ takes $U_1$ to $U_2$, but we cannot assume that $f$ takes $U_1^\varepsilon$ to $U_2^\varepsilon$. In particular, we do not have a vertical map in the second column in the diagram. Likewise, we cannot use $f$ directly to compare the Thom classes $\tau_{M_1}$ and $\tau_{M_2}$. To make up for this, we will replace $f$ by an embedding. A key tool in the proof will be the properties of composed tubular embeddings, as detailed in Appendix A.

Pick an embedding $e : M_1 \rightarrow D^k$, for some $k$ large. Then the map $(f, e) : M_1 \rightarrow M_2 \times D^k$ is an embedding too and we can consider the composition of embeddings

$$\tau_{M_1} : \Delta_{M_1} \rightarrow M_1 \times M_1 \xrightarrow{(id, f, e)} \Delta_{M_1} \rightarrow M_1 \times M_2 \times D^k.$$ (1.1)

To this situation, we can associate a diagram

$$\begin{array}{ccc}
TM_1 & \xrightarrow{p_1} & N_2 = M_1 \times N(f, e) \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\Delta_{M_1}} & M_1 \times M_1
\end{array}$$

where $\nu_1 := e_{M_1} : TM_1 \rightarrow M_1 \times M_1$ is the tubular neighborhood of $\Delta_1$ defined as above, with associated Thom class $\tau_{M_1} \in C^n(M_1 \times M_1, U_1^\varepsilon)$, and where $N(f, e)$ is the normal bundle of $(f, e)$ with $\nu_e : N(f, e) \rightarrow M_2 \times D^k$ a chosen tubular embedding. Pick a Thom class $\tau_e \in C^k(M_2 \times D^k, (\nu_e N(f, e)^c)$ for $\nu_e$. Then

$$1 \times \tau_e \in C^k(M_1 \times M_2 \times D^k, (\nu_2 N_2)^c)$$

is a Thom class of $\nu_2$.

Applying Proposition A.1, we now have the following:

**Proposition 1.1.** Consider the sequence of embeddings $M_1 \xrightarrow{\Delta_{M_1}} M_1 \times M_1 \xrightarrow{(id, f, e)} M_1 \times M_2 \times D^k$ and let $\nu_1, \nu_2$ and $\tau_1 = \tau_{M_1}, \tau_2 = 1 \times \tau_e$ be the embeddings and Thom classes chosen above. Then the following hold:

1. The bundle $N_3 := TM_1 \oplus N(f, e)$ is isomorphic to the normal bundle of the composed embedding $(id, f, e) : M_1 \rightarrow M_1 \times M_2 \times D^k$.

2. There is an isomorphism $\hat{\nu}_1 : N_3 \cong p_2^{-1}(U_1) \subset M_1 \times N(f, e)$, as bundles over $TM_1 \cong U_1$, with the property that

$$\nu_3 := \nu_2 \circ \hat{\nu}_1 : N_3 \rightarrow M_1 \times M_2 \times D^k$$

is a tubular embedding for $(id, f, e)$. 


(3) The class
\[ \tau_3 := p_2^* \tau_{M_1} \cup (1 \times \tau_e) \in C^{n+k}(M_1 \times M_2 \times D^k, N^k_3) \]
is a Thom class for the tubular embedding \( \nu_3 \), and the diagram
\[
\begin{array}{ccc}
C_{*+k}(M_1 \times M_2 \times D^k) & \xrightarrow{[1 \times \tau_e \cap]} & C_*(M_1 \times M_1) \\
\downarrow{[\tau_3 \cap]} & & \downarrow{[\tau_{M_1} \cap]} \\
C_{*-n}(M_1) & & \\
\end{array}
\]
commutes, where we use the notation \([\tau \cap] : C_*(X) \to C_*(Y)\) for maps that are compositions of the form
\[ [\tau \cap] : C_*(X) \to C_*(X, U^c) \xrightarrow{\tau} C_*(U) \to C_*(Y) \]
for \( U \) a tubular neighborhood of \( X \) inside \( Y \) with associated Thom class \( \tau \).

The embedding \((\text{id}, f, e) : M_1 \to M_1 \times M_2 \times D^k\) is also equal to the composition of embeddings
\[
(1.3) \quad M_1 \xleftarrow{(\text{id}, f)} M_1 \times M_2 \xrightarrow{(\text{id}, e \circ \pi)} M_1 \times M_2 \times D^k.
\]
for \((\text{id}, e \circ \pi) : M_1 \times M_2 \to M_1 \times M_2 \times D^k\) taking \((m, m')\) to \((m, m', e(m))\). Under these circumstances, there is an associated diagram of tubular embeddings
\[
\begin{array}{ccc}
M_1 \times M_2 & \xrightarrow{f^*TM_2} & M_1 \times M_2 \times \mathbb{R}^k \\
\downarrow{\tilde{\nu}_1} & & \downarrow{\tilde{\nu}_2} \\
M_1 & \xleftarrow{(\text{id}, f)} & M_1 \times M_2 \xrightarrow{(\text{id}, e \circ \pi)} M_1 \times M_2 \times D^k,
\end{array}
\]
where we define
\[ \tilde{\nu}_1(m, V) = (m, \exp_f(m) V) \quad \text{and} \quad \tilde{\nu}_2(m, m', x) = (m, m', e(m) + x) \]
after identifying \( \mathbb{R}^k \) with the vectors of length at most \( \varepsilon \), where we assume that \( \varepsilon \) is small enough for this to make sense. Note that \( \tilde{\nu}_1 \) has image
\[ U_{1,2} := \{(m, m') \in M_1 \times M_2 \mid |f(m) - m'| < \varepsilon\} = (f \times 1)^{-1}(U_2) \]
for \( U_2 \subset M_2 \times D^k \) as above.

**Lemma 1.2.** The map \( f \times 1 \) induces a map of pairs \( f \times 1 : (M_1 \times M_2, U_1^{1,2}) \to (M_2 \times M_2, U_2^{1,2}) \) and, if \( f \) is a degree 1 map, the class \((f \times 1)^* \tau_{M_2} \in C^n(M_1 \times M_2, U_1^{1,2})\) is a Thom class for the tubular neighborhood \((f^*TM_2, \tilde{\nu}_1)\).

**Proof.** The Thom class \( \tau \in C^n(M_1 \times M_2, U_1^{1,2}) \) is characterized by the property \( \tau \cap [M_1 \times M_2] = [M_1] \), which holds for \( \tau = (f \times 1)^* \tau_{M_2} \) by naturality of the cap product, under the assumption that \( \deg(f) = 1 \).

For the tubular neighborhood \((M_1 \times M_2, \mathbb{R}^k, \tilde{\nu}_2)\), we choose the class
\[ (1 \times \eta) \in C^k(M_1 \times M_2 \times D^k, \tilde{\nu}_2(M_1 \times M_2 \times \mathbb{R}^k)^c) \]
as Thom class, for \( \eta \in C^k(D^k, \partial D^k) \) a chosen representative of the fundamental class. Note that this is indeed a Thom class as the embedding \((\text{id}, e \circ \pi)\) is isotopic to the embedding \((\text{id}, 0 \circ \pi)\), replacing \( e \) by the constant map at \( 0 \in D^k \), and hence their Thom class identify under the map induced by the canonical inclusion
\[ j : (M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) \xrightarrow{\cong} (M_1 \times M_2 \times D^k, \tilde{\nu}_2(M_1 \times M_2 \times \mathbb{R}^k)^c). \]

Applying Proposition A.1 to this new factorization of the embedding \((\text{id}, f, e)\), we get...
Proposition 1.3. Consider the sequence of embeddings $M_1 \overset{(id,f)}{\hookrightarrow} M_1 \times M_2 \overset{(id,e)\circ \pi}{\hookrightarrow} M_1 \times M_2 \times D^k$ and let $\tilde{v}_1, \tilde{v}_2$ and $\tilde{\tau}_1 = (f \times 1)^*\tau_{M_2}, \tilde{\tau}_2 = 1 \times \eta$ be the embeddings and associated Thom classes chosen above. Assume $\deg(f) = 1$. Then the following hold:

1. The bundle $\tilde{\mathcal{N}}_3 := f^*TM_2 \oplus \mathbb{R}^k$ is isomorphic to the normal bundle of the composed embedding $(id, f, e) : M_1 \hookrightarrow M_1 \times M_2 \times D^k$.
2. There is an isomorphism $\tilde{\nu}_1 : \tilde{\mathcal{N}}_3 \cong \tilde{\nu}^{-1}_2(U_{1,2}) \subset M_1 \times M_2 \times \mathbb{R}^k$ as bundles over $f^*TM_2 \cong U_{1,2}$, with the property that
   \[ \tilde{\nu}_2 := \tilde{\nu}_2 \circ \tilde{\nu}_1 : \tilde{\mathcal{N}}_3 \longrightarrow M_1 \times M_2 \times D^k \]
   is a tubular embedding for $(id, f, e)$.
3. The class
   \[ \tilde{\tau}_3 := \tilde{\nu}_2 \circ \tilde{\nu}_1 \circ \tilde{\tau}_3 : \tilde{\mathcal{N}}_3 \longrightarrow M_1 \times M_2 \times D^k \]
   is a Thom class for the tubular embedding $\tilde{\nu}_3$ and the diagram
   \[ C_+(M_1 \times M_2 \times D^k) \longrightarrow C_+(M_1 \times M_2) \]
   commutes, where we again use the notation $[\tau \cap]$ for maps that are compositions of the form
   \[ [\tau \cap] : H_*(X) \to H_*(X, U^c) \to H_*(U) \to Y \]
   for $U$ a tubular neighborhood of $X$ inside $Y$ with associated Thom class $\tau$.

We have given two explicit tubular neighborhoods $(N_3, \nu_3)$ and $(\tilde{N}_3, \tilde{\nu}_3)$ of the embedding $(id, f, e)$, coming from two different ways of considering it as a composition of embeddings. The bundles $N_3$ and $\tilde{N}_3$ are isomorphic, as they are both isomorphic to the normal bundle of the embedding. More than that, the uniqueness theorem for tubular neighborhoods implies that the embeddings are necessarily isotopic, giving the following result:

Proposition 1.4. Let $(N_3, \nu_3), \tau_3$ and $(\tilde{N}_3, \tilde{\nu}_3), \tilde{\tau}_3$ be the tubular neighborhoods and Thom classes for the embedding $(id, f, e)$ obtained in Propositions 1.1 and 1.3, and pick an isomorphism $\phi : \tilde{N}_3 \to N_3$. There exists a diffeomorphism $h : M_1 \times M_2 \times D^k \longrightarrow M_1 \times M_2 \times D^k$ such that

1. $h$ restricts to the identity on $M_1 \overset{(id,f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$ and $M_1 \times M_2 \times \partial D^k$,
2. $h$ is isotopic to the identity relative to $M_1 \overset{(id,f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$ and to $M_1 \times M_2 \times \partial D^k$,
3. $h \circ \tilde{\nu}_3 = \nu_3 \circ \phi : \tilde{N}_3 \longrightarrow M_1 \times M_2 \times D^k$, and thus $h(\tilde{\nu}_3\tilde{N}_3) = \nu_3 N_3$ and $h((\tilde{\nu}_3\tilde{N}_3)^c) = (\nu_3 N_3)^c$.
4. $[h^*\tau_3] = [\tilde{\tau}_3] \in H^{n+k}(M_1 \times M_2 \times D^k, \tilde{\nu}_3(\tilde{N}_3)^c)$.

