

# STRING TOPOLOGY IN THREE FLAVOURS

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*Dedicated to Dennis Sullivan on the occasion of his 80th birthday.*

ABSTRACT. We describe two major string topology operations, the Chas-Sullivan product and the Goresky-Hingston coproduct, from geometric and algebraic perspectives. The geometric construction uses Thom-Pontrjagin intersection theory while the algebraic construction is phrased in terms of Hochschild homology. We give computations of products and coproducts on lens spaces via geometric intersection, and deduce that the coproduct distinguishes 3-dimensional lens spaces. Algebraically, we describe the structure these operations define together on the Tate-Hochschild complex. We use rational homotopy theory methods to sketch the equivalence between the geometric and algebraic definitions for simply connected manifolds and real coefficients, emphasizing the role of configuration spaces. Finally, we study invariance properties of the operations, both algebraically and geometrically.

## 1. INTRODUCTION

String topology is concerned with algebraic structures defined by intersecting, concatenating, and cutting families of paths and loops in a manifold  $M$ . It began with Chas and Sullivan's construction of an intersection type product on  $H_*(LM)$ , the homology of the space  $LM = \text{Map}(S^1, M)$  of all loops in  $M$ , also known as the free loop space of  $M$  [13]. The loop product induces a Lie bracket on  $H_*^{S^1}(LM)$ , the  $S^1$ -equivariant homology of  $LM$ , generalizing an earlier construction of Goldman for curves on surfaces [36].

Over the last twenty years, string topology has branched out to many corners of mathematics:

- It has an algebraic counterpart in Hochschild homology through the Jones [47] and Goodwillie [37] isomorphisms

$$H^*(LM; \mathbb{F}) \cong HH_*(C^*(M; \mathbb{F}), C^*(M; \mathbb{F})) \text{ and } H_*(LM) \cong HH_*(C_*(\Omega M), C_*(\Omega M))$$

for  $\mathbb{F}$  a field,  $\Omega M$  the based loop space of  $M$  and where  $M$  is assumed to be simply connected for the first isomorphism, see e.g. [24, 64, 30, 67, 59];

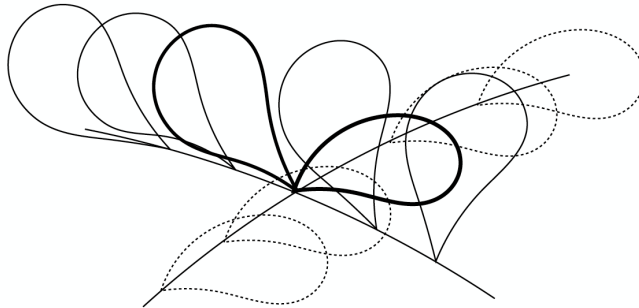
- It has a symplectic interpretation through the Viterbo [79] isomorphism (with appropriate coefficients)

$$H_*(LM) \cong FH_*(T^*M)$$

with target the Floer homology of the cotangent bundle of  $M$ , see e.g. [3, 4, 22];

- Rich families of string operations have been defined, in particular using the algebraic model of string topology, with for instance BV structures, Lie bialgebras, 2-dimensional field theories of various flavours, and more, see e.g. [35, 27, 77, 51, 50, 81];
- String topology has been used to study closed geodesics on Riemannian manifolds through Morse theory on the energy functional, see e.g. [38, 42];
- String operations can be defined instead on the loop space  $LBG$  for  $G$  a Lie group, or more generally on the loop space of stacks, see [14, 41, 8] and see e.g. [55, 39] for applications to group homology.

We will not be able to cover all aspects of string topology in this note and will instead focus on a few highlights that, we hope, illustrate the richness of the subject. In particular, we will restrict our attention to the original loop product of Chas and Sullivan and its “dual”, the Goresky-Hingston coproduct. We will describe these two operations geometrically as well as algebraically, and use methods from rational homotopy theory to compare the two descriptions, where the role of configuration spaces will be emphasized. The geometric aspect of string topology will be illustrated through computations of loop products and coproducts via intersections of geometric cycles in examples from lens spaces. Algebraically, we will see that the two operations together



define a single product on the Tate-Hochschild complex, defined below, and are encoded by the data of a Manin triple. Finally, we will address the question of invariance of the product and coproduct.

We describe now in more detail the content of this text. Throughout,  $M$  will be a closed oriented manifold of dimension  $n$ , and homology is with  $\mathbb{Z}$ -coefficients unless otherwise stated.

**Intersection products.** Recall that the classical intersection product

$$\bullet : H_p(M) \otimes H_q(M) \rightarrow H_{p+q-n}(M)$$

can be computed by geometric intersection for transverse cycles: if  $A, B \in H_*(M)$  are homology classes represented by smooth transversally embedded submanifolds, then their product  $A \bullet B$  is given by the geometric intersection  $A \cap B$  of the cycles. The original idea behind the Chas-Sullivan product is to define a product on  $H_*(LM)$  by likewise transversally intersecting two families of loops in  $M$  at their basepoints, which is an intersection in  $M$ , and concatenating loops at the locus of intersection. This results in a graded commutative and associative product

$$\wedge : H_p(LM) \otimes H_q(LM) \rightarrow H_{p+q-n}(LM),$$

that is, by construction, compatible with the intersection product under the evaluation map  $\text{ev}_0 : LM \rightarrow M$ . We will refer to the Chas-Sullivan product as the *loop product*.

Following ideas going back to Cohen-Jones [24], we give in Section 2.2 a formal definition of this product by lifting the definition of the classical intersection product phrased in terms of a Thom-Pontrjagin construction for the diagonal embedding  $\Delta : M \rightarrow M \times M$ .

The Goresky-Hingston coproduct [38], also considered by Sullivan [75] and referred to as the *loop coproduct* here, has the form

$$\vee : H_p(LM, M) \rightarrow H_{p-n+1}(LM \times LM, LM \times M \cup M \times LM).$$

The idea of the coproduct is, given a family of loops, to look for all the self-intersections in the family of the form  $\gamma(0) = \gamma(t)$ , for  $\gamma$  a loop and  $t \in I$  is a time coordinate, and then cut. Following Hingston-Wahl [43], we show that it can be defined using a simple variant of the definition of the loop product. The operation is most naturally a relative operation because the interval  $I$  has non-trivial boundary; see Remark 2.3 for non-relative versions of the coproduct.

The loop product and coproduct can be diagrammatically described as

$$\begin{array}{ccc} LM \times LM \xleftarrow{\quad \bar{\quad} \quad} \text{Fig}(8) \xrightarrow{\text{concat}} LM & & LM \times I \xleftarrow{\quad \bar{\quad} \quad} \mathcal{F} \xrightarrow{\text{cut}} LM \times LM \\ \text{ev}_0 \times \text{ev}_0 \downarrow & & \text{ev}_0 \times \text{ev}_t \downarrow \\ M \times M \xleftarrow{\quad \Delta \quad} M & & M \times M \xleftarrow{\quad \Delta \quad} M \end{array}$$

where the middle spaces  $\text{Fig}(8) \cong LM \times_M LM$  and  $\mathcal{F} \subset LM \times I$  are the subspaces where the desired intersection holds, and where the dashed arrows are “intersection products” that are only defined on homology (or on chains). In Sections 4.1 and 4.2, we will formulate the data used from  $M$  to define these intersection products in terms of an *intersection context* (see

Definition 4.7). Our preferred intersection context associated to a manifold  $M$  will be

$$\begin{array}{ccc} UTM & \longrightarrow & FM_2 \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times M, \end{array}$$

where  $FM_2$  is the configuration space of two points in  $M$  and  $UTM$  the unit tangent bundle of  $M$ .

**Geometric computations.** Just like the intersection product  $\bullet$  can be computed by geometric intersection for nice enough cycles, the loop product and coproduct can be computed by a direct intersection for cycles that are appropriately transverse. This is made precise in Proposition 2.4, following [43], and illustrated through the computation of the loop product and coproduct of a family of classes generating  $H_3(L\mathcal{L}_{p,q})$ , for  $\mathcal{L}_{p,q}$  a 3-dimensional lens spaces; see Propositions 2.5 and 2.8. As an application of the computation, we prove the following

**Theorem A** (Theorem 2.11). *The loop coproduct distinguishes non-homeomorphic 3-dimensional lens spaces.*

This result is an extension of a computation of the first author in [66], used in that paper to show that the loop coproduct is not homotopy invariant; see below for more details about the invariance properties of the loop product and coproduct.

**String topology algebraically.** Assume now that  $M$  is a simply connected closed manifold. The isomorphism  $HH_*(C^*(M; \mathbb{F}), C^*(M; \mathbb{F})) \cong H^*(LM; \mathbb{F})$  mentioned above, actually holds independently of the fact that  $M$  is a manifold. However, the algebraic structure of the Hochschild complex becomes much richer once one inputs that  $H^*(M)$  satisfies Poincaré duality, or in other words that it is a *Frobenius algebra* (see Definition 3.2). In the above isomorphism, we can replace  $C^*(M; \mathbb{F})$  by any algebra  $A$  quasi-isomorphic to it in the category of dg algebras. By a theorem of Lambrechts-Stanley, it is possible to find a model  $A$  for the rational cochains  $C^*(M; \mathbb{Q})$  that has the structure of a (strict) commutative dg Frobenius algebra compatible with the Frobenius structure on  $H^*(M; \mathbb{Q})$  (see Theorem 3.4 and Example 3.5). The relevant consequence for us is that:

*The algebraic structure of the Hochschild chains or cochains of dg Frobenius algebras reflects rational string topology.*

For Frobenius algebras, we indeed have an isomorphism between the linear dual of the Hochschild chain complex  $C_*(A, A)$  and the Hochschild cochain complex  $C^*(A, A)$ , so both complexes are relevant (see Remark 3.10).

There is a wealth of literature on the algebraic structure of the Hochschild chains and cochains of Frobenius algebras, including algebraic versions of the product and coproduct just described, see e.g. [24, 64, 2, 31] for the loop product and [2, 54] for the loop coproduct, or e.g. [77, 51, 50, 52, 81] for larger structures encompassing both, or [54, 80] for a prop of universal operations on the Hochschild complex of symmetric or commutative Frobenius algebras. (See also [9] in the present volume.)

It turns out that the loop product identifies with the classical cup product on Hochschild cochains [30], while the loop coproduct becomes the following product on *relative* Hochschild chains (see Definition 4.1):

**Theorem B.** [67] *Let  $A$  be a dg Frobenius algebra model for  $C^*(M; \mathbb{R})$ . Under a relative version of the Jones isomorphism  $H^*(LM; \mathbb{R}) \cong HH_*(C^*(M; \mathbb{R}), C^*(M; \mathbb{R})) \cong HH_*(A, A)$ , the linear dual of the loop coproduct is given on cochains by the formula*

$$(\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) * (\overline{b_1} \otimes \cdots \otimes \overline{b_q} \otimes b_{q+1}) = \sum_i \pm \overline{b_1} \otimes \cdots \otimes \overline{b_{q+1}} e_i \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} f_i,$$

where  $\Delta(1) = \sum_i e_i \otimes f_i \in A \otimes A$  represents the Thom class of the diagonal in  $M \times M$ . (See Example 3.3 and Definition 3.15).

This result is stated as Theorem 4.2 in the present paper, and we give a sketch proof of the result in Section 4.4.

In Section 3.6, we will focus on the following aspect of the algebraic structure defined by the algebraic product and coproduct:

**Theorem C.** [72] *The algebraic product and coproduct extend to define together a single  $A_\infty$ -structure on the Tate-Hochschild complex*

$$\mathcal{D}^{*,*}(A, A) = \dots \xrightarrow{\partial_h} s^{1-k}C_{-1,*}(A, A) \xrightarrow{\partial_h} s^{1-k}C_{0,*}(A, A) \xrightarrow{\gamma} C^{0,*}(A, A) \xrightarrow{\delta_h} C^{1,*}(A, A) \xrightarrow{\delta_h} \dots$$

that is compatible with the natural pairing between Hochschild chains and cochains and with an extension of Connes' operator  $B$  to the Tate-Hochschild complex. On cohomology, the product is graded commutative, and  $H^*(\mathcal{D}^*(A, A))$  identifies, as an algebra, with the endomorphism algebra of  $A$  in the singularity category of  $A$ - $A$ -bimodules (see Remark 3.13).

Here the Tate-Hochschild complex “glues together” the Hochschild chains and cochains along the map  $\gamma$  that can be thought of as an Euler characteristic, constructed using the Frobenius structure of  $A$ , see Section 3.4 for a complete definition of this complex. In Remark 3.18, we give a description of this structure in terms of Manin triples, and this implies a form of infinitesimal bialgebra compatibility between the Goresky-Hingston coproduct and the Chas-Sullivan loop product. Note that Cieliebak-Hingston-Oancea have given a geometric version of the above Tate construction, including its algebra structure, using Rabinowitz-Floer homology, a theory that combines symplectic homology and cohomology via a “V-shaped” Hamiltonian [17, 23, 21, 20]. Theorem C is stated as Theorem 3.17 in the text.

The Tate-Hochschild complex satisfies the following strong invariance property, that is a consequence of the interpretation in terms of the singularity category:

**Theorem D.** [72] *If two simply connected symmetric dg Frobenius algebras are quasi-isomorphic as dg associative algebras, then their Tate-Hochschild cohomologies are isomorphic as algebras.*

This result is stated as Theorem 3.19 in the text. A direct consequence of the result is that the algebraic version of the loop coproduct is a homotopy invariant in the simply connected setting (see Corollary 3.20).

**Naturality and invariance.** One of the original motivations of Chas and Sullivan in studying free loop spaces was to understand what characterizes the algebraic topology of manifolds and to construct algebraic invariants that could detect beyond the homotopy type; in Sullivan's own words to us

“...it is the question that has fascinated me since grad school: what is the algebraic chain level meaning of a space being a combinatorial or smooth manifold?”

The particular instance of this question we will address here is the following: a homotopy equivalence  $M \xrightarrow{\simeq} N$  induces an isomorphism  $H_*(LM) \xrightarrow{\simeq} H_*(LN)$ , and likewise on relative homology, and one can ask whether this induced map respects the loop product or coproduct. We summarize in the following result what is known about the question:

**Theorem E.** (1) [25] *The Chas-Sullivan product on  $H_*(LM)$  is invariant under homotopy equivalences of manifolds  $M \xrightarrow{\simeq} N$ .*  
 (2) ([72] and [67]) *The Goresky-Hingston coproduct on  $H_*(LM; \mathbb{R})$  is invariant under homotopy equivalences of simply connected manifolds  $M \xrightarrow{\simeq} N$ .*  
 (3) [66] *The Goresky-Hingston coproduct on  $H_*(LM)$  is not homotopy invariant in general.*

Alternative proofs of part (1) of the theorem were given by [40, 30, 26]. We give here a sketch proof of this result, in Theorem 4.11, stated in terms of homotopy invariance of general intersection products. Part (2) of the theorem is a direct consequence of combining Theorems B and D, while part (3) is a consequence of Theorem A.

The essential difference between the loop product and coproduct is that the loop coproduct uses a *relative intersection product*, and the proof of homotopy invariance of intersection product does not extend to proving the relative result. The article [66] suggests that the failure of

invariance of the loop coproduct is related to Reidemeister torsion, which is compatible with Theorem A. See also [44] for conditions on the homotopy equivalences under which the coproduct is invariant.

A non-invariance result was earlier obtained by Basu for a modified version of the coproduct [7]. Naef used the lens spaces  $\mathcal{L}_{1,7}$  and  $\mathcal{L}_{2,7}$  in [66] to show non-homotopy invariance of the coproduct on homology. The very same lens spaces were used by Longoni-Salvatore in [58] to show that the configuration space of two points in a manifold is likewise not a homotopy invariant of the manifold. Although we do not directly relate these two computations of non-homotopy invariance, we have already seen above that the configuration space of two points is an important ingredient in the definition of the loop coproduct, being part of the data needed to define the corresponding (relative) intersection product, see Sections 4.1 and 4.2.

The Lie bialgebra structure at the level of  $S^1$ -equivariant homology is a homotopy invariant for simply connected manifolds by [66]. The recent paper [19] proves that homotopy invariance over the reals is also satisfied for a chain level version of the Lie bialgebra structure (also known as  $IBL_\infty$ -algebra) in the case of 2-connected manifolds. It is so far unknown whether the chain level Lie bialgebra structure on  $S^1$ -equivariant chains (or a chain level version of the coalgebra structure in the non-equivariant case) is a homotopy invariant for simply connected manifolds.

*Organisation of the paper.* In Section 2, after recalling the Thom-Pontrjagin definition of the intersection product, we give a chain level definition of the loop product and coproduct. Section 2.3 gives the computations of the loop products and coproducts on  $H_3(L\mathcal{L}_{p,q})$  for 3-dimensional lens spaces  $\mathcal{L}_{p,q}$ . The coproduct computation is used in Section 2.4 to show that the loop coproduct is not homotopy invariant. Then Section 2.5 gives an alternative definition of the loop coproduct as a relative version of the so-called “trivial coproduct”, the coproduct on the loop space that only looks for basepoint self-intersections at time  $t = \frac{1}{2}$ . This definition will be used in Section 4 to show the equivalence between the algebraic and geometric descriptions of the coproduct.

Section 3 is concerned with the algebraic version of string topology. It starts with recalling and setting in context the concepts of Frobenius algebras, Hochschild chains and cochains. Section 3.4 then gives the definition of the Tate-Hochschild complex of a dg Frobenius algebra. The loop product and coproduct are defined algebraically in Section 3.5 as products on the Hochschild cochains and chains respectively. These two products are assembled to a single product on the Tate-Hochschild complex in Section 3.6, where it is also interpreted in the language of Manin triples. The invariance of the product on the Tate-Hochschild complex is stated at the end of the section.