Proof. Uniqueness of tubular neighborhoods (see eg. [16, Chap 4, Thm 5.3]) gives that the tubular embeddings $\tilde{\nu}_3$ and $\nu_3 \circ \phi$ are isotopic relative to the embedding $M_1 \overset{(id,f,e)}{\hookrightarrow} M_1 \times M_2 \times D^k$. Let $H : \tilde{N}_3 \times I \longrightarrow M_1 \times M_2 \times D^k$ be such an isotopy with $H(\cdot, 0) = \tilde{\nu}_3$ and $H(\cdot, 1) = \nu_3 \circ \phi$. We will also denote by $H$ its restriction to a closed disc subbundle $D\tilde{N}_3 \subset \tilde{N}_3$, and consider it as an isotopy of embeddings of $\tilde{\nu}_3(D\tilde{N}_3)$ inside $M_1 \times M_2 \times D^k$. By the isotopy extension theorem (see eg. [16, Chap 8, Thm 1.3]), there exists an isotopy of diffeomorphisms
   \[ \tilde{H} : (M_1 \times M_2 \times D^k) \times I \longrightarrow M_1 \times M_2 \times D^k \]
such that $\tilde{H}(-, 0)$ is the identity on $M_1 \times M_2 \times D^k$ and such that $\tilde{H}$ restricts to $H$ on $\tilde{\nu}_3(D\tilde{N}_3) \times I$. As $\tilde{H}$ is only non-trivial in a neighborhood of the image of $H$, we can assume that it fixes $M_1 \times M_2 \times \partial D^k$. Now by construction, $h := \tilde{H}(-, 1)$ is a diffeomorphism of $M_1 \times M_2 \times D^k$ satisfying (1)–(3) in the statement. Because $h$ is a diffeomorphism compatible with the tubular embeddings, it pulls back the Thom class to a Thom class. Unicity of the Thom class in cohomology gives the last part of the statement. \qed
Consider now the diagram

\[ \begin{array}{cccccc}
H_*(M_1 \times M_1) & \xrightarrow{[\tau_{M_1}]} & H_*(M_1) \\
\downarrow{1 \times f_*} & & \uparrow{f_*} \\
H_*(M_1 \times M_2) & \xrightarrow{[\tau_{M_2}]} & H_*(M_2)
\end{array} \]

\[ \begin{array}{cccc}
H_{n+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) & \xrightarrow{\tau \cap \eta \cap} & H_* \to H_{n+k}(M_1) \\
\text{commutes} & & \text{commutes} \\
\text{by naturality of the cap product. So what} & & \\
\tau & \text{for} & d \text{a degree} \\
\end{array} \]

Invariance of the intersection product will follow from the commutativity of this diagram once we have also shown that the maps \([1 \times \tau_\cap] \) and \([1 \times \eta \cap] \) are isomorphisms. This last fact is a consequence of the homology suspension theorem for \([1 \times \eta \cap] \), and, for \([1 \times \tau_\cap] \), will follow from the commutativity of (1) in the diagram and the fact that \(f \) is a homology isomorphism. \(^2\)

We analyse the diagram step by step. The top triangle commutes by Proposition 1.1, and the middle triangle by Proposition 1.3. The bottom square commutes by the naturality of the cap product. So what we need to show is that the triangles labeled (1) and (2) commute. To prove commutativity (1), we use the following lemma:

**Lemma 1.5.** Let \(i : (M_2 \times D^k, M_2 \times \partial D^k) \to (M_2 \times D^k, \nu_e(N(f,e))^c) \) be the inclusion of pairs. If \(f \) is a degree \(d \) map, then

\[ i^*[\tau_e] = \deg(f)[1 \times \eta] \in H^k(M_2 \times D^k, M_2 \times \partial D^k) \]

for \(\tau_e \) the Thom class of \((N(f,e), \nu_e)\) and \(\eta \in C^k(D^k, \partial D^k)\) a generator as above.

**Proof.** Consider the diagram

\[ \begin{array}{cccccc}
H^*(N(f,e), \partial N(f,e)) & \xrightarrow{\nu_e^*} & H^*(M_2 \times D^k, \nu_e(N(f,e))^c) & \xrightarrow{i^*} & H^*(M_2 \times D^k, M_2 \times \partial D^k) \\
\cap [N(f,e)] & \cong & \cap [M_2 \times D^k] & \cong & \cap [M_2 \times D^k] \\
H_{n+k-*}(N(f,e)) & \xrightarrow{\nu_e \cap} & H_{n+k-*}(M_2 \times D^k) & \xrightarrow{i_* = \text{id}} & H_{n+k-*}(M_2 \times D^k)
\end{array} \]

where the vertical maps cap with the fundamental class in each case and \(\nu_e^* \) is an isomorphism by excision. The two squares commute by naturality of the cap product, given that \([M_2 \times D^k] = i_*[M_2 \times D^k] = \nu_e[N(f,e)] \in H^*(M_2 \times D^k, N(f,e))^c\). Hence in homology we have a commuting diagram

\[ \begin{array}{cccccc}
[M_1] & \xrightarrow{H^k(M_2 \times D^k, \nu_e(N(f,e))^c)} & \xrightarrow{i^*} & H^k(M_2 \times D^k, M_2 \times \partial D^k) \\
\cong & \cap [N(f,e)] & \cong & \cap [M_2 \times D^k] \\
H_n(N(f,e)) & \xrightarrow{\nu_e \cap} & H_n(M_2 \times D^k) \\
\cong & \cong & \cong
\end{array} \]

where the first square commutes by the commutativity of the previous diagram, and the bottom square commutes as it comes from a commuting square in spaces. Now by definition of the Thom classes, \([\tau_e] \) is taken to \([M_1] \) along the left vertical maps, and \([1 \times \eta] \) to \([M_2] \) along the right vertical maps as \(1 \times \eta \) is a Thom class for the trivial \(D^k\)-bundle on \(M_2\). On the other hand \(f \) takes \([M_1] \) to \(\deg(f)[M_2] \) along the bottom horizontal map, from which the result follows.

\[ \text{Note that the map } [1 \times \tau_\cap] \text{ in the diagram is not directly a case of a Thom isomorphism (because the source of the map is not correct), and thus we cannot deduce that it is an isomorphism from the Thom isomorphism theorem.} \]
Proposition 1.6. Assume that $f$ is a degree 1 map. The sub-diagram labeled (1) in diagram (1.4) commutes.

Proof. Spelling out the maps, the diagram is the following:

$$
\begin{array}{ccc}
H_*(M_1 \times M_1) & \xrightarrow{\tau_{M_1} \cap} & H_*(M_2) \\
\downarrow 1 \times f_* & & \downarrow \nu_2 * \\
H_*(M_1 \times M_2) & \xrightarrow{\tau_{M_2} \cap} & H_*(M_1 \times M_2 \times D^k, \partial N_2)^c\\
\end{array}
\xrightarrow{1 \times \tau_c \cap} \begin{array}{ccc}
H_{+k}(N_2, \partial N_2) & \xrightarrow{\tau_{M_1} \cap} & H_{+k}(M_1 \times M_2 \times D^k, \nu_2(N_2)^c) \\
\downarrow 1 \times \tau_c \cap & & \downarrow 1 \times \iota_* \\
H_{+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) & \xrightarrow{\tau_{M_2} \cap} & H_{+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k)
\end{array}
$$

The interior of the left square commutes because it comes from the a commuting diagram in spaces, and hence the exterior also commutes as the two maps on the top are inverse to each other. By Lemma 1.5, we have that $i^*[1 \times \tau_c] = [1 \times \eta]$. Hence the bottom right triangle commutes by naturality of the cap product, and the top triangle commutes by definition of the maps. This proves the claim.

Proposition 1.7. Let $f : M_1 \to M_2$ be a homology isomorphism between closed manifolds. Then $f$ induces a ring map $H_*(M_1) \to H_*(M_2)$, with ring structure given by the intersection product, up to the sign $\deg(f)$, that is the diagram

$$
\begin{array}{ccc}
H_*(M_1 \times M_1) & \xrightarrow{\tau_{M_1} \cap} & H_{*-n}(M_1) \\
\downarrow f_* \times f_* & & \downarrow f_* \\
H_*(M_2 \times M_2) & \xrightarrow{\tau_{M_2} \cap} & H_{*-n}(M_2)
\end{array}
$$

commutes up to the sign $\deg(f) = \pm 1$.

We repeat here that the is not the optimal result for invariance of the intersection product, and absolutely not the optimal proof!

Proof. Given that $f$ is a homotopy equivalence, it has degree $\pm 1$. If $\deg(f) = -1$, changing the orientation of $M_2$, so that the degree of $f$ becomes 1, changes the intersection product of $M_2$ by a sign $(-1)$. Hence it is enough the consider the case where $\deg(f) = 1$.

We need to show that Diagram (1.4) commutes, and that the two maps $[1 \times \tau_c \cap]$ and $[1 \times \eta \cap]$ are isomorphisms. We have already seen that the top and middle triangles commute by Propositions 1.1 and 1.3, and the bottom square by the naturality of the cap product. The commutativity of the triangle labeled (1) is given by Proposition 1.6. For the triangle labeled (2), commutativity follows from Proposition 1.4. Indeed, the map $h$ in the proposition gives a commutative diagram

$$
\begin{array}{ccc}
H_{+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) & \xrightarrow{\tau_{M_2} \cap} & H_{+k}(M_1 \times M_2 \times D^k, N_3) \\
\downarrow h & & \downarrow h \\
H_{+k}(M_1 \times M_2 \times D^k, \tilde{N}_3) & \xrightarrow{\tau_{M_2} \cap} & H_{+k}(\tilde{N}_3, H_{+k}(M_1 \times M_2 \times D^k, \tilde{N}_3)
\end{array}
$$

because $h$ is isotopic to the identity (for the first triangle), pulls back $\tau_3$ to $\tilde{\tau}_3$ (for the middle square), and fixes $M_1$ (for the last square). Hence Diagram (1.4) commutes. Now the map $[1 \times \eta \cap]$ is defined by the right side of the diagram

$$
\begin{array}{ccc}
H_{+k}(M_1 \times M_2 \times D^k, M_1 \times M_2 \times \partial D^k) & \xrightarrow{1 \times \eta \cap} & H_{+k}(M_1 \times M_2 \times D^k, \nu_2(M_1 \times M_2 \times \mathbb{R}^k)^c) \\
\downarrow 1 \times \eta \cap & & \downarrow 1 \times \eta \cap \\
H_*(M_1 \times M_2 \times D^k) & \xrightarrow{\times \eta \cap} & H_*(M_1 \times M_2 \times D^k, \nu_2(M_1 \times M_2 \times \mathbb{R}^k)
\end{array}
$$