Section 4 takes a closer look at the “intersection products” that appear in the definition of the loop product and coproduct. After revisiting the definitions of the loop product and coproduct in Section 4.1, the notion of *intersection context* is defined in Section 4.2, a data one can construct intersection and relative intersection products from. The naturality and invariance properties of such intersection products are discussed in Section 4.3. Finally, Section 4.4 gives a sketch proof of the equivalence between the algebraic and geometric coproduct (Theorem 4.2) using an intersection context featuring the configuration space of two points in  $M$  and its real model [11, 45].

**Acknowledgements.** The authors would like to thank Dennis Sullivan for being a constant source of mathematical inspiration and for encouraging us to compare and combine our different approaches to the question of invariance of the loop coproduct, with the goal of illuminating the conceptual tensions in our (at times contradictory!) results. We see this paper as a first step in this direction.

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## 2. STRING TOPOLOGY VIA GEOMETRIC INTERSECTION

Let  $M$  be a closed oriented manifold of dimension  $n$ , and pick a Riemannian metric on  $M$ . The loop space  $LM = \text{Map}(S^1, M)$  is homotopy equivalent to the space  $\Lambda M$  of  $H^1$ -loops on which the energy functional is defined:

$$LM \simeq \Lambda M \xrightarrow{E} \mathbb{R}, \quad \text{where } E(\gamma) = \int_{S^1} |\gamma'(t)|^2 dt.$$

The critical points of the energy are precisely the closed geodesics. Given that the energy is nice enough to do Morse theory, it follows that the homology  $H_*(LM) \cong H_*(\Lambda M)$  “knows”, or even “is build out of” closed geodesics. (See e.g., [69] for a survey of Morse theory on the free loop space.)

As a graded abelian group,  $H_*(LM)$  depends only on the homotopy type of  $M$ , whereas the closed geodesics depend on  $M$  as a Riemannian manifold. This naturally leads to the question of whether there is some additional structure on  $H_*(LM)$  that depends on a more refined structure than just the homotopy type of  $M$ . When  $M$  is a closed manifold, its homology satisfies Poincaré duality, and this duality takes the cup product of  $H^*(M)$  to the *intersection product*:

$$H_p(M) \otimes H_q(M) \xrightarrow{\bullet} H_{p+q-n}(M).$$

The lifts of the intersection product given by the Chas-Sullivan product

$$H_p(LM) \otimes H_q(LM) \xrightarrow{\wedge} H_{p+q-n}(LM)$$

and Goresky-Hingston coproduct

$$H_p(LM, M) \xrightarrow{\vee} H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M)$$

briefly described in the introduction, give a potential answer to the above question. Following ideas of Cohen-Jones [24] as implemented in [43], we explain here how both operations can be defined on chains as direct lifts of the intersection product, by using a chain-level definition of the intersection product in terms of a Thom-Pontrjagin construction. Section 2.3 will give example computations, obtained from intersecting geometric cycles, from which we will be able

to deduce in Section 2.4 that the coproduct does detect more than the homotopy type. Finally, Section 2.5 will give an alternative definition of the coproduct.

Note that homology in this section will always mean homology with integral coefficients:  $H_*(-) := H_*(-; \mathbb{Z})$ , and the same for cohomology.

**2.1. The intersection product as a Thom-Pontrjagin construction.** The normal bundle of the diagonal embedding  $\Delta : M \hookrightarrow M \times M$  is isomorphic to the tangent bundle  $TM$ . Identifying  $TM \cong TM_\epsilon$  with its subbundle of *small vectors*, i.e. vectors of length at most  $\epsilon \ll \rho$  for  $\rho$  the injectivity radius, the map

$$\nu_M : TM \hookrightarrow M \times M \quad \text{defined by} \quad \nu_M(x, V) = (x, x + \exp_x V)$$

is an explicit tubular neighborhood for  $\Delta$ , with image the  $\epsilon$ -neighborhood of the diagonal

$$\nu_M : TM \xrightarrow{\cong} U_M = \{(x, y) \in M \times M \mid |x - y| < \epsilon\}.$$

Under this identification, the bundle projection map  $TM \rightarrow M$  becomes the retraction  $r : U_M \rightarrow M$  defined by  $r(x, y) = x$ . We let

$$(2.1) \quad \tau_M \in C^n(M \times M, M \times M \setminus M) \xleftarrow{\sim} C^n(TM, TM \setminus M)$$

denote the image of a cochain representative for the Thom class for  $TM$ , where  $M \subset M \times M$  is the diagonal, and the arrow is the map  $\nu_M^*$ , which is a quasi-isomorphism by excision.

Out of this data, we can give the following chain level description of the intersection product on  $H_*(M)$ :

$$(2.2) \quad \bullet : C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \xrightarrow{[\tau_M \cap]} C_{p+q-n}(U_M) \xrightarrow{r} C_{p+q-n}(M),$$

where the middle map is the following composition:

$$(2.3) \quad [\tau_M \cap] : C_*(M \times M) \rightarrow C_*(M \times M, M \times M \setminus M) \xrightarrow{\sim} C_*(U_M, U_M \setminus M) \xrightarrow{\tau_M \cap} C_{*-n}(U_M),$$

with the middle map being a homotopy inverse to excision, as can be obtained, for example, by subdividing simplices. (To be precise, this definition differs by a sign from the intersection product defined as the Poincaré dual of the cup product, see eg. [43, Proposition B.1].)

An important property of the intersection product, for computational purposes, is that it can indeed be computed by geometric intersection for homology classes that can be represented by transverse embedded submanifolds: if  $A, B \subset M$  are embedded transverse submanifolds of  $M$ , with  $[A] \in H_p(M)$  and  $[B] \in H_q(M)$  the corresponding homology classes, then

$$[A] \bullet [B] = [A \cap B] \in H_{p+q-n}(M).$$

See eg. [10, VI Theorem 11.9].

**2.2. Definition of the product and coproduct as lifts of the intersection product.**

Let  $ev_0 : LM \rightarrow M$  denote the evaluation at 0. The Chas-Sullivan product  $\wedge$  being a lift of the intersection product  $\bullet$  means that both products should fit in a commutative diagram of the form

$$(2.4) \quad \begin{array}{ccc} H_p(LM) \otimes H_q(LM) & \xrightarrow{\wedge} & H_{p+q-n}(LM) \\ \text{ev}_0 \otimes \text{ev}_0 \downarrow & & \downarrow \text{ev}_0 \\ H_p(M) \otimes H_q(M) & \xrightarrow{\bullet} & H_{p+q-n}(M) \end{array}$$

We explain now how this can be achieved simply by “pulling back” all the ingredients of the above definition of the intersection product to the loop space along the evaluation map  $ev_0 \times ev_0$ .

Recall from above the  $\epsilon$ -neighborhood  $U_M$  of the diagonal in  $M \times M$  and define

$$U_{CS} = (e \times e)^{-1}U_M = \{(\gamma, \lambda) \in LM \times LM \mid |\gamma(0) - \lambda(0)| < \epsilon\}.$$

The retraction  $r : U_M \rightarrow M$  lifts to a retraction

$$R_{CS} : U_{CS} \longrightarrow \text{Fig}(8) = \{(\gamma, \lambda) \in LM \times LM \mid \gamma(0) = \lambda(0)\} = (ev_0 \times ev_0)^{-1}(M \subset M \times M)$$

by concatenating with a geodesic stick to connect the loops so that they form a “figure 8”:

$$R_{CS}(\gamma, \lambda) = (\gamma, \overline{\gamma(0)\lambda(0)} \star \lambda \star \overline{\lambda(0)\gamma(0)})$$

where, for  $x, y \in M$  with  $|x - y| < \rho$ ,  $\overline{xy}$  denotes the unique minimal geodesic path  $[0, 1] \rightarrow M$  from  $x$  to  $y$ , which is possible by our choice of  $\epsilon$ , and  $\star$  is the concatenation of paths.\* See also Figure 1(a).

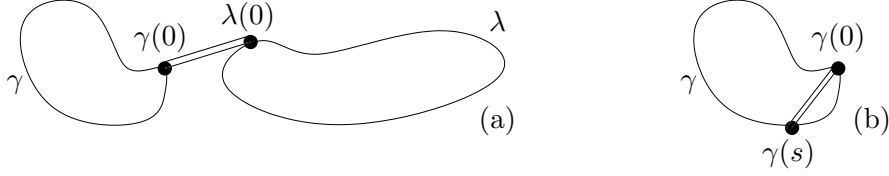


FIGURE 1. The retraction maps  $R_{CS}$  and  $R_{GH}$ .

Pulling back our representative of the Thom class  $\tau_M$  along the evaluation map gives a cochain

$$\tau_{CS} := (e \times e)^* \tau_M \in C^*(LM \times LM, \text{Fig}(8)^c).$$

Together,  $U_{CS}$ ,  $R_{CS}$  and  $\tau_{CS}$  are all the ingredients we need to define the desired product:

**Definition 2.1.** The following sequence of chain maps is a chain model for the Chas-Sullivan product:

$$(2.5) \quad \wedge: C_p(LM) \otimes C_q(LM) \xrightarrow{\times} C_{p+q}(LM \times LM) \xrightarrow{[\tau_{CS} \cap]} C_{p+q-n}(U_{CS}) \\ \xrightarrow{R_{CS}} C_{p+q-n}(\text{Fig}(8)) \xrightarrow{\text{concat}} C_{p+q-n}(LM),$$

where, just as in (2.2), the middle map is the composition of an homotopy inverse to excision followed by the capping map.

Naturality of the maps gives that the resulting homology product on the homology  $H_*(LM)$  makes Diagram (2.4) commute. And it is shown in [43, Proposition 2.4] that this simple minded chain description of the Chas-Sullivan product agrees in homology with the definition of Cohen-Jones [24] given in terms of a tubular neighborhood of the figure 8 space  $\text{Fig}(8)$  inside  $LM \times LM$ .

The coproduct can be defined completely analogously, replacing the evaluation map  $\text{ev}_0 \times \text{ev}_0: LM \times LM \rightarrow M \times M$  by the evaluation map

$$e_I: LM \times I \rightarrow M \times M \quad \text{defined by } e_I(\gamma, s) = (\gamma(0), \gamma(s)).$$

Indeed, setting

$$U_{GH} = e_I^{-1} U_M = \{(\gamma, s) \in LM \times I \mid |\gamma(0) - \gamma(s)| < \epsilon\},$$

we again have a retraction map

$$R_{GH}: U_{GH} \longrightarrow \mathcal{F} = \{(\gamma, s) \in LM \times I \mid \gamma(0) = \gamma(s)\} = e_I^{-1}(M \subset M \times M)$$

by concatenating with a geodesic stick to force a self-intersection:

$$R_{CS}(\gamma, s) = (\gamma[0, s] \star \overline{\gamma(s)\gamma(0)} \star_s \overline{\gamma(0)\gamma(s)} \star \gamma[s, 0], s)$$

where we choose the parametrization of the concatenated loop so that it exactly passes through  $\gamma(0)$  at time  $s$ ; this is possible even if  $s = 0$  or  $1$  as in that case  $\gamma(0) = \gamma(s)$  to begin with and the geodesic sticks are thus length 0. See also Figure 1(b).

We can consider the sequence of maps

$$C_p(LM) \xrightarrow{\times I} C_{p+1}(LM \times I) \xrightarrow{[\tau_{GH} \cap]} C_{p+1-n}(U_{GH}) \xrightarrow{R_{GH}} C_{p+1-n}(\mathcal{F}) \xrightarrow{\text{cut}} C_{p+q-n}(LM \times LM).$$

totally analogous to the maps (2.5) defining the product above. The only new issue that arises in the coproduct compared to the product is that the first map in the sequence, crossing with an interval, is not a chain map because the interval has non-trivial boundary. This corresponds to the fact that the operation is now parametrized by an interval  $I$ . To obtain an induced

\*See eg., [43, Sec 1.2] for a definition of an associative concatenation.



operation on homology, we need to appropriately kill the resulting “boundary operation” at the endpoints of the interval. The simplest way to do this is to consider the operation as a relative operation, noting that, when  $s = 0$  or  $1$ , the above sequence of maps creates a left or right constant loop.

**Definition 2.2.** The following sequence of chain maps is a chain model for the Goresky-Hingston-Sullivan coproduct:

(2.6)

$$\begin{aligned} \vee : C_p(LM, M) &\xrightarrow{\times I} C_{p+1}(LM \times I, LM \times \partial I \cup M \times I) \xrightarrow{[\tau_{\text{GH}} \cap]} C_{p+1-n}(U_{\text{GH}}, LM \times \partial I \cup M \times I) \\ &\xrightarrow{R_{\text{GH}}} C_{p+1-n}(\mathcal{F}, LM \times \partial I \cup M \times I) \xrightarrow{\text{cut}} C_{p+q-n}(LM \times LM, M \times LM \cup LM \times M) \end{aligned}$$

This sequence of maps now indeed induces a well-defined degree  $1-n$  coproduct on  $H_*(LM, M)$ :

$$\vee : H_p(LM, M) \longrightarrow H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M);$$

if we work with field coefficients, the target is isomorphic to  $H_*(LM, M)^{\otimes 2}$ . It is shown in [43, Proposition 2.12] that this chain level description of the Goresky-Hingston-Sullivan coproduct agrees with the definition given in [38] using a tubular neighborhood of  $\mathcal{F}$  inside  $LM \times I$  away from the boundary  $LM \times \partial I$ , together with a limit argument reach to the boundary.

Applying the evaluation map  $e_I$  gives a diagram of the same form as Diagram (2.4), with the coproduct replacing the Chas-Sullivan product on the top row, but now with intersection product *relative to*  $M$  on the bottom row, which is a trivial operation! Hence there is no formal way in which the homology loop coproduct is a lift of the homology intersection product. We will however see in Section 2.3 that the coproduct still can be computed by an appropriate geometric intersection, for nice enough geometric cycles, away from the “trivial self-intersections” coming from constant loops or from the intersection times  $s = 0$  and  $s = 1$ .

**Remark 2.3** (Lifting the coproduct to a non-relative operation). There exists several ways to lift the coproduct  $\vee$  to a non-relative operation.

- (1) One such lift is the *extension by zero* of [43, Sec 4], that uses the splitting  $H_*(LM) \cong H_*(LM, M) \oplus H_*(M)$  coming from the inclusion of the constant loops and the evaluation  $\text{cst} : M \rightleftharpoons LM : \text{ev}_0$ , declaring the coproduct to be zero on constant loops.
- (2) If the Euler characteristic of the manifold is zero, one can instead use a nowhere vanishing vector field  $\bar{v}$  to define such an extension, by replacing the diagonal  $M \subset M \times M$  in the above definition of the coproduct, with the homotopy equivalent subspace  $\Delta_{\bar{v}}M = \{(m, \exp_m \bar{v}_m) \in M \times M \mid m \in M\}$ . Indeed, if the vector field has no zeros, the coproduct will then automatically be trivial at the special points with  $s = 0$  or  $s = 1$ . See also [67, Sec 3.4] for an analogous definition of a lifted coproduct in the  $\chi(M) = 0$  case, using instead a lift of the Thom class.

If the Euler characteristic is not zero, one can instead pick a vector field vanishing only in the neighborhood of a single point, which will yield a coproduct in reduced homology of the loop space instead, corresponding to what we will see in the algebraic version of the coproduct, see Definition 3.15.

- (3) The following variant of the previous idea has been described for the case of surfaces in [53]. Instead of attaching the non-vanishing vector field to the manifold  $M$  one can attach it to the loop. That is one considers loops in the unit tangent bundle of  $M$ . In the case of surfaces, such loops can be identified with regular homotopy classes of immersed curves. Moreover, in case the surface has a non-vanishing vector field, the above construction is recovered by using that every homotopy class of a loop in a surface has a unique representative as an immersed loop with rotation number 0 with respect to the vector field. This is the point of view taken in [6].
- (4) As we will see in Section 3.6 in the algebraic context, following the paper [72] (see [17, 23] for a geometric version), the loop product and coproduct together define a single (non-relative) product on the *Tate-Hochschild complex*, a complex that combines both the chains and cochains of the loop space, attached together using the Euler class

(see Section 3.4). When the Euler characteristic of the manifold vanishes, the Tate complex splits and this recovers a non-relative cohomology product, dual to the homology coproduct.

**2.3. Computation via geometric intersections.** Recall that two smooth maps  $f : X \rightarrow M$  and  $g : Y \rightarrow M$  are *transverse* if for every  $x, y$  such that  $f(x) = m = g(y)$ , we have  $f_*T_xX + g_*T_yY = T_mM$ . Because the product and coproduct are defined as lifts of the intersection product along evaluation maps, they can both be computed by geometric intersection, under appropriate transversality assumptions on the cycles representing the homology classes:

**Proposition 2.4.** [43, Propositions 3.1 and 3.7]

- (1) If  $Z_1 : \Sigma_1 \rightarrow LM$  and  $Z_2 : \Sigma_2 \rightarrow LM$  are smooth cycles with the property that the maps  $\text{ev}_0 \circ Z_1 : \Sigma_1 \rightarrow M$  and  $\text{ev}_0 \circ Z_2 : \Sigma_2 \rightarrow M$  are transverse, then the loop product

$$Z_1 \wedge Z_2 = (Z_1 \star Z_2)|_{\Sigma_1 \times_{\text{ev}_0} \Sigma_2} \in H_*(LM)$$

is the concatenation of the loops of  $Z_1$  and  $Z_2$  along the locus of basepoint-intersections  $\Sigma_1 \times_{\text{ev}_0} \Sigma_2 \subset \Sigma_1 \times \Sigma_2$ , oriented as stated in [43].