This diagram commutes because the embeddings $(\text{id}, e \circ \pi)$ and $(\text{id}, 0 \circ \pi)$ are isotopic. And thus $[1 \times \eta \cap]$ is an isomorphism by the left part of the diagram using the homology suspension theorem. It follows that $[1 \times \tau_c \cap]$ is then also necessarily an isomorphism by the commutativity of the triangle labeled (1) and the fact that $f$ induces an isomorphism in homology, which completes the proof. \qed
2. Homotopy invariance of the string coproduct: plan of proof

We start by recalling the definition of the string coproduct, as given in [15]. Let \( \Lambda_i := \Lambda M_i \) denote the free loop space of \( M_i \), for \( i = 1, 2 \). There is an evaluation map

\[ e_I : \Lambda_i \times I \to M_i \times M_i \]

defined by \( e_I(\gamma, s) = (\gamma(0), \gamma(s)) \). Recall from the previous section the neighborhoods \( U_1 \subset M_1 \times M_1 \) and \( U_2 \subset M_2 \times M_2 \) of the diagonals. For \( i = 1, 2 \), let

\[ U_{GH,i}^- = e_I^{-1}(U_i) = \{(\gamma, s) \in \Lambda_i \times I \mid |\gamma(0) - \gamma(s)| < \varepsilon \} \subset \Lambda_i \times I. \]

Then \( U_{GH,i}^- \) is a neighborhood of the self-intersecting loops

\[ F_i = \{ (\gamma, s) \in \Lambda_i \times I \mid \gamma(0) = \gamma(s) \} = e_I^{-1}(\Delta(M_i)). \]

Pulling back along the evaluation map \( e_I \), we get classes

\[ e_I^*\tau M_i \in C^n(\Lambda_i \times I, (U_{GH,i}^-)^c) \xleftarrow{e_I} C^n(M_i \times M_i, U_i^c) \ni \tau M_i \]

Let

\[ B_i = (M_i \times I) \cup (\Lambda_i \times \partial I) \subset F_i \subset \Lambda_i \times I. \]

The relative coproduct for \( \Lambda_i \) is the composition

\[ \vee : H_*(\Lambda_i, M_i) \xrightarrow{\times I} H_{*-1}(\Lambda_i \times I, B_i) \to H_{*-1}(\Lambda_i \times I, B_i \cup (U_{GH,i}^-)^c) \cong H_{*-1}(U_{GH,i}^- \cup \Lambda_i \times I) \]

\[ \xrightarrow{e_I^*\tau M_i, \cap} H_{*+1-n}(U_{GH,i}^- \cup \Lambda_i \times I) \xrightarrow{R_i} H_{*+1-n}(F_i, B_i) \xrightarrow{\text{cut}} H_{*+1-n}(\Lambda_i \times \Lambda_i, M_i \times \Lambda_i \cup \Lambda_i \times M_i) \]

where the map \( R_i \) is a retraction map defined by adding a geodesic stick in \( M_i \), see [15, Sec 1.5], and the cut map cuts at time \( s \).

Consider the maps \( M_i \xrightarrow{i} \Lambda_i \xrightarrow{e} M_i \) including \( M_i \) as the constant loops and evaluating at 0. They define a canonical splitting \( H_*(\Lambda_i) \cong H_*(M_i) \oplus H_*(\Lambda_i, M_i) \), and similarly for \( H_*(\Lambda \times \Lambda) \). The lifted coproduct

\[ \tilde{\vee} : H_*(\Lambda_i) \to H_{*-1-n}(\Lambda_i \times \Lambda_i) \]

is the extension by 0 on \( H_*(M) \) of the relative coproduct \( \vee \); see [15, Thm 3.3]. The following result show that it is equivalent to prove homotopy invariance for \( \vee \) or \( \tilde{\vee} \), and we will in the present paper work with the relative version \( \tilde{\vee} \).

**Lemma 2.1.** Let \( f : M_1 \to M_2 \) be a continuous map. Then the map \( \Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2) \) respects the lifted coproduct \( \tilde{\vee} \) in the sense of Theorem A, i.e.

\[ \tilde{\vee} \circ \Lambda f = (\Lambda f \times \Lambda f) \circ \tilde{\vee} : H_*(\Lambda_1) \to H_*(\Lambda_2 \times \Lambda_2) \]

if and only if the map \( \Lambda f_* : H_*(\Lambda_1, M_1) \to H_*(\Lambda_2, M_2) \) respects the unlifted coproduct

\[ H_*(\Lambda_1, M_1) \xrightarrow{\vee} H_*(\Lambda_1 \times \Lambda_1, M_i \times \Lambda_i \cup \Lambda_i \times M_i), \]

i.e.

\[ \vee \circ \Lambda f = (\Lambda f \times \Lambda f) \circ \vee : H_*(\Lambda_1, M_1) \to H_*(\Lambda_2 \times \Lambda_2, M_2 \times \Lambda_2 \cup \Lambda_2 \times M_2). \]

**Proof.** The result follows from the fact that a continuous map \( f : M_1 \to M_2 \) induces a map \( \Lambda f_* : H_*(\Lambda_1) \to H_*(\Lambda_2) \) that respects the splitting \( H_*(\Lambda_i) \cong H_*(M_i) \oplus H_*(\Lambda_i, M_i) \) and the corresponding splitting of \( H_*(\Lambda_i \times \Lambda_i) \) (see [15, Lem 3.2]) as it commutes with both the inclusion of the constant loops \( i \) and the evaluation \( e \). \( \square \)

Following ideas of Gruher-Salvatore [12] who gave a proof of homotopy invariance for the Chas-Sullivan product, we will use intermediate spaces:

\[ W_{1,2} = \{(m, m', \gamma, s) \in M_1^2 \times M_2 \times I \mid f(m) = \gamma(0), f(m') = \gamma(s)\} \]

\[ \Lambda_{1,2} = \{(m, \gamma) \in M_1 \times M_2 \mid f(m) = \gamma(0)\} \]
together with the following “evaluation” maps
\[ e_1: W_{11,2} \rightarrow M_1 \times M_1 \]
\[ e_1: \Lambda_{1,2} \times I \rightarrow M_1 \times M_2 \]
\[ \tilde{e}_1: \Lambda_{1,2} \times I \times D^k \rightarrow M_1 \times M_2 \times D^k \]
defined by setting \( e_1(m, m', \gamma, s) = (m, m') \) and \( e_1(m, \gamma, s) = (m, \gamma(s)) \) respectively, with \( \tilde{e}_1 = e_1 \times D^k \).
The spaces \( W_{11,2} \) and \( \Lambda_{1,2} \) will play the roles of \( M_1 \times M_1 \) and \( M_1 \times M_2 \) in the proof of homotopy invariance of the intersection product given in the previous section.

The map \( \Lambda f \) factors as a composition
\[ \Lambda_1 \xrightarrow{f'} \Lambda_{1,2} \xrightarrow{f''} \Lambda_2, \]
with \( f' \) and \( f'' \) defined by setting \( f'(\gamma) = (\gamma(0), f \circ \gamma) \) and \( f''(m, \gamma) = \gamma. \) Also \( f' \times I \) factors as
\[ \Lambda_1 \times I \xrightarrow{f'_I} W_{11,2} \xrightarrow{f'_I} \Lambda_{1,2} \times I, \]
with \( f'_I(\gamma, s) = (\gamma(0), \gamma(s), f \circ \gamma, s) \) and \( f'_I(m, m', \gamma, s) = (m, \gamma(s), s) \). Let
\[ \mathcal{F}_{1,2} := \left\{ (m, \gamma, s) \in M_1 \times \Lambda_2 \times I \mid f(m) = \gamma(0) = \gamma(s) \right\} \subset \Lambda_{1,2} \times I \]
be the subspace of self-intersecting loops. Note that the maps \( f' \times I \) and \( f'' \times I \) restrict to maps
\[ \mathcal{F}_1 \xrightarrow{f' \times I} \mathcal{F}_{1,2} \xrightarrow{f'' \times I} \mathcal{F}_2. \]

We will prove homotopy invariance using the relative coproduct. We will use the following subspaces:
\[ B_{1,1,2} = \left\{ (m, m, \gamma, s) \in W_{11,2} \mid \gamma = [f(m)] \text{ or } s \in \{0, 1\} \right\} \]
\[ B_{1,2} = \left\{ (m, \gamma, s) \in \Lambda_{1,2} \times I \mid \gamma = [f(m)] \text{ or } s \in \{0, 1\} \right\} \]
\[ B_{1,2} = \left\{ (m, \gamma, s, e(m)) \in \Lambda_{1,2} \times I \times D^k \mid \gamma = [f(m)] \text{ or } s \in \{0, 1\} \right\} \]
Note that they are all homeomorphic to \( B_{1,2} = M_1 \times I \cup \Lambda_{1,2} \times \partial I \).

The spaces and maps introduced fit into a commutative diagram of maps and embeddings

Here \( N_2 = M_1 \times \nu(f, e) \), \( N_3 = TM_1 \oplus \nu(f, e) \) and \( \tilde{N}_3 = f^*TM_2 \oplus \mathbb{R}^k \) are the normal bundles of Propositions 1.1 and 1.3, that embed inside \( M_1 \times M_2 \times D^k \) via the maps \( \nu_2, \nu_3 \) and \( \tilde{\nu}_3 \) respectively. We write
\[ X_2 := \tilde{e}_1^{-1}(\nu_2 N_2), \quad X_3 := \tilde{e}_1^{-1}(\nu_3 N_3), \quad \tilde{X}_3 := \tilde{e}_1^{-1}(\tilde{\nu}_3 \tilde{N}_3) \text{ in } \Lambda_{1,2} \times I \times D^k \]
for their inverse images along the map \( \tilde{e}_1 \). The sets \( U_1, U_2 \) are the neighborhoods of the diagonal defined above, with inverse images \( U_{GH}^1 = e_1^{-1}(U_1) \) and \( U_{GH}^2 = e_1^{-1}(U_2) \) in \( \Lambda_1 \times I \) and \( \Lambda_2 \times I \) respectively.
Our lift of Diagram (1.4), which we will analyse piece by piece below, is the following:

(2.1)

\[
\begin{array}{c}
\Lambda_1 \times I \\
\downarrow f' \downarrow \\
W_{11,2} \\
\downarrow (f', e) \circ f' \downarrow \\
X_2 \\
\downarrow j \downarrow \\

\end{array}
\]

where the terms in lines 1, 2, 4 and 5 are relative to the appropriate subspaces \( B_1, B_{11,2}, \bar{B}_{1,2} \) and \( B_2 \), while the term in the middle of the square is relative to \( \Lambda_{1,2} \times I \times D^k \cup \bar{B}_{1,2} \). (The relative parts will be made explicit below when we study each subdiagram.)

To obtain homotopy invariance, we will pre-compose this diagram with crossing with the interval, and post-compose with the cut maps. Before doing that, we need to check that

1. The maps in Diagram 2.1 are well-defined.
2. Each subdiagram in Diagram 2.1 commutes.
3. The maps \( p, [\bar{\epsilon}]_1(1 \times \tau_\alpha \cap) \) and \( [\bar{\epsilon}]_1(1 \times \eta_\alpha \cap) \) in Diagram 2.1 are isomorphisms.