- (2) If  $Z : (\Sigma, \Sigma_0) \rightarrow (LM, M)$  is a smooth relative cycle with the property that the restriction of  $e_I \circ (Z \times I) : \Sigma \times I \rightarrow M \times M$  to  $(\Sigma \setminus \Sigma_0) \times (0, 1)$  is transverse to the diagonal, then

$$\vee Z = \text{cut} \circ (Z \times I)|_{\overline{\Sigma_\Delta}} \in H_*(LM \times LM, M \times LM \cup LM \times M)$$

for  $\overline{\Sigma_\Delta}$  the closure in  $\Sigma \times I$  of the locus of basepoint self-intersecting loops  $\Sigma_\Delta \subset (\Sigma \setminus \Sigma_0) \times (0, 1)$ , oriented as stated in [43].

We illustrate this proposition here through a loop product and coproduct computation for lens spaces. The coproduct computation will be used in Section 2.4 to show that the coproduct is not homotopy invariant, following [66].

Let  $S^3$  be the 3-sphere, considered as the unit sphere in  $\mathbb{C}^2$ . We will write elements of  $S^3$  in spherical coordinates as tuples  $(\underline{r}, \underline{\theta}) = ((r_1, \theta_1), (r_2, \theta_2))$  with  $\theta_i \in \mathbb{R}/\mathbb{Z}$  and  $r_i \geq 0$ , satisfying  $r_1^2 + r_2^2 = 1$ . The lens space  $\mathcal{L}_{p,q}$ , for  $p, q$  coprime, is the quotient of  $S^3$  by the relation

$$((r_1, \theta_1), (r_2, \theta_2)) \sim ((r_1, \theta_1 + \frac{1}{p}), (r_2, \theta_2 + \frac{q}{p})).$$

This relation comes from the action of the torus  $S^1 \times S^1$  on  $S^3 \subset \mathbb{C}^2$  rotating each coordinate, where we have picked a particular subgroup  $\mathbb{Z}/p$  inside  $S^1 \times S^1$ . Note that there is a residual torus action on the lens space:

$$\begin{aligned} \alpha : (S^1 \times S^1) \times \mathcal{L}_{p,q} &\rightarrow \mathcal{L}_{p,q}, \\ ((s, t), (\underline{r}, \underline{\theta})) &\mapsto ((r_1, \theta_1 + \frac{s}{p}), (r_2, \theta_2 + \frac{sq}{p} + t)). \end{aligned}$$

In particular, given a point  $(\underline{r}, \underline{\theta})$  on the lens space, and a pair of integers  $(\ell, m)$ , we get a loop  $t \mapsto (\ell t, mt)$  in the torus, and hence a loop  $\gamma_{\underline{r}, \underline{\theta}}^{\ell, m}$  in the lens space based at  $(\underline{r}, \underline{\theta})$  by composing with the action. Explicitly, the loop  $\gamma_{\underline{r}, \underline{\theta}}^{\ell, m} : S^1 \rightarrow \mathcal{L}_{p,q}$  is defined by

$$\gamma_{\underline{r}, \underline{\theta}}^{\ell, m}(t) = ((r_1, \theta_1 + \frac{\ell t}{p}), (r_2, \theta_2 + \frac{q\ell t}{p} + mt)).$$

As the action is continuous, varying the startpoint  $(\underline{r}, \underline{\theta})$  of the loop gives a family of loops  $\rho_{\ell, m} : \mathcal{L}_{p,q} \rightarrow L\mathcal{L}_{p,q}$  parametrized by the lens space itself, and hence for each pair of integers  $(\ell, m)$  a class

$$\rho_{\ell, m} \in H_3(L\mathcal{L}_{p,q}),$$

where we use the same notation  $\rho_{\ell, m}$ . Note that each class  $\rho_{\ell, m}$  is non-trivial as it maps to the fundamental class of  $\mathcal{L}_{p,q}$  under the evaluation map

$$\text{ev}_0 : H_3(L\mathcal{L}_{p,q}) \longrightarrow H_3(\mathcal{L}_{p,q}).$$

We will here compute the loop products and coproducts of these classes.

We consider first the product:

$$\wedge : H_3(L\mathcal{L}_{p,q}) \otimes H_3(L\mathcal{L}_{p,q}) \longrightarrow H_{3+3-3}(L\mathcal{L}_{p,q}) = H_3(L\mathcal{L}_{p,q}).$$

**Proposition 2.5.** *The Chas-Sullivan loop product of the classes  $\rho_{\ell,m} \in H_3(L\mathcal{L}_{p,q})$  defined above, is given by summing the indices:*

$$\rho_{\ell_1,m_1} \wedge \rho_{\ell_2,m_2} = \rho_{\ell_1+\ell_2,m_1+m_2}.$$

*Proof.* The cycles  $\rho_{\ell,m} : \mathcal{L}_{p,q} \rightarrow L\mathcal{L}_{p,q}$  are smooth cycles parametrized  $\mathcal{L}_{p,q}$ . To apply Proposition 2.4, we need to check that the maps

$$\mathcal{L}_{p,q} \xrightarrow{\rho_{\ell_i,m_i}} L\mathcal{L}_{p,q} \xrightarrow{\text{ev}_0} \mathcal{L}_{p,q}$$

are transverse. But for each  $(\ell_i, m_i)$ , this composition is the identity on the lens space, so the maps are certainly transverse, and the locus of basepoint-intersections is the diagonal  $\Delta\mathcal{L}_{p,q} \subset \mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$ . The product is thus explicitly given by

$$\rho_{\ell_1,m_1} \wedge \rho_{\ell_2,m_2} = (\rho_{\ell_1,m_1} \star \rho_{\ell_2,m_2})|_{\Delta\mathcal{L}_{p,q}} : \mathcal{L}_{p,q} \equiv \Delta\mathcal{L}_{p,q} \longrightarrow L\mathcal{L}_{p,q}$$

for  $\star$  the concatenation of the loops in the image at their common basepoint. At each point  $(r, \theta)$  in  $\mathcal{L}_{p,q}$ , we are thus left to compute the concatenation  $\gamma_{r,\theta}^{\ell_1,m_1} \star \gamma_{r,\theta}^{\ell_2,m_2}$  which is exactly the image under the torus action of the concatenation of the loops  $(\ell_1, m_1)$  and  $(\ell_2, m_2)$  in the torus. This concatenation in the torus is homotopic to the loop  $(\ell_1 + \ell_2, m_1 + m_2)$  (corresponding to the fact that  $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ ) and hence the above product is homotopic the loop  $\gamma_{r,\theta}^{\ell_1+\ell_2,m_1+m_2}$ . As this homotopy originates in the torus, it defines a continuous homotopy over the lens space. It follows that the Chas-Sullivan product of such classes is as claimed.  $\square$

The coproduct of homology classes of degree 3 in  $L\mathcal{L}_{p,q}$  is a map

$$\vee : H_3(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q}) \longrightarrow H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q}).$$

For the classes  $\rho_{\ell,m}$ , it will given in terms of  $\beta$ -classes in the target, that we describe now.

Let  $\lambda : S^1 \rightarrow \mathcal{L}_{p,q}$  be the loop defined by  $\lambda(t) = ((1, \frac{t}{p}), 0)$ , tracing the points  $(r, \theta) \in \mathcal{L}_{p,q}$  with  $r_2 = 0$ . This is a generator of  $\pi_1\mathcal{L}_{p,q} \cong \mathbb{Z}/p$ . Note that  $\lambda = \gamma_{((1,0),0)}^{1,0}$  is the evaluation of the class  $\rho_{1,0}$  at  $((1, 0), 0) \in \mathcal{L}_{p,q}$ . In particular, it is freely homotopic to  $\gamma_{(0,(1,0))}^{1,0}$ , the evaluation of  $\rho_{1,0}$  at  $(0, (1, 0))$ , where we note that  $\gamma_{(0,(1,0))}^{1,0} = (\lambda')^{\star q}$  for  $\lambda' : S^1 \rightarrow \mathcal{L}_{p,q}$  defined by  $\lambda'(t) = (0, (1, \frac{t}{p}))$ , the loop tracing the points  $(r, \theta)$  with  $r_1 = 0$ .

The coproduct of the classes  $\rho_{\ell,m}$  will be given by applying the cut map to families of figure eights

$$\beta_{k,k'} : S^1 \rightarrow L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q},$$

based at the points of  $\lambda$  and defined by  $\beta_{k,k'}(t) = [s \mapsto ((1, \frac{t+ks}{p}), 0)] \star [s \mapsto ((1, \frac{t+k's}{p}), 0)]$ . We denote likewise by  $\beta_{k,k'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q})$  the associated homology class.

Just like the loop  $\lambda$  is freely homotopic to  $\lambda'^{\star q}$ , the family of figure eights  $\beta_{k,k'}$  is freely homotopic to a family loops with basepoints parametrized by  $\lambda'^{\star q}$ :

**Lemma 2.6.** *Let  $\beta'_{k,k'} : S^1 \rightarrow L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}$  be the family of figure eights based at the points of  $\lambda'$  defined by  $\beta'_{k,k'}(t) = [s \mapsto (0, (1, \frac{t+ks}{p}))] \star [s \mapsto (0, (1, \frac{t+k's}{p}))]$ . Then*

$$\beta_{k,k'} = q\beta'_{qk,qk'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}).$$

*Proof.* An explicit free homotopy  $\beta_{k,k'} \simeq_H \beta'_{qk,qk'}$  between the families of loops is given by setting  $H(\tau, t) = [s \mapsto ((\tau, \frac{t+ks}{p}), (1 - \tau, \frac{q(t+ks)}{p}))] \star [s \mapsto ((\tau, \frac{t+k's}{p}), (1 - \tau, \frac{q(t+k's)}{p}))]$ .  $\square$

**Lemma 2.7.** *We have that*

$$(1) \beta_{k,k'} = \beta_{h,h'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}) \text{ if and only if } k = h \pmod{p} \text{ and } k' = h' \pmod{p}.$$

(2) *The relative classes*

$$\{\beta_{k,k'}\}_{\substack{0 < k < p \\ 0 < k' < p}} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q})$$

are linearly independent over  $\mathbb{Z}_p$ .

*Proof.* The evaluation at 0 takes the family of figure eights  $\beta_{k,k'}$  to the generator  $\lambda$  of  $\pi_1(\mathcal{L}_{p,q}) \cong \mathbb{Z}/p$ . Hence the map  $H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}) \rightarrow H_1(\mathcal{L}_{p,q})$  projecting on the first component and evaluating at 0, takes  $\beta_{k,k'}$  to the generator of  $H_1(\mathcal{L}_{p,q})$ . In particular, each class  $\beta_{k,k'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q})$  is non-trivial.

Note now that  $\beta_{k,k'}$  has image in the component  $(k \bmod p, k' \bmod p)$  of the space  $L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}$ , as each loop  $[s \mapsto ((1, \frac{t+ks}{p}), 0)]$  is homotopic to  $\lambda^{*k}$ . Given that the classes are non-zero,  $\beta_{k,k'} = \beta_{h,h'}$  thus necessarily requires that  $k = h \bmod p$  and  $k' = h' \bmod p$ . The converse follows from the fact that any homotopy  $\lambda^{*p} \simeq *$  extends continuously over such a family of loops, using the residual torus action to push it along  $\lambda$ , proving that  $\beta_{k,k'} = \beta_{k+np, k'+mp}$  in homology for any  $n, m \in \mathbb{N}$ , which proves (1).

Finally, by the above,  $\beta_{k,k'}$  is non-zero in relative homology precisely when  $k$  and  $k'$  are not equal to 0 mod  $p$ , as  $\beta_{0,k'}$  and  $\beta_{k,0}$  being trivial in relative homology. And the classes are linearly independent as they live in different components.  $\square$

We are now ready to compute the coproduct of  $\rho$ -classes, where we will assume that  $\ell$  and  $m$  are positive for simplicity.

**Proposition 2.8.** *The coproduct of the class  $\rho_{\ell,m} \in H_3(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q})$  with  $\ell, m \geq 0$  is given by the formula*

$$\vee \rho_{\ell,m} = \sum_{\substack{0 < t < \ell \\ t, (\ell-t) \neq 0 \pmod p}} \beta_{t, \ell-t} + q' \sum_{\substack{0 < t < q\ell + pm \\ t, (\ell-tq') \neq 0 \pmod p}} \beta_{tq', \ell-tq'}$$

where  $q'$  is the multiplicative inverse of  $q$  mod  $p$ .

Using the previous lemma, one deduces that the coproduct of  $\rho$ -classes is non-trivial most of the time.

*Proof.* To compute the coproduct  $\vee \rho_{\ell,m}$  by geometric intersection applying Proposition 2.4, we need the map

$$\mathcal{L}_{p,q} \times (0, 1) \xrightarrow{\rho_{\ell,m} \times \text{id}} L\mathcal{L}_{p,q} \times (0, 1) \xrightarrow{e_I} \mathcal{L}_{p,q} \times \mathcal{L}_{p,q},$$

where  $e_I$  evaluates the loops at 0 and  $s \in (0, 1) \subset I$ , to be transverse to the diagonal embedding  $\Delta : \mathcal{L}_{p,q} \rightarrow \mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$  after removing the locus of constant loops. In the present case, either  $(\ell, m) = (0, 0)$  in which case all loops are constant, with  $\rho_{(0,0)} = 0$  in homology relative to the constant loops, or  $(\ell, m) \neq (0, 0)$  and the cycle has no constant loop in its image. So we can assume  $(\ell, m) \neq (0, 0)$  and work with the parametrizing pair  $(\Sigma, \Sigma_0) = (\mathcal{L}_{p,q}, \emptyset)$  for our relative cycle.

To achieve transversality, we represent the homology class of  $\rho_{\ell,m}$  by the homotopic family  $\tilde{\rho}_{\ell,m} : \mathcal{L}_{p,q} \rightarrow L\mathcal{L}_{p,q}$  defined by  $\tilde{\rho}_{\ell,m}(r, \underline{\theta}) = \tilde{\gamma}_{r, \underline{\theta}}^{\ell, m}$  for  $\tilde{\gamma}_{r, \underline{\theta}}^{\ell, m} : S^1 \rightarrow \mathcal{L}_{p,q}$  the loop based at  $(r, \underline{\theta})$  given by

$$\tilde{\gamma}_{r, \underline{\theta}}^{\ell, m}(t) = \left( (\tilde{r}_1(t), \theta_1 + \frac{\ell t}{p}), (\tilde{r}_2(t), \theta_2 + \frac{q\ell + pm}{p}t) \right),$$

where  $(\tilde{r}_1(t), \tilde{r}_2(t))$  is a deformation of  $(r_1, r_2)$  with  $(\tilde{r}_1(t), \tilde{r}_2(t)) = (r_1, r_2)$  only when  $r_1$  or  $r_2 = 0$ , or when  $t = 0$  or 1. Such a deformation can be obtained by e.g., interpolating back and forth between the identity on  $r_1$  at times  $t = 0$  and 1 and  $r_1^2$  at  $t = \frac{1}{2}$ , with  $\tilde{r}_2(t) = \sqrt{1 - \tilde{r}_1(t)^2}$ .

The map  $e_I \circ (\tilde{\rho}_{\ell,m} \times \text{id})|_{\mathcal{L}_{p,q} \times (0,1)}$  intersects the diagonal whenever a loop  $\tilde{\gamma}_{r, \underline{\theta}}^{\ell, m}$  has a self-intersection  $\tilde{\gamma}_{r, \underline{\theta}}^{\ell, m}(0) = \tilde{\gamma}_{r, \underline{\theta}}^{\ell, m}(t)$  for some  $t \in (0, 1)$ . Such self-intersections can only happen when  $r_1 = 0$  or  $r_2 = 0$ , as otherwise  $\tilde{r}_i(t) \neq r_i(0)$ , and when

$$\begin{cases} 0 < t = \frac{a}{\ell} < 1 & \text{if } r_2 = 0 \\ 0 < t = \frac{b}{q\ell + pm} < 1 & \text{if } r_1 = 0 \end{cases}$$

for some  $a, b \in \mathbb{N}$ . That is the locus of self-intersections of  $\tilde{\rho}_{\ell, m} \times \text{id} |_{\mathcal{L}_{p, q} \times (0, 1)}$  is

$$\Sigma_{\Delta} = \lambda \times I_1 \cup \lambda' \times I_2 \subset \mathcal{L}_{p, q} \times (0, 1)$$

for  $I_1 = \{\frac{1}{\ell}, \dots, \frac{\ell-1}{\ell}\}$  and  $I_2 = \{\frac{1}{q\ell+pm}, \dots, \frac{q\ell+pm-1}{q\ell+pm}\}$ , and  $\lambda, \lambda'$  the loops parametrizing the points with  $r_2 = 0$  and  $r_1 = 0$  respectively, as above.

We need to check that these self-intersections are transverse to the diagonal. This can be checked in local coordinates  $(r_1, \theta_1, \theta_2, t) = (z, \theta_2, t) \in \mathbb{C} \times \mathbb{R}^2$  around points with  $r_1 = 0$ , and similarly with coordinates  $(\theta_1, r_2, \theta_2, t) = (\theta_1, z, t)$  when  $r_2 = 0$ . In those coordinates, the function  $e_I \circ (\tilde{\rho}_{\ell, m} \times \text{id})$  has the form

$$(z, \theta, t) \mapsto ((z, \theta), (e^{2\pi i \alpha t} r(t) z, \theta + \beta t))$$

where  $r(t)$  is a function so that  $r(t) = 1$  only for  $t = 0, 1$ . It is transversal to all the diagonals

$$\Delta_k = (z, \theta, e^{2\pi i \frac{k}{p}} z, \theta + \frac{kq}{p}) \quad \text{resp.} \quad (z, \theta, e^{2\pi i \frac{k}{p}} z, \theta + \frac{kq'}{p})$$

because the zeros of the functions  $f_k(z, \theta, t) = ((e^{2\pi i \alpha t} r(t) - e^{2\pi i \frac{k}{p}})z, \beta t - \frac{kq'}{p})$  are transversal.

Indeed, away from  $t = 0, 1$  the factor  $(e^{2\pi i \alpha t} r(t) - e^{2\pi i \frac{k}{p}})$  is never zero, so, up to translation,  $f_k$  has the form  $f_k(z, \theta, t) = (a(t)z, \beta t)$  for  $0 \neq a(t) \in \mathbb{C}$  and  $\beta > 0$ , either equal to  $\frac{\ell}{p}$  or to  $\frac{q\ell+pm}{p}$ .