3. Analysis of the subdiagrams in Diagram 2.1

In this section, we analyse each of the 12 subdiagrams in Diagram 2.1, showing that the maps are well-defined, that they commute, and that appropriate maps are invertible.

3.1. Diagram (1). The maps in Diagram (1) are

\[
\begin{array}{c}
f'_1: \Lambda_1 \times I \longrightarrow W_{11,2}, \quad f'_1(\gamma, s) = (\gamma(0), \gamma(s), f \circ \gamma, s). \\
(f'_2, e): W_{11,2} \longrightarrow X_2 \subset \Lambda_{1,2} \times I \times D^k, \quad (f'_2, e)(m, m', \gamma, s) = (m, \gamma(s), e(m')).
\end{array}
\]

Here we need to check that \((f'_2, e)\) indeed has image inside \( X_2 = \bar{\epsilon}^{-1}_1(\nu_2 N_2) \). This follows from the fact that

\[
\bar{\epsilon} \circ (f'_2, e)(m, m', \gamma, s) = (m, \gamma(s), e(m')) = (m, f(m'), e(m')) \in (id \times (f, e))M_1 \times M_1 \subset M_1 \times M_2 \times D^k
\]

in fact lies on the image of the zero section of \( N_2 = M_1 \times N(e, f) \).

We need to further check that the maps are maps of pairs

\[
\begin{array}{c}
f'_1: (\Lambda_1 \times I, B_1) \longrightarrow (W_{11,2}, B_{11,2}) \\
(f'_2, e): (W_{11,2}, B_{11,2}) \longrightarrow (X_2, \bar{B}_{1,2})
\end{array}
\]

which is immediate as constant loops are mapped to constant loops in both cases, the time parameter \( s \) is unchanged, and, if \((m, m', \gamma, s) \in B_{11,2}\), then we necessarily have \( m = m' \) so that the last coordinate of \((f'_2, e)(m, m', \gamma, s)\) is equal to \( e(m) \) in that case and lies in \( \bar{B}_{1,2} \). Hence the maps are well-defined.

Diagram (1) commutes by definition.
3.2. **Diagram (2).** Recall from Proposition 1.1 the maps

\[
\begin{array}{ccc}
N_2 \subset M_1 	imes M_1 \xleftarrow{p_2} N_2 = M_1 \times N(f,e)^c \xrightarrow{\nu_3} M_1 \times M_2 \times D^k.
\end{array}
\]

Proposition 1.1(2) gives that \(\hat{\nu}_1 N_1 = p_2^{-1}(U_1)\) as subsets of \(N_2\), and thus also \(\hat{\nu}_1(N_3)^c = p_2^{-1}(U_1^c) \subset N_2\).

We will use this to prove the following:

**Lemma 3.1.** Subdiagram (2) in Diagram (2.1)

\[
(3.1) \quad H_*(\Lambda_1 \times I, B_1) \xrightarrow{\epsilon_1^{I\tau_{M_1} \cap}} H_*(\Lambda_1 \times I, (U_{GH,1})^c \cup B_1) \xrightarrow{\epsilon_1^{I\tau_{M_1} \cap}} H_{*-n}(U_{GH,1}, B_1)
\]

is well-defined and commutes.

**Proof.** We have already seen above that the map \((f'_2, e) \circ f'_1\) takes \((\Lambda_1 \times I, B_1)\) to \((X_2, \bar{B}_{1,2})\). We need to check that it takes \(U_{GH,1}\) to \(X_3\) and also \((U_{GH,1})^c\) to \(X_3^c\). This follows directly from the above remarks, and the fact that the diagram

\[
\begin{array}{ccc}
U_{GH,1}, U_{GH,1}^c & \subset & \Lambda_1 \times I \xrightarrow{f'_1} W_{11,2} \xrightarrow{(f'_2, e)} \Lambda_{1,2} \times I \times D^k \supset X_3, X_3^c \\
& \xrightarrow{\epsilon_1} & \\
U_1, U_1^c & \subset & M_1 \times M_1 \xleftarrow{p_2} N_2 = M_1 \times N(f,e) \xrightarrow{p_2} M_1 \times M_2 \times D^k \supset \nu_3 N_3, \nu_3 N_3^c
\end{array}
\]

commutes, where the middle vertical map takes \((m, m', e, s)\) to \((m, m', e(m'))\) in the zero-section of \(N_2\).

Now the left square in Diagram (3.1) clearly commutes. For the right square, the above gives in particular a commuting diagram of pairs

\[
\begin{array}{ccc}
(A_1 \times I, U_{GH,1}^c) \xrightarrow{f'_1} (W_{11,2}, e^{-1}(U_1^c)) \xrightarrow{(f'_2, e)} (X_2, X_3^c) & \subset & \Lambda_{1,2} \times I \times D^k \\
& \xrightarrow{\epsilon_1} & \\
(A_1 \times I, M_1, U_1^c) \xleftarrow{p_2} (M_1 \times \nu_3 N_3^c) & \subset & M_1 \times M_2 \times D^k
\end{array}
\]

where we identified the homeomorphic pairs \((N_2, \hat{\nu}_1 N_3^c)\) and \((\nu_2 N_2, \nu_3 N_3^c)\). Hence the bottom right horizontal map in Diagram (3.1), capping with \(\epsilon_1^* \tau_{M_1}\), is well-defined. Moreover, the commutativity of the diagram gives that

\[
((f'_2, e) \circ f'_1)^* \epsilon_1^* \tau_{M_1} = \epsilon_1^* \tau_{M_1} \in C^n(\Lambda_1 \times I, U_{GH,1}^c),
\]

which, together with the naturality of the cap product, gives the commutativity of the right square in (3.1).

\[\Box\]

3.3. **Diagrams (3) and (4).** Recall that the map \(f'' : \Lambda_{1,2} \rightarrow \Lambda_2\) is the forgetful map, taking \((m, \gamma)\) to \(\gamma\). And \(p : \Lambda_{1,2} \times I \times D^k \rightarrow \Lambda_{1,2} \times I\) is the projection. Note that \(p\) takes \(\bar{B}_{1,2}\) to \(B_{1,2}\) and \(f'' \times I\) takes \(B_{1,2}\) to \(B_2\).
Diagram (3), which completely explicitly is the diagram

\[ H_*(\Lambda_1 \times I, B_1) \xleftarrow{\partial D^k} H_*(\Lambda_1 \times I \times D^k, B_1) \xrightarrow{\partial D^k} H_*(\Lambda_1 \times I, B_1) \]

commutes by definition of the maps.

For Diagram (4), we need to check that \((f'' \times I) \circ p : \Lambda_1 \times I \times D^k \to \Lambda_2 \times I \times D^k \) takes \(X_3\) to \(U^2\), and likewise for their complements. We do this as part of the following lemma.

**Lemma 3.3.** Subdiagram (4) in (2.1)

\[ H_*(\Lambda_1 \times I \times D^k, \bar{B}) \xrightarrow{(f'' \times I)_{OP}} H_*(\Lambda_2 \times I \times D^k, \bar{B}) \]

is well-defined and commutes.

**Proof.** Consider the diagram

\[ \begin{array}{ccc}
\hat{X}_3, \bar{X}_3 & \subset & \Lambda_1 \times I \times D^k \\
\downarrow \hat{\nu}_3, (\hat{\nu}_3, \bar{\nu}_3) & \subset & M_1 \times M_2 \times D^k \\
\hat{p}_2, \bar{p}_2 & \subset & U_1, U_2 \\
\end{array} \]

The top part of the diagram is easily seen to commute. For the bottom part, the left triangle commutes by definition of \(\hat{\nu}_3\) (see Proposition 1.3) and the right triangle as \(\hat{p}_2 : M_1 \times M_2 \times \mathbb{R}^k \to M_1 \times M_2\) is the projection and \(\hat{p}_2(m, m', x) = (m, m', e(m) + x)\) does not affect the first two coordinates. Hence \(\bar{\nu}_3 = e^{-1}(\bar{\nu}_3)\) is taken to \(U^2\) by the horizontal composition, and also their complements. Hence the vertical maps in Diagram (4) are well-defined.

Now the commutativity of the diagram also gives that the Thom classes are relative classes in the chain complexes \(C^*(\Lambda_1 \times I \times D^k, \bar{X}_3)\) and \(C^*(\Lambda_2 \times I, (U^2)^c)\) respectively, compatible under the map \((f'' \times I) \circ p\). Hence Diagram (4) commutes by naturality of the cap product. \(\square\)

**3.4. Diagram (5).** Let

\[ \begin{array}{c}
\hat{i} : (\Lambda_1 \times I \times D^k, (\Lambda_1 \times I \times \partial D^k) \cup \bar{B}_1) \to (\Lambda_1 \times I \times D^k, \bar{X}_3 \cup \bar{B}_1) \\
\hat{j} : (X_2, \bar{B}_1) \to (\Lambda_1 \times I \times D^k, \bar{B}_1) \\
\end{array} \]

denote the inclusions of pairs.

**Lemma 3.3.** Assume \(f\) is a degree 1 map. Then subdiagram (5) in Diagram (2.1)

\[ H_{*+k}(\Lambda_1 \times I \times D^k, (\Lambda_1 \times I \times \partial D^k) \cup \bar{B}_1) \xrightarrow{\partial D^k} H_{*+k}(\Lambda_1 \times I \times D^k, \bar{B}_1) \]

is well-defined and commutes.
Here, subdiagram (5) in Diagram (2.1) is the rightmost triangle of the above diagram; the leftmost triangle is there to make the diagonal map more explicit, that is the left most triangle commutes by definition, and the commutativity of the right triangle is equivalent to the commutativity of the square.

Proof. We have a commuting diagram
\[
\begin{array}{ccc}
\Lambda_{1,2} \times I \times D^k \times (\Lambda_{1,2} \times I \times D^k) \\
\downarrow \tilde{e}_t \downarrow \\
(\Lambda_{1,2} \times I \times D^k, X^\nu_0) \\
\end{array}
\]
for \(i : (M_2 \times D^k) \to (M_2 \times D^k, \nu_2(N_2)^c)\) the inclusion, inducing the map \(M_2 \times D^k\). From the commutativity of diagram, it follows that
\[
i^* \tilde{e}_t^* [1 \times \tau_n] = \tilde{e}_t^* [1 \times \eta] \in H^k(\Lambda_{1,2} \times I \times D^k, \Lambda_{1,2} \times I \times D^k).
\]

Now diagram (5), even more precisely, is the diagram
\[
\begin{array}{ccc}
H_{*+k}(\Lambda_{1,2} \times I \times D^k) \\
\downarrow i \\
H_{*+k}(\Lambda_{1,2} \times I \times D^k, X^\nu_0) \\
\downarrow j \\
H_*(X_2, \partial X_2 \cup \tilde{B}_{1,2}) \\
\end{array}
\]
where each subdiagram commutes by naturality of the cap product.

The map \(\tilde{e}_t^* [1 \times \eta] \cap\) is the homology suspension isomorphism. By the commutativity of the diagram, invertibility of the map \([\tilde{e}_t^* [1 \times \eta] \cap]\) is thus equivalent to the invertibility of the map \(j\). This will be shown below, when analysing diagram (9).