Applying Proposition 2.4, it now follows that the coproduct

$$\vee \tilde{\rho}_{\ell, m} = [\text{cut} \circ (\tilde{\rho}_{\ell, m} \times I) |_{\overline{\Sigma_{\Delta}}}]$$

where  $\overline{\Sigma_{\Delta}}$  is the closure inside  $\mathcal{L}_{p, q} \times I$  of  $\Sigma_{\Delta}$ , with  $\Sigma_{\Delta}$  oriented so that the isomorphism

$$T_{(r, \theta, t)}(\mathcal{L}_{p, q} \times I) \cong N\Delta\mathcal{L}_{p, q} \oplus T_{(r, \theta, t)}\Sigma_{\Delta},$$

coming from transversality, is orientation preserving.<sup>†</sup> Our computation above shows that  $\overline{\Sigma_{\Delta}} = \Sigma_{\Delta}$  is the disjoint union of circles  $\lambda \times I_1 \cup \lambda' \times I_2 \subset \mathcal{L}_{p, q} \times (0, 1)$ . Given that the sign depends on choices and conventions, we only give here the important part of the sign computation for us, namely that it is independent of  $t \in I_1 \cup I_2$ , and independent of  $\ell, m$ .

Orient  $T_{(r, \theta, t)}(\mathcal{L}_{p, q} \times I)$  around  $r_1 = 0$  as  $\mathbb{R}^4 \langle r_1, \theta_1, \theta_2, t \rangle$ . Then we have  $T_{(r, \theta, t)}(\mathcal{L}_{p, q} \times I) \cong -\mathbb{R}^3 \langle r_1, \theta_1, t \rangle \oplus T\Sigma_{\Delta} \langle \theta_2 \rangle$  at the intersections with  $r_1 = 0$ . Around  $r_2 = 0$ , we then have  $T_{(r, \theta, t)}(\mathcal{L}_{p, q} \times I) \cong \mathbb{R}^4 \langle r_2, \theta_2, \theta_1, t \rangle$  as  $r_2 = \sqrt{1 - r_1^2}$  is orientation preserving, and hence likewise  $T_{(r, \theta, t)}(\mathcal{L}_{p, q} \times I) \cong -\mathbb{R}^3 \langle r_2, \theta_2, t \rangle \oplus T\Sigma_{\Delta} \langle \theta_1 \rangle$ . And in local coordinate  $(z, \theta, t)$ , the map considered has the form  $(z, \theta, t) \mapsto ((z, \theta), (c(t)z, \theta + \beta))$ , independently of the point of  $\Sigma_{\Delta}$ .

Finally, we have that

$$\text{cut} \circ (\tilde{\rho}_{\ell, m} \times I) |_{\lambda \times I_1 \cup \lambda' \times I_2} = \left( \sum_{t=1}^{\ell-1} \beta_{t, \ell-t} + \sum_{t=1}^{q\ell+pm-1} \beta'_{t, q\ell+pm-t} \right)$$

as a family of pairs of loops. The result thus follows from Lemmas 2.6 and 2.7.  $\square$

**2.4. Homotopy invariance.** A diffeomorphism  $f : M \xrightarrow{\cong} N$  induces an isomorphism  $Lf_* : H_*(LM) \xrightarrow{\cong} H_*(LN)$ , and likewise for relative homology, that preserves both the loop product and coproduct, as all their defining ingredients are identified by diffeomorphisms. It is natural to ask whether only assuming that  $f$  is a homotopy equivalence could be enough for the induced isomorphism  $Lf_*$  to preserve the loop product and coproduct. Note that if  $f$  satisfies the even weaker assumption of being a degree 1 map, then  $f_* : H_*(M) \rightarrow H_*(N)$  already preserves the intersection product, see eg. [10, VI, Proposition 14.2].

The following two results show that the answer to the above question is yes for the product, and no for the coproduct.

**Theorem 2.9.** [25] (see also [26, 40, 30]) *Let  $f : M \rightarrow N$  be a degree 1 homotopy equivalence between two closed oriented manifolds. Then  $Lf_* : H_*(LM) \rightarrow H_*(LN)$  is an isomorphism of algebras with respect to the Chas-Sullivan product.*

<sup>†</sup>In our conventions,  $N\Delta M$  is oriented so that  $\tau_M \cap [M \times M] = [M]$ , for  $\tau_M$  the corresponding Thom class.

The main ingredient of the proof of this theorem is sketched in Section 4.3 (see Theorem 4.11), where we will revisit the question of invariance of the loop product and coproduct after going through a deeper analysis of their defining ingredients.

In the meanwhile, as noted by the first author in [66], the computations presented in Section 2.3 can already be used to show that the loop coproduct is not homotopy invariant:

**Theorem 2.10.** [66] *Let  $f : \mathcal{L}_{7,1} \rightarrow \mathcal{L}_{7,2}$  be a homotopy equivalence and  $\rho_{1,0} \in H_3(L\mathcal{L}_{7,1})$  be as in Section 2.3. Then*

$$0 = f(\vee(\rho_{1,0})) \neq \vee(f(\rho_{1,0})) \in H_1(L\mathcal{L}_{7,2} \times L\mathcal{L}_{7,2}, \mathcal{L}_{7,2} \times L\mathcal{L}_{7,2} \cup L\mathcal{L}_{7,2} \times \mathcal{L}_{7,2}).$$

*In particular, the loop coproduct  $\vee$  is not preserved by  $f$ .*

The lens spaces  $\mathcal{L}_{7,1}$  and  $\mathcal{L}_{7,2}$  are the simplest examples of lens spaces that are homotopy equivalent, but not simple homotopy equivalent. They were also used in [58] to prove that the configuration space of two points in a manifold is not a homotopy invariant of the manifold. In Section 4.2, we will see that the same configuration of two points plays an important role in the definition of the loop coproduct.

*Proof.* The class  $\rho_{1,0} \in H_3(L\mathcal{L}_{7,1})$  has trivial coproduct by Proposition 2.8 as  $\ell = q = 1$  and  $m = 0$ . (This also follows, using [43, Theorem 3.10], from the fact that  $\rho_{1,0}$  is a family of simple loops whenever  $q = 1$ .)

We need to compute the coproduct of the image  $f(\rho_{1,0})$ . The free loop space  $L\mathcal{L}_{7,q}$  has 7 components, and each component  $L_\ell\mathcal{L}_{7,q}$  has  $H_3(L_\ell\mathcal{L}_{7,q}) \cong \mathbb{Z} \oplus \mathbb{Z}/7$  (see [66, Sec 2.1]). From Lemma 2.7 and Proposition 2.8, one can deduce that e.g. the classes  $\rho_{\ell,0}$  and  $\rho_{\ell,1}$  generate  $H_3(L_\ell\mathcal{L}_{p,q})$ . Now [65, Lemma 6.9] tells us that, because  $f$  is a homotopy equivalence,  $\rho_{1,0}$  has image in  $L_\ell\mathcal{L}_{7,2}$  for  $\ell = 2$  or  $5$ . Hence  $f(\rho_{1,0}) = a\rho_{\ell,0} + (1-a)\rho_{\ell,1}$  for some  $a \in \{0, \dots, p-1\}$ , with  $\ell = 2$  or  $5$ . Now Proposition 2.8 for  $\ell = 2$  shows that  $\vee\rho_{2,0} = 5\beta_{1,1} + 4(\beta_{4,5} + \beta_{5,4})$  while  $\vee\rho_{2,1} = 2\beta_{1,1} + \beta_{4,5} + \beta_{5,4} + 4(\beta_{3,6} + \beta_{6,3})$ . And one checks readily that there is no such  $a$  such that  $\vee(a\rho_{\ell,0} + (1-a)\rho_{\ell,1}) = 0$ . A similar computation rules out the possibility in the case  $\ell = 5$ .  $\square$

Combining the invariance of the corresponding (co)product in algebra (see Theorem 3.19), with the fact that the algebraic model indeed models the loop coproduct (see Theorem 4.2), it follows that, when working over real coefficients and with simply connected manifolds, the coproduct is homotopy invariant, as stated in Theorem E. On the other hand, the above computation can be extended to show the following:

**Theorem 2.11.** *A degree 1 homotopy equivalence  $f : \mathcal{L}_{p,q_1} \rightarrow \mathcal{L}_{p,q_2}$  between two 3-dimensional spaces such that  $Lf_* : H_*(L\mathcal{L}_{p,q_1}, \mathcal{L}_{p,q_1}) \rightarrow H_*(L\mathcal{L}_{p,q_2}, \mathcal{L}_{p,q_2})$  preserves the loop coproduct of degree 3 classes is homotopic to a homeomorphism.*

*Proof.* Suppose  $f$  is such a homotopy equivalence. Let  $\rho_{1,0} \in H_3(L\mathcal{L}_{p,q_1})$  be as above. We will just as above compare  $f(\vee(\rho_{1,0}))$  with  $\vee(f(\rho_{1,0}))$ .