3.5. Diagrams (6) and (7).

Lemma 3.4. Subdiagrams (6) and (7) in Diagram (2.1)

\[
\begin{array}{ccc}
H_{*+n}(X_3, \tilde{B}_{1,2}) \\
\downarrow [\tilde{e}_t \tau_3(\cap)] \\
H_{*+k}(\Lambda_{1,2} \times I \times D^k, \tilde{B}_{1,2}) \\
\downarrow [\tilde{e}_t^* (\cap)] \\
H_*(X_2, \tilde{B}_{1,2}) \\
\end{array}
\]

are well-defined and commute.

Proof. Recall from Propositions 1.1 and 1.3 that the classes \(\tau_3 \in C^{n+k}(M_1 \times M_2 \times D^k, \nu_3(N_3)^c)\) and \(\tilde{\tau}_3 \in C^{n+k}(M_1 \times M_2 \times D^k, \nu_3(N_3)^c)\) are Thom classes for the bundles \(N_3\) and \(N_3\). The maps \([\tilde{e}_t^* \tau_3(\cap)]\) and \([\tilde{e}_t^* \tilde{\tau}_3(\cap)]\) are well-defined as \(X_3, X_3\) are by definition the inverse images of \(N_3, \tilde{N}_3\) under the map \(e_I\), and likewise for their complements.

The triangles are the lift to \(\Lambda_{1,2} \times I \times D^k\), along the map \(\tilde{e}_t\), of the triangles occurring in Propositions 1.1 (3) and 1.3 (3). Made completely explicit, these triangles take the form of Diagram (4.3), used to prove that they commute. The input in that proof is the relations \(\tau_3 := p_2^* \tau_{M_1} \cup (1 \times \tau_n) \in C^{n+k}(M_1 \times M_2 \times D^k, N_3^c)\) and \(\tilde{\tau}_3 := p_2^*(f \times 1)^* \tau_{M_2} \cup (1 \times \eta) \in C^{n+k}(M_1 \times M_2 \times D^k, \tilde{N}_3^c)\). Pulling back the classes along \(e_I\), we get the analogous relations in the case considered here:

\[
e^* \tau_3 := e^* \tau_{M_1} \cup e^* (1 \times \tau_n) \in C^{n+k}(\Lambda_{1,2} \times I \times D^k, e^* (N_3^c))
\]

and
\[
e^* \tilde{\tau}_3 := e^* (f \times 1)^* \tau_{M_2} \cup e^* (1 \times \eta) \in C^{n+k}(\Lambda_{1,2} \times I \times D^k, e^* \tilde{N}_3^c))
\]
which gives the commutativity of the diagrams by the same proof.

3.6. The map \( \hat{h} \) and Diagram (8). We start by defining the map \( \hat{h} \) of Diagram (8).

Recall from Proposition 1.4 the diffeomorphism \( h : M_1 \times M_2 \times D^k \rightarrow M_1 \times M_2 \times D^k \), homotopic to the identity, that identified the tubular neighborhoods \( N_3 \) and \( \bar{N}_3 \) fixing their 0-section \( M_1 \). We want to lift \( h \) to a map \( \hat{h} : \Lambda_{1,2} \times I \times D^k \rightarrow \Lambda_{1,2} \times I \times D^k \), likewise homotopic to the identity, and now identifying \( X_3 = \tilde{e}_t^{-1}(\nu_3 N_3) \) and \( \bar{X}_3 = \tilde{e}_t^{-1}(\nu_3 \bar{N}_3) \) and fixing \( F_{1,2} \):

\[
\begin{array}{ccc}
F_{1,2} & \xrightarrow{\tilde{e}} & \Lambda_{1,2} \times I \times D^k \\
\downarrow & & \downarrow \hat{h} \\
M_1 & \xrightarrow{id \times (f,e)} & M_1 \times M_2 \times D^k \\
\downarrow & & \downarrow \nu_3 \bar{N}_3 \\
\end{array}
\]

for \( \tilde{e} \) defined by \( \tilde{e}(m, \gamma, s) = (m, \gamma, s, e(m)) \).

**Proposition 3.5.** There is a continuous map

\[
\hat{h} : (\Lambda_{1,2} \times I \times D^k, \Lambda_{1,2} \times I \times \partial D^k \cup \tilde{B}_{1,2}) \rightarrow (\Lambda_{1,2} \times I \times D^k, \Lambda_{1,2} \times I \times \partial D^k \cup \tilde{B}_{1,2})
\]

such that

1. The evaluation map \( \tilde{e}_1 : \Lambda_{1,2} \times I \times D^k \rightarrow M_1 \times M_2 \times D^k \) intertwines the maps \( \hat{h} \) and \( h \), for \( h \) the diffeomorphism of Proposition 1.4(2). In particular, \( \hat{h}(X_3) \subset \bar{X}_3 \) and \( \hat{h}(X_3^\circ) \subset \bar{X}_3^\circ \);
2. \( \hat{h} \) fixes \( \tilde{e}(F_{1,2}) \subset \Lambda_{1,2} \times I \times D^k \);
3. \( \hat{h} \) is homotopic to the identity relative to \( \Lambda_{1,2} \times I \times \partial D^k \cup \tilde{e}(F_{1,2}) \).

Note that \( \tilde{B}_{1,2} \) is a subset of \( \tilde{e}(F_{1,2}) \).

**Proof.** From Proposition 1.4 we have that \( h : M_1 \times M_2 \times D^k \rightarrow M_1 \times M_2 \times D^k \) is isotopic to the identity through an isotopy \( \hat{H} \). Writing \( h = (h_1, h_2, h_3) \) for the components of \( h \), we define

\[
\hat{h}(m, \gamma, s, x) = (h_1(m, \gamma(s), x), \gamma', s', h_3(m, \gamma(s), x))
\]

with

\[
\gamma' = \delta_0^{-1} \ast \gamma[0,s] \ast \delta_1 \ast s' \ast \delta_1^{-1} \ast \gamma[s,1] \ast \delta_0
\]

where \( \ast \) denotes the optimal concatenation of paths and \( \ast s' \) the concatenation at time \( s' \), see [15, Sec 1.2]. Here \( \delta_0 \) is the path from \( f(m) = \gamma(0) \) to \( \gamma'(0) = f(h_1(m, \gamma(s), x)) \) defined using the first component of the isotopy \( \hat{H} = (\hat{H}_1, \hat{H}_2, \hat{H}_3) \):

\[
\delta_0(t) = f(\hat{H}_1(m, \gamma(s), x, t)),
\]

and \( \delta_1 \) is the path from \( \gamma(s) \) to \( \gamma'(s) = h_2(m, \gamma(s), x) \) obtained using the second component of \( \hat{H} \):

\[
\delta_1(t) = \hat{H}_2(m, \gamma(s), x, t).
\]

Recall that \( D^k \) is a small disc of radius \( \varepsilon \), so that \( |x - e(m)| < \frac{1}{2} \). The time \( s' \) is defined by

\[
s' = \begin{cases} 
\max(|x - e(m)|, s) & s \leq \frac{1}{2} \\
\min(1 - |x - e(m)|, s) & s \geq \frac{1}{2}.
\end{cases}
\]

We first check that \( \hat{h} \) is well-defined. The time \( s' \) is defined so that \( s' = 0 \) is only possible if \( s = 0 \) and \( x = e(m) \) and \( s' = 1 \) is only possible if \( s = 1 \) and \( x = e(m) \). In both cases, \( \delta_0 \) and \( \delta_1 \) are trivial paths as \( h \) and \( \hat{H} \) both fix points of the form \( (m, f(m), e(m)) \) by (1),(2) in Proposition 1.4. Hence the concatenation at time \( s' \) is well-defined. Also, we have constructed \( \gamma' \) so that \( (h_1(m, \gamma(s), x), \gamma') \in \Lambda_{1,2} \) as \( \gamma'(0) = f(h_1(m, \gamma(s), x)) \). So \( \hat{h} \) does have image inside \( \Lambda_{1,2} \times I \times D^k \). Because \( \hat{h} \) fixes \( M_1 \times M_2 \times \partial D^k \), we have that \( h_3(m, \gamma(s), x) = x \) whenever \( x \in \partial D^k \), and hence \( \hat{h} \) takes \( \Lambda_{1,2} \times I \times \partial D^k \) to itself. And because \( \hat{h} \) and \( \hat{H} \) fix the points \( (m, f(m), e(m)) \subset M_1 \times M_2 \times D^k \), and \( x = e(m) \) for such points, we have that \( \hat{h} \)
restricts to the identity on \( \hat{e}(F_{1,2}) \), namely the subspace of elements \((m, \gamma, s, e(m))\) with \( \gamma(s) = f(m) \). In particular, it takes \( \hat{B}_{1,2} \) to itself.

We now check the properties of \( \hat{h} \). The diagram
\[
\begin{array}{c}
\Lambda_{1,2} \times I \times D^k \xrightarrow{\hat{h}} \Lambda_{1,2} \times I \times D^k \\
\downarrow \varepsilon_i \downarrow \varepsilon_i \\
M_1 \times M_2 \times D^k \xrightarrow{\hat{h}} M_1 \times M_2 \times D^k
\end{array}
\]
commutes as \( \varepsilon_i \circ \hat{h}(m, \gamma, s, x) = (h_1(m, \gamma(s), x), \gamma'(s'), h_3(m, \gamma(s), x)) \) and \( \gamma'(s) = h_2(m, \gamma(s), x) \). As \( \hat{h} \) takes \( \nu_3 N_3 \) to \( \hat{\nu}_3 \hat{N}_3 \) (Proposition 1.4 (1)), it follows that \( \hat{h} \) takes \( X_3 \) to \( \hat{X}_3 \), and likewise for the complements.

Finally, the isotopy \( \hat{H} \) together with reparametrization induces a homotopy between \( \hat{h} \) and the identity, which retracts the added paths \( \delta_0 \) and \( \delta_1 \). This isotopy restricts to an isotopy on \( \Lambda_{1,2} \times I \times \partial D^k \) by reparametrization of the paths only, since \( \hat{H} \) restricts to the identity on \( M_1 \times M_2 \times \partial D^k \), and fixes \( \hat{e}(F_{1,2}) \) where \( \hat{h} \) is the identity already.

**Lemma 3.6.** Subdiagram (8) in Diagram (2.1)
\[
H_{s+k}(\Lambda_{1,2} \times I \times D^k, \Lambda_{1,2} \times I \times D^k \cup \hat{B}_{1,2}) \xrightarrow{\varepsilon_i \tau_n [\gamma]} H_{s-n}(X_3, \hat{B}_{1,2}) \xrightarrow{\hat{h}} H_{s-n}(\hat{X}_3, \hat{B}_{1,2})
\]
is well-defined and commutes.

**Proof.** This follows from (1),(3) in Proposition 3.5 together with (4) in Proposition 1.4 and the naturality of the cap product with respect to the map \( \hat{h} \approx \text{id} \). \( \square \)

**3.7. Diagram (9) and invertibility of \([\hat{\varepsilon}_j^* (1 \times \tau_e) \cap \gamma]\).** Recall that \( p : \Lambda_{1,2} \times I \times D^k \to \Lambda_{1,2} \times I \) is the projection onto the first two factors. Note that Diagram (9) commutes on the space level as the composition \( p \circ j \circ (f_2', e) \) is precisely the map \( f_2' \). Note also that \( p \) induces an isomorphism
\[
p : H_*(\Lambda_{1,2} \times I \times D^k, \hat{B}_{1,2}) \xrightarrow{\approx} H_*(\Lambda_{1,2} \times I, B_{1,2})
\]
as \( p : \Lambda_{1,2} \times I \times D^k \to \Lambda_{1,2} \times I \) is a homotopy equivalence taking \( \hat{B}_{1,2} \to B_{1,2} \). (It is actually a homotopy equivalence relative to the subspaces, as can be seen by choosing the map \( q : \Lambda_{1,2} \times I \to \Lambda_{1,2} \times I \times D^k \) define by \( q(m, \gamma, s) = (m, \gamma, s, e(m) \text{ as homotopy inverse.}) \)

To show that the map \( \hat{j} \) induces an isomorphism in homology, which in turn will give the invertibility of \([\hat{\varepsilon}_j^* (1 \times \tau_e) \cap \gamma]\), we show here that the remaining maps in Diagram (9) do induce isomorphisms in homology.

**Lemma 3.7.** The map \((f_2', e) : W_{11,2} \to X_2 \subset \Lambda_{1,2} \times I \times D^k\) is a homotopy equivalence relative to the subspaces \( B_{11,2} \) and \( \hat{B}_{1,2} \).

**Proof.** We define a map \( r : X_2 \to W_{11,2} \) as follows. Given a point \((m, \gamma, s, x) \in X_2 = \hat{e}_i^{-1}(N_2)\), we have that \((m, \gamma(s), x) \in N_2 = M_1 \times N(f, e)\), that is \((\gamma(s), x)\) is the image of an element \((m', W) \in N(f, e)\) under the embedding \( \nu_e : N(f, e) \to M_2 \times D^k \). Define a path \( \delta_W : [0, 1] \to M_2 \) by setting
\[
\delta_W(t) = \nu_e(m', tW)_{M_2},
\]
to be the first component of \( \nu_e(m', tW) \in M_2 \times D^k \). Then \( \delta_W(0) = f(m') \) and \( \delta_W(1) = \gamma(s) \). Recall that \( D^k \) is a disc of radius \( \varepsilon \). In particular, \(|x - e(m)| < \varepsilon \) is small. Define
\[
s' = \begin{cases} 
\max(|x - e(m)|, s) & s \leq \frac{1}{2} \\
\min(1 - |x - e(m)|, s) & s \geq \frac{1}{2} 
\end{cases}
\]
and
\[
\gamma' = \gamma[0, s] \ast \delta_W^{-1} \ast s', \delta_W \ast \gamma[s, 1]
\]
Finally, set \( r(m, \gamma, s, x) = (m, m', \gamma', s') \).

The path \( \gamma' \) is well-defined because \( s' = 0 \) or \( s' = 1 \) is only possible if \( x = e(m) \) and \( s = 0 \) or \( s = 1 \) accordingly, which itself implies that \( \gamma(s) = \gamma(0) = f(m) \), so that \( (\gamma(s), x) \) lies on the 0-section of \( N(f, e) \) and \( W = 0 \) is a trivial vector. It follows that \( \delta_W \) is a constant path in that case, with \( s' = s \) and \( \gamma' = \gamma \).

By construction, \( (m, m', \gamma', s') \in W_{1,1,2} \).

Note also that \( r \) maps \( B_{1,2} \) to \( B_{1,1,2} \). Indeed, if \((m, \gamma, s, x) = (m, \gamma, s, e(m)) \) with \( \gamma = [f(m)] \) or \( s = 0 \) or \( 1 \), then \( \gamma(s) = f(m) \) giving again that \((\gamma(s), x)\) lies on the 0-section of \( N(f, e) \), so that \( \delta_W \) is a constant path in that case, with \( \gamma' = \gamma \). Also \( x = e(m) \) so we also have \( s' = s \).

We have that \( r \circ (f_2', e) \simeq \text{id} \). Indeed, if \((m, m', \gamma, s) \in W_{1,1,2} \), then \((f_2', e)(m, m', \gamma, s) = (m, \gamma, s, e(m')) \) also has the property that \((\gamma(s), x) = (f(m'), e(m')) \) lies on the zero section, so \( \delta_W \) is a trivial path. Now setting \( s'(t) = ts + (1 - t)s' \) and \( \gamma'(t) = \gamma[0, s] \ast s'(t) \) \( \gamma[s, 1] \) defines the required homotopy (by reparametrization) from \((m, m', \gamma, s) \) to \((m, m', \gamma', s') \).

Finally, we consider the composition \((f_2', e) \circ r \). It takes a tuple \((m, \gamma, s, x) \) to \((m, \gamma', s', e(m')) \) for \( m' \), \( \gamma' \) and \( s' \) defined as above. We define a homotopy \( H \) to the identity by setting

\[
H(m, \gamma, s, x, t) = (m, \gamma'_t, ts + (1 - t)s', tx + (1 - t)e(m'))
\]

with

\[
\gamma_t = \gamma[0, s] \ast \delta_W[0, 1 - t]^{-1} \ast ts' + (1 - t)s \delta_W[0, 1 - t] \ast \gamma[s, 1].
\]

This homotopy preserves \( B_{1,2} \) at all times as \( \delta_W \) is a trivial path with \( m = m' \), \( s = s' \) and \( x = e(m) = e(m') \) in this case.

\[\Box\]

**Proposition 3.8.** Assume that \( f \) is a homotopy equivalence. Then the map \( f_2' : W_{1,1,2} \to \Lambda_{1,2} \times I \) is a homotopy equivalence, taking \( B_{1,1,2} \) homeomorphically onto \( B_{1,2} \). In particular, it induces an isomorphism \( H_*(W_{1,1,2}, B_{1,1,2}) \cong H_*(\Lambda_{1,2} \times I, B_{1,2}) \).

**Proof.** By assumption, \( f : M_1 \to M_2 \) is a homotopy equivalence. We pick a homotopy inverse \( g : M_2 \to M_1 \) and homotopies \( h_1 : M_1 \times I \to M_1 \) from \( g \circ f \) to the identity and \( h_2 : M_2 \times I \to M_2 \) from \( f \circ g \) to the identity such that

\[
f \circ h_1 \simeq_K h_2 \circ (f \times 1) : M_1 \times I \to M_2 \quad \text{relative to} \quad M_1 \times \partial I
\]

and likewise \( g \circ h_2 \simeq h_1(g \times 1) \) relative to \( M_2 \times \partial I \), which is possible by Vogt’s lemma [22]. Here

\[
K : M_1 \times I \times I \to M_2
\]

denotes the first of these two homotopies.

Define a map \( G : \Lambda_{1,2} \times I \to W_{1,1,2} \) by setting \( G(m, \gamma, s) = (g \circ f(m), g \circ \gamma(s), f \circ g \circ \gamma, s) \). As \( \gamma(0) = f(m) \), we have that \( f \circ g \circ \gamma(0) = f \circ g \circ f(m) \) and \( G \) does have image in \( W_{1,1,2} \).

We have that \( f_2' \circ G(m, \gamma, s) = (g \circ f(m), f \circ g \circ \gamma, s) \). We will use concatenation of the homotopy \( h_2(\gamma, -) \) from \( f \circ g \circ \gamma \) to \( \gamma \) and the homotopy \( K \) from \( f \circ h_1(m, -) \) to \( h_2(f(m), -) \) to define a homotopy \( L \) between \( f_2' \circ G \) and the identity. Define \( L : \Lambda_{1,2} \times I \times I \to \Lambda_{1,2} \times I \) by

\[
L(m, \gamma, s, t) = (h_1(m, t), K(m, t, -) \ast h_2(\gamma, t) \ast K(m, t, -)^{-1}, s)
\]

where \( K(m, t, -) : I \to M_2 \) is the path from \( h_1(m, t) \) to \( h_2(f(m), t) \) defined at time \( u \) by \( K(m, t, u) \).

(See Figure 1.) As \( h_2(\gamma, t) \) is a loop based at \( h_2(f(m), t) \), we see that the concatenation is well-defined. Also, the resulting loop is based at \( f(h_1(m, t)) \), as required. So \( L \) is well-defined. As \( K \) is a homotopy relative to \( M_1 \times \partial I \), we have that \( K(m, t, -) \) is a trivial path when \( t = 0 \) or 1, giving that \( L \) is a homotopy between \( f_2' \circ G \) and the identity.

The other way around, we have that \( G \circ f_2'(m, m', \gamma, s) = (g \circ f(m), g \circ f(m'), f \circ g \circ \gamma, s) \) as \( \gamma(s) = f(m') \). Similarly to the previous case, concatenating \( h_2(\gamma, -) \) with \( K \) from \( f \circ h_1(m, -) \) to \( h_2(f(m), -) \) as well as \( K \) from \( f \circ h_1(m, -) \) to \( h_2(f(m'), -) \) gives a homotopy between \( G \circ f_2' \) and the identity. Explicitly, define \( L' : W_{1,1,2} \times I \to W_{1,1,2} \) by

\[
L'(m, m', \gamma, s, t) = (h_1(m, t), h_1(m', t), \gamma'_t(t), s'(t))
\]

with

\[
\gamma'_t = K(m, t, -) \ast h_2(\gamma[0, s], t) \ast K(m', t, -)^{-1} \ast s', K(m', t, -) \ast h_2(\gamma[s, 1], t) \ast K(m, t, -)^{-1}
\]
and
\[ s'(t) = s + (\frac{1}{2} - s) \min(t, 1 - t) \varepsilon. \]

The time \( s' = s'(t) \) has the property that \( s'(t) = s \) if \( t = 0 \) or \( t = 1 \), allowing us to recover our original maps at time 0 and 1, while \( 0 < s'(t) < 1 \) when \( 0 < t < 1 \) so that the concatenation is well-defined, as \( K(m, t, -) \) and \( K(m', t, -) \) will in general be non-trivial paths when \( 0 < t < 1 \).

As above, \( \gamma'(t) \) is a loop based at \( f(h_1(m, t)) \), and \( \gamma'(t) \) now passes through \( f(h_1(m', t)) \) at time \( s' \), as required to have a well-defined element of \( W_{11.2} \). As \( \gamma(0) = f(m) \) and \( \gamma(s) = f(m') \), we see that the concatenations defining \( \gamma'(t) \) make sense: \( K(m, t, -) \) is a path from \( f(h_1(m, t)) \) to \( h_2(f(m), t) \) and \( K(m', t, id) \) from \( f(h_1(m', t)) \) to \( h_2(f(m'), t) \). Also, these two paths are trivial at time \( t = 0 \) and \( t = 1 \), giving that \( \gamma'(0) = f \circ g \circ \gamma \) while \( \gamma'(1) = \gamma \). Hence \( L' \) is a homotopy between \( G \circ f_2 \) and the identity.

Hence \( \tilde{f}_2 : W_{11.2} \to \Lambda_{1.2} \times I \) is a homotopy equivalence. Moreover it restricts to an homeomorphism \( B_{11.2} \to B_{1.2} \) (as \( m = m' \) for points of \( B_{11.2} \)). The result then follows from considering the long exact sequences of homology groups and using the 5-lemma. \( \square \)

**Corollary 3.9.** The map \( \tilde{j} : H_*(X_2, \tilde{B}_{1.2}) \to H_*(\Lambda_{1.2} \times I \times D^k, \tilde{B}_{1.2}) \) is an isomorphism.

**Proof.** This follows from Lemma 3.7 and Proposition 3.8 together with the commutativity of Diagram (9) and the fact that the map \( p \) is a homotopy equivalence. \( \square \)

### 3.8. The retractions: Diagrams (10), (11) and (12)

Recall from [15, Sec 1.5] that, for \( i = 1, 2 \), the retraction maps
\[ R_i : U_{GH,i} \to F_i \]

are defined by adding a geodesic stick to the loop so it self-intersects at time \( s \): \( R_i(\gamma, s) = (\gamma', s) \) with \( \gamma' = \gamma[0, s] \ast \gamma(s), \gamma(0) \ast s, \gamma(0) \ast \gamma(s) \ast \gamma[s, 1] \), with \( \overline{xy} \) denoting the geodesic from \( x \) to \( y \). Note that the concatenation at time \( s \) does indeed make sense even when \( s = 0 \) or 1.

We define a similar map
\[ R_{1,2} : X_3 \to F_{1,2} \]
by setting
\[ R_{1,2}(m, \gamma, s, x) = (m, \gamma', s') \quad \text{with} \quad \gamma' = \gamma[0, s] \ast \delta^{-1} \ast \delta \ast \gamma[s, 1] \]
for \( s' \) given, as earlier, by
\[
s' = \begin{cases} 
\max(|x - e(m)|, s) & s \leq \frac{1}{2} \\
\min(1 - |x - e(m)|, s) & s \geq \frac{1}{2}
\end{cases}
\]
and \( \delta \) a path from \( \gamma(0) \) to \( \gamma(s) \) defined as follows. By assumption, \((m, \gamma(s), x) \in \nu_3(N_3)\) and by Proposition 1.1(2), \( \nu_3(N_3) \cong \nu_2(p_2^{-1}(U_1)) \). Explicitly, this means that \((m, \gamma(s), x) = \nu_3(m, V, W) = \nu_2(m, m', W')\) for \( V \in T_mM_1 \) and \( W \in N(f, e)_m, m' = \exp_m(V) \) and \( W' \in N(f, e)_{m'} \) the parallel transport of \( W \) along \( V \). Now let \( \delta = \delta_1 \ast \delta_2 \) with \( \delta_1(t) = f(\exp_m(tV)) \) the image in \( M_2 \) of the geodesic from \( m \) to \( m' = \exp_m(V) \) in \( M_1 \), and \( \delta_2(t) = \nu_2(m, m', tW')_{M_2} \), the path from \( f(m') \) to \( \gamma(s) \) defined by \( \nu_2 \).

The path \( \gamma' \) is well-defined as \( s' = 0 \) can only happen when \( s = 0 \) and \( x = e(m) \), in which case \( \delta \) is trivial as \((m, \gamma(s), x) = (m, f(m), e(m))\) is on the zero-section of \( N_2 \), and likewise for \( s' = 1 \). By construction, \((m, \gamma', s') \) is in \( \mathcal{F}_{1,2} \). Likewise, if \((m, \gamma, s, x) \in \mathcal{B}_{1,2}\), we have that \((m, \gamma(s), x) = (m, f(m), e(m))\) is again on the zero-section of \( N_2 \), with \( s' = s \) and the path \( \delta \) trivial. Hence \( R_{1,2} \) is a relative map
\[
R_{1,2}: (X_3, \mathcal{B}_{1,2}) \rightarrow (\mathcal{F}_{1,2}, B_{1,2}).
\]

Finally, we define
\[
\bar{R}_{1,2}: \bar{X}_3 \rightarrow \mathcal{F}_{1,2}
\]
by setting
\[
\bar{R}_{1,2}(m, \gamma, s, x) = (m, \gamma', s) \quad \text{with} \quad \gamma' = \gamma[0, s] \ast \gamma(s), \gamma(0) \ast s \ast \gamma(s) \ast \gamma[s, 1],
\]
which makes sense as \((m, \gamma(s), x) \in \bar{N}_3 \cong p_2^{-1}(U_{1,2})\) by Proposition 1.3, so that \( |f(m) - \gamma(s)| < \varepsilon \). Note that this is well-defined also when \( s = 0 \) or 1. Note also that if \((m, \gamma, s, x) \in \mathcal{B}_{1,2}\), we have that \( \gamma(0) = \gamma(s) \) so that \( \bar{R}_{1,2} \) defines a relative map
\[
\bar{R}_{1,2}: (\bar{X}_3, \mathcal{B}_{1,2}) \rightarrow (\mathcal{F}_{1,2}, B_{1,2}).
\]

**Lemma 3.10.** The maps \( R_{1,2} \) and \( \bar{R}_{1,2} \) are homotopy inverses, relative to \( \mathcal{B}_{1,2} \) and \( B_{1,2} \), to the inclusions \( l: \mathcal{F}_{1,2} \hookrightarrow X_3 \) and \( \bar{l}: \mathcal{F}_{1,2} \hookrightarrow \bar{X}_3 \) defined by \( l(m, \gamma, s) = (m, \gamma, s, e(m)) = \bar{l}(m, \gamma, s) \).

**Proof.** The fact that \( R_{1,2} \circ l = id \) and \( \bar{R}_{1,2} \circ \bar{l} = id \) follows from the fact that for points in the image of \( \mathcal{F}_{1,2} \) in both cases, the paths \( \delta \) and \( \gamma(s), \gamma(0) \) added by \( R_{1,2} \) and \( \bar{R}_{1,2} \) respectively are trivial, with \( s' = s \) in the first case, by the same computations showing that \( R_{1,2} \) and \( \bar{R}_{1,2} \) map \( \mathcal{B}_{1,2} \) to \( B_{1,2} \).

The other way around, we need to check that \( l \circ R_{1,2} \simeq id \) and \( \bar{l} \circ \bar{R}_{1,2} \simeq id \). In the first case, a homotopy is given by first retracting \( \delta_2 \), then \( \delta_1 \), scaling along the vectors \( V \) and \( W' \), and finally reparametrizing the path back to its original parametrization. For \( \bar{R}_{1,2} \), the homotopy just retracts the added geodesic stick. As the retracted paths and geodesics were trivial for elements of \( \mathcal{B}_{1,2} \), with no reparametrization, we see that the homotopy is relative to that subspace in each case. \( \square \)

**Proposition 3.11.** Subdiagram (11) in Diagram (2.1) commutes.

**Proof.** By the lemma, inverting \( R_{1,2} \) and \( \bar{R}_{1,2} \), it is equivalent to show that the diagram
\[
\begin{array}{ccc}
H_*(X_3, \mathcal{B}_{1,2}) & \xleftarrow{l} & H_*(\mathcal{F}_{1,2}, B_{1,2}) \\
\downarrow \bar{h} & & \downarrow \bar{l} \\
H_*(\bar{X}_3, \mathcal{B}_{1,2}) & \xleftarrow{\bar{l}} & H_*(\mathcal{F}_{1,2}, B_{1,2})
\end{array}
\]
commutes, which holds on the space level as \( l \) and \( \bar{l} \) are the same map, and \( \bar{h} \) fixes \( \mathcal{F}_{1,2} \). \( \square \)
Proposition 3.12. Subdiagram (10) in Diagram (2.1)

\[
\begin{array}{c}
H_*(U_{GH,1}, B_1) \xrightarrow{R_1} H_*(F_1, B_1) \\
(f'_2, e) \circ f'_1 \downarrow \quad \downarrow f' \times t \\
H_*(X_3, \tilde{B}_{1,2}) \xrightarrow{R_{1,2}} H_*(F_{1,2}, B_{1,2})
\end{array}
\]

commutes.

Note that we have already checked in Lemma 3.1 that the left vertical map is well-defined, so we already know that the diagram is well-defined.

Proof. The diagram commutes on the space level up to a reparametrization of the paths. Indeed, for \((\gamma, s) \in U_{GH,1}\), the tuple \((f'_2, e) \circ f'_1)(\gamma, s) = (m, f \circ \gamma, s, e(m'))\), for \(m = \gamma(0)\) and \(m' = \gamma(s)\), has the property that \(\tilde{e}_I(m, f \circ \gamma, s, e(m')) = (m, f(m'), e(m'))\) lies in the zero-section of \(M_1 \times N(f, e)\). It follows that the vector \(W' \in N(e, f)_{m'}\) in the definition of \(R_{1,2}\), defined so that \((m, \exp_{m'}(W')) = (m, f(m'), e(m'))\), is trivial. Hence the path \(\delta_2\) in the definition of \(R_{1,2}\) is the constant path at \((m, f(m'), e(m'))\), and \(\delta = \delta_1 \ast \delta_2 = \delta_1 = f(\exp_m(V))\), for \(V \in TM_1\) the vector satisfying that \(\exp_m(V) = m'\). In particular, \(\delta\) is the image in \(M_2\) of the geodesic in \(M_1\) from \(\gamma(0)\) to \(\gamma(s)\), the same geodesic stick occurring in the definition of \(R_1\). So going along the bottom part of the diagram, \((\gamma, s)\) is taken to \((m, \gamma', s')\), with \(\gamma' = f \circ \gamma[0, s] \ast \delta^{-1} \ast s' \ast f \circ \gamma[s, 1]\), while along the top of the diagram it is taken to \((m, f \circ \tilde{\gamma}, s)\) for \(f \circ \tilde{\gamma} = \gamma[0, s] \ast \delta^{-1} \ast s' \ast \gamma[s, 1]\) with \(f \circ \tilde{\gamma} = \tilde{\gamma}\). Hence the path components \(\gamma'\) and \(f \circ \tilde{\gamma}\) in the two compositions differ only up to a reparametrization. The time components \(s\) and \(s'\) also need not agree, as \(s'\) need not be equal to \(s\) because the value of \(s'\) depends on the difference \(e(m) - x\), with \(x = e(m')\) not in general equal to \(e(m)\). The homotopy between the two compositions linearly interpolates between \(s'\) and \(s\), while reparametrizing the paths. This is well-defined as \((1-t)s' + ts\) only takes the value 0 or 1 at time \(t = 1\) if \(s = 0\) or 1. \(\Box\)

Recall that \(f'' : \Lambda_{1,2} \rightarrow \Lambda_2\) is the forgetful map defined by setting \(f''(m, \gamma) = \gamma\).

Proposition 3.13. Subdiagram (12) in Diagram (2.1)

\[
\begin{array}{c}
H_*(\tilde{X}_3, \tilde{B}_{1,2}) \xrightarrow{R_{1,2}} H_*(F_{1,2}, B_{1,2}) \\
(f'' \times I) \circ p \downarrow \quad \downarrow f'' \times t \\
H_*(U_{GH,2}, B_2) \xrightarrow{R_2} H_*(F_2, B_2)
\end{array}
\]

commutes.

Note that we have already seen that each map in the diagram is well-defined.

Proof. The diagram commutes on the space level: a point \((m, \gamma, s, x) \in \tilde{X}_3\) is taken by \(\tilde{R}_{1,2}\) to \((m, \gamma', s, x)\), with \(\gamma'\) obtained from \(\gamma\) by adding a geodesic stick from \(\gamma(0)\) to \(\gamma(s)\). The triple \((m, \gamma', s)\) is then taken to \((\gamma', s)\) by \(f'' \times I\). The other composition first forgets \(m\) and \(x\), taking \((m, \gamma, s, x)\) to \((\gamma, s)\), and then produces the same pair \((\gamma', s)\), which was constructed independently of \(m\) and \(x\). \(\Box\)

3.9. Proof of Theorem A. Suppose \(f : M_1 \rightarrow M_2\) is a homotopy equivalence between two closed Riemannian manifolds \(M_1\) and \(M_2\). Then \(f\) has degree \(\pm 1\). If \(f\) has degree \(-1\), change the orientation of \(M_2\) so that it becomes a degree 1 map. This change of orientation of \(M_2\) has the effect of changing the sign of the Thom class \(\tau_{M_2}\), and hence multiplies the old coproduct by \((-1)\). This reduces the question to the case of a degree 1 homotopy equivalence. From Lemma 2.1, for the commutativity of the two diagrams, it is enough to show that \(f\) respects the uplifted coproduct \(\vee\).

By Sections 3.1–3.8, we know that each subdiagram in Diagram (2.1) commutes. The maps labelled \(p\) and \(\tilde{e}_x'(1 \times \eta) \cap\) are isomorphisms. By Corollary 3.9, so is the map \(j\), which implies, by the commutativity of Subdiagram (5), that so is the map \([\tilde{e}_x'(1 \times \tau_x) \cap]\). Hence the maps going the wrong way in the diagram...
(marked with double arrows in Diagram (2.1)) are all invertible. The commutativity of each of the subdiagrams implies the commutativity of the outside square. Now consider the diagram

\[
\begin{array}{cccc}
H_*(\Lambda_1, M_1) & \times I & H_*(\Lambda_1 \times I, B_1) & \xrightarrow{R_1 \circ \kappa_1 \tau_{M_1}} \ H_*(\mathcal{F}_1, B_1) & \xrightarrow{\text{cut}} \ H_*(\Lambda_1 \times \Lambda_1, M_1 \times \Lambda_1 \cup \Lambda_1 \times M_1) \\
& \downarrow & & \downarrow & \\
H_*(\Lambda_2, M_2) & \times I & H_*(\Lambda_2 \times I, B_2) & \xrightarrow{R_2 \circ \kappa_2 \tau_{M_2}} \ H_*(\mathcal{F}_2, B_2) & \xrightarrow{\text{cut}} \ H_*(\Lambda_2 \times \Lambda_2, M_2 \times \Lambda_2 \cup \Lambda_2 \times M_2)
\end{array}
\]

The middle square commutes as it identifies with the outside of Diagram (2.1), and the left and right squares commute by the naturality of the cross and cut maps. Homotopy invariance of the coproduct follows since the top and bottom compositions are the definition of the coproduct of \(\Lambda_1\) and \(\Lambda_2\).

The statement in cohomology follows from the universal coefficient theorem as the coproducts are maps induced in homology by chain maps, and the cohomology products are the maps in cohomology induced by these same chain maps (see [15, Def 1.6]).

**Appendix A. Composing tubular embeddings**

Given a composition of embeddings \(M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3\) and choices of tubular neighborhoods for each embedding, we will construct here an associated tubular neighborhood for the composed map and study its properties.

**Proposition A.1.** Consider

\[
\begin{array}{ccc}
N_1 & \xrightarrow{\nu_1} & N_2 \\
p_1 & & p_2 \\
M_1 & \xrightarrow{f_1} & M_2 \\
\end{array}
\]

with \(f_1, f_2\) codimension \(k\) and \(l\) embeddings, \(N_1, N_2\) the associated normal bundles and \(\nu_1, \nu_2\) chosen tubular neighborhoods. Then

1. The bundle \(N_{12} := N_1 \oplus f_1^* N_2 \longrightarrow M_1\) is isomorphic to the normal bundle of \(f_2 \circ f_1\).

2. There is an isomorphism

\[
\begin{array}{ccc}
N_{12} & \xrightarrow{\nu_1} & N_2 \\
p & & p_2 \\
M_1 & \xrightarrow{f_1} & M_2 \\
\end{array}
\]

as bundles over \(N_1\), where \(p\) is the natural projection, with the property that

\[
\nu_{12} := \nu_2 \circ \hat{\nu}_1 : N_{12} \longrightarrow N_2|_{N_1} \longrightarrow M_3
\]

is a tubular neighborhood for \(f_2 \circ f_1 : M_1 \hookrightarrow M_3\).

3. Let \(\tau_1 \in C^k(M_2, \nu_1(N_1)^c)\) and \(\tau_2 \in C^k(M_3, \nu_2(N_2)^c)\) be Thom classes for \((N_1, \nu_1)\) and \((N_2, \nu_2)\). Let \(p_{12}^* \tau_1\) be the image of \(\tau_1\) along the maps

\[
\begin{array}{ccc}
C^k(M_2, \nu_1(N_1)^c) & \xrightarrow{p_{12}^*} & C^k(N_2, \nu_1(N_1)^c) & \xrightarrow{\nu_2} & C^k(M_3, \nu_{12}(N_{12})^c),
\end{array}
\]

with the second map is the isomorphism induced by \(\nu_2\) and excision. Then

\[
\tau_{12} := p_{12}^* \tau_1 \cup \tau_2 \in C^{k+l}(M_3, \nu_{12}(N_{12})^c)
\]
is a Thom class for the tubular embedding $\nu_{12}$. Moreover, the diagram
\[
\begin{array}{c}
C_*(M_3) \xrightarrow{[\tau_2 \cap]} C_*(M_2) \\
\xrightarrow{[\tau_{12} \cap]} C_*(M_1)
\end{array}
\]
commutes, where
\[
[\tau_1 \cap]: C_*(M_2) \to C_*(M_2, \nu_1(N_1)^c) \xrightarrow{\tau_1 \cap} C_{*-k}(N_1) \to C_{*-k}(M_1)
\]
\[
[\tau_2 \cap]: C_*(M_3) \to C_*(M_3, \nu_2(N_2)^c) \xrightarrow{\tau_2 \cap} C_{*-l}(N_2) \to C_{*-l}(M_2)
\]
\[
[\tau_{12} \cap]: C_*(M_3) \to C_*(M_3, \nu_{12}(N_{12})^c) \xrightarrow{\tau_{12} \cap} C_{*-l}(N_{12}) \to C_{*-l}(M_1)
\]
are the maps induced by capping with the respective Thom classes.

To prove the proposition, we will, as in the rest of the paper, pick Riemannian metrics. This allows us for example to consider the normal bundle of an embedding as a subbundle of the tangent bundle of the target, as we explain now: Let
\[
f: M \hookrightarrow M'
\]
be an embedding of manifolds, with $M$ compact and $M'$ a complete Riemannian manifold. Let $p: N \to M$ be the normal bundle. The metric of $M'$ gives an identification of $N$ with a subbundle of $TM'|_M$:
\[(A.2) \quad N \cong TM^\perp =: \{(m, W) : m \in M, W \in T_{f(m)}M' \text{ and } W \perp f_*T_mM\}
\]
and thus splits the exact sequence
\[0 \to TM \to TM' \to N \to 0\]
of bundles over $M$. We will identify $N$ with this subbundle, with the notation of $(A.2)$.

The following lemma is an elementary consequence of the inverse function theorem.

**Lemma A.2.** Let $N \to M$ be the normal bundle of the embedding $f: M \to M'$ as above. Suppose that $\nu: N \to M'$ has the following properties:

(i) $\nu(m, 0) = f(m)$ for all $m \in M$.

(ii) The map $d\nu: T_{(m, 0)}N \to T_{f(m)}M'$ is surjective.

Then there exists $\varepsilon > 0$ so that the restriction of $\nu$ to $N_\varepsilon$ is a tubular embedding of $M$ in $M'$, for $N_\varepsilon = \{(m, W) \in N \mid |W| < \varepsilon\}$.

**Proof of Proposition A.1.** Choose a Riemannian metric on $M_3$. Let $M_1$ and $M_2$ carry the induced metrics. Statement (1) follows from (suppressing the inclusion maps $f_1, f_2$):
\[N = TM_1^\perp \subset TM_3; \quad N_1 = TM_1^\perp \subset TM_2; \quad N_2 = TM_2^\perp \subset TM_3,
\]
so that for each $m \in M_1$,
\[N_m \cong (N_1)_m \oplus (N_2)_{f(m)}.
\]
As a consequence we can and will identify
\[N \cong N_{12} \cong \left\{(m, V, W) \mid m \in M_1, \ V \in T_{f_1(m)}M_2, \ V \perp df_1T_mM_1, \text{ and } W \in T_{f_1(m)}M_3, \ W \perp df_2T_{f_1(m)}M_2\right\}.
\]
To prove (2), define $\tilde{\nu}_1: N_{12} \to N_2$ by
\[\tilde{\nu}_1(m, V, W) = (\nu_1(m, V), W')
\]
where $W' \in (N_2)_{\nu_1(m,tV)}$ is the vector obtained from $W$ by parallel transport along the path $\nu_1(m,tV)$. The inverse is given by parallel transporting back, which gives the claimed isomorphism by uniqueness of such transports.

We then apply the lemma in the case $M = M_1$, $M' = M_3$, $f = f_2 \circ f_1$, and $\nu = \nu_{12}$. We need to check condition (ii) of the lemma. Fix $m \in M_1$. Because $\nu_1 : N_1 \to M_2$ is a tubular embedding, the image of $d\nu_1 : T_{(m,0)}N_1 \to T_{M_2}$ is surjective, and the tangent space to $M_2$ in $M_3$ is in the image of $d\nu_{12} =: d(\nu_2 \circ \nu_1)$. Now let $W \in N_2$; that is, $W \in T_{M_3}$, with $W \perp T_{M_2}$. We find by definition $\nu_1(m,0,W) = (m,W) \in N_2$ and $\nu_{12}(m,0,W) = \nu_2 \circ \nu_1(m,0,W) = (m,W)$. (When $V = 0$, the parallel transport is trivial.) Since $T_{M_2}$ is spanned by $T_{m}M_2$ and $d\nu_2 N_2$, we conclude that

$$\nu_{12} : T_{(m,0)}N_1 \to T_{m}M_3$$

is surjective.

To show the commutativity of the diagram in statement (3), we decompose the diagram as

\[(A.3) \quad C_\ast(M_3) \xrightarrow{\tau_2 \cap} C_\ast(M_2, N_2^\ast) \xrightarrow{c} C_\ast(N_1, \partial N_1) \xrightarrow{p_1} C_\ast(N_1)\]

where we suppressed the inclusions for readability. Commutativity follows from the naturality of the cap product and the compatibility of the cup and cap products. The fact that $\tau_3$ is the Thom class of the composition is a consequence of the commutativity of the diagram, and the fact that both $\tau_1$ and $\tau_2$ are Thom classes, so that capping with them take fundamental classes to fundamental classes. \qed

\textbf{References}