The class  $f(\rho_{1,0})$  lies in  $H_3(L_\ell\mathcal{L}_{p,q_1})$  for some  $\ell$  satisfying  $q_1 = \ell^2 q_2 \pmod{p}$ , because  $f$  is a degree 1 homotopy equivalence, with  $f$  inducing multiplication by  $\ell$  on  $\pi_1$ , see e.g. [65, Theorem 6.11], where  $0 < q_1, q_2, \ell < p$ . We want to show that  $f$  is homotopic to a homeomorphism. By [65, Lemma 6.8], it is enough to check that the two lens spaces are homeomorphic, which happens precisely if either  $q_1 q_2 = \pm 1 \pmod{p}$  or  $q_1 = \pm q_2 \pmod{p}$ , see [65, Theorem 1.3]. We may assume without loss of generality that  $q_2 \neq 1$ .

To avoid confusion, denote by  $\bar{\rho}_{\ell,0}, \bar{\rho}_{\ell,1} \in H_3(L\mathcal{L}_{p,q_2})$  the classes in the second lens space, and likewise for the  $\beta$ -classes. As argued above for  $\mathcal{L}_{7,2}$ , we have that  $\bar{\rho}_{\ell,0}, \bar{\rho}_{\ell,1}$  generate  $H_3(L_\ell\mathcal{L}_{p,q_2})$ , so we know that  $f(\rho_{1,0}) = (1-a)\bar{\rho}_{\ell,0} + a\bar{\rho}_{\ell,1}$  for some  $a \in \{0, \dots, p-1\}$ .

The answer of the computations  $f(\vee(\rho_{1,0}))$  and  $\vee(f(\rho_{1,0}))$  will be in terms of the classes  $\bar{\beta}_{k,\ell-k} \in H_1(L\mathcal{L}_{p,q_2} \times L\mathcal{L}_{p,q_2})$  (or the corresponding relative homology group). As these classes only depends on the parameter  $k$ , we will denote them by  $[k]$  below, for better readability. Note also that  $f(\beta_{k,1-k}) = \ell\bar{\beta}_{\ell k, \ell-\ell k}$  as  $f$  is multiplication by  $\ell$  on  $\pi_1$ .

From our computation above, we have that

$$\vee \rho_{1,0} = q'_1 \sum_{0 < t < q_1} \beta_{tq'_1, 1-tq'_1}$$

so in the above notation,

$$f(\vee \rho_{1,0}) = \ell q'_1 \sum_{0 < t < q_1} [t\ell q'_1] = \ell' q'_2 \sum_{0 < t < q_1} [t\ell' q'_2]$$

using that  $\ell q'_1 = \ell' q'_2$  for the second equality. On the other hand,

$$\begin{aligned} \vee((1-a)\bar{\rho}_{\ell,0} + a\bar{\rho}_{\ell,1}) &= \sum_{0 < t < \ell} [t] + q'_2 \sum_{\substack{0 < t < q_2\ell \\ t, (\ell-tq'_2) \neq 0 \pmod p}} [tq'_2] + aq'_2 \sum_{\substack{q_2\ell \leq t < q_2\ell+p \\ t, (\ell-tq'_2) \neq 0 \pmod p}} [tq'_2] \\ &= \sum_{0 < t < \ell} [t] + q'_2 \sum_{0 < t < d} [tq'_2] + (a+c)q'_2 \sum_{0 < t < p} [t] \end{aligned}$$

where  $q_2\ell = cp + d$ , for some  $0 < d < p$ .

The equality  $\vee(f(\rho_{1,0})) - f(\vee(\rho_{1,0})) = 0$  holds precisely if all possible terms  $[s]$  appear with coefficient a multiple of  $p$ . A necessary condition for that is that the terms  $[s]$  all appear in

$$\sum_{0 < t < \ell} [t] + q'_2 \sum_{0 < t < d} [tq'_2] - \ell' q'_2 \sum_{0 < t < q_1} [t\ell' q'_2]$$

with the same total coefficient. Consider the sets

$$\begin{aligned} A &= \{t \mid 0 < t < \ell\} \\ B &= \{tq'_2 \mid 0 < t < d\} \\ C &= \{t\ell' q'_2 \mid 0 < t < q_1\}. \end{aligned}$$

*Case 1:*  $A = \emptyset$ , or equivalently  $\ell = 1$ . Then  $q_1 = \ell^2 q_2 = q_2 \pmod p$  and  $f$  is a homeomorphism.

*Case 2:*  $B = \emptyset$ . Then  $\ell q_2 = 1 \pmod p$  (as  $d = 1$ ), so that  $q_1 = \ell^2 q_2 = \ell$ . But then  $q_1 q_2 = 1 \pmod p$ , which also gives that  $f$  is a homeomorphism.

*Case 3:*  $C = \emptyset$  with  $A, B \neq \emptyset$ . So  $q_1 = 1$ , and either  $A = B$  or  $A = B^c$ . If  $A = B$ , then  $\ell = d = \ell q_2 \pmod p$ , giving  $q_2 = 1$ . If  $A = B^c$ , we would need  $q'_2 = 1$  for the coefficients to agree. In both cases, this contradicts our assumption that  $q_2 \neq 1$ .

*Case 4:*  $A, B, C \neq \emptyset$ . If the sets are disjoint, we need the three coefficients to be equal, giving in particular  $q'_2 = 1$  which contradicting again  $q_2 \neq 1$ . The case  $A = B$  is ruled out above. If  $A = C$ , then  $\ell = q_1 = \ell^2 q_2 \pmod p$ , giving  $\ell q_2 = 1 \pmod p$ , i.e.  $d = 1$  contradicting that  $B$  is non-empty. And if  $B = C$ ,  $q_2\ell = q_1 = \ell^2 q_2 \pmod p$ , giving  $\ell = 1$ , contradicting that  $A \neq \emptyset$ . We are now left with the case when all three sets intersect, but none are equal. In that case, we need all sums of coefficients to agree:  $1 + q'_2 = 1 - \ell' q'_2 = q'_2 - \ell' q'_2$  modulo  $p$ , implying in particular  $q'_2 = 1 \pmod p$ , again a contradiction.  $\square$

**2.5. The good and the bad coproduct.** The coproduct we have described looks for self-intersections of the form  $\gamma(t) = \gamma(0)$  in families of loops  $\gamma$  where  $t \in I$  is any time along the interval. One could instead define a coproduct  $\vee_{\frac{1}{2}}$  that only looks for self-intersections at time  $t = \frac{1}{2}$ . This leads to a rather trivial coproduct, as noted by Tamanoi in [76]. Indeed, the coproduct  $\vee_{\frac{1}{2}}$  is homotopic to the coproduct  $\vee_0$  that looks for (“left-trivial”) self-intersections at  $t = 0$ , i.e. of the form  $\gamma(0) = \gamma(0)$ , or likewise to the coproduct  $\vee_1$  looking for (“right-trivial”) self-intersection at  $t = 1$  only. Combining these two facts can be used to show that the coproduct  $\vee_{\frac{1}{2}}$  is only non-trivial on the fundamental class of  $[M]$ , considered as a family of constant loops, and only when  $\chi(M) \neq 0$ , with  $\vee_{\frac{1}{2}}[M] = (-1)^n \chi(M) [\{*\} \times \{*\}] \in H_0(LM \times LM)$  (see e.g., [43, Lemma 4.5]). In fact, the “good” coproduct  $\vee$  that we have worked with here can be thought of as a secondary operation, coming from these two reasons that  $\vee_{\frac{1}{2}}$  is trivial, homotoping it to its  $t = 0$  or  $t = 1$  versions.

One way to formulate this relationship between the two coproduct is as follows: the coproduct  $\vee$  can be defined as a *relative version* of the coproduct  $\vee_{\frac{1}{2}}$ , as we explain now. This form of definition first appeared in [38, Section 9], in the definition of the dual cohomology product.

The coproduct  $\vee_{\frac{1}{2}}$  is defined just like  $\vee$  but replacing the evaluation  $e_I$  by the map  $\text{ev}_{0, \frac{1}{2}} = (\text{ev}_0, \text{ev}_{\frac{1}{2}}) : LM \rightarrow M \times M$ . Denoting  $\text{Fig}(8) = \text{ev}_{0, \frac{1}{2}}^{-1}(\Delta M) \subset LM$  is the space of “figure eights”, i.e. loops  $\gamma$  with a self-intersection  $\gamma(0) = \gamma(\frac{1}{2})$ , and  $U_\epsilon(\text{Fig}(8)) = \text{ev}_{0, \frac{1}{2}}^{-1}(U_M)$  its  $\epsilon$ -neighborhood, we have

$$\vee_{\frac{1}{2}} : H_*(LM) \xrightarrow{(\text{ev}_{0, \frac{1}{2}})^* \tau_M \cap} H_*(U_\epsilon(\text{Fig}(8))) \xrightarrow{R_{\frac{1}{2}}} H_*(\text{Fig}(8)) \xrightarrow{\text{cut}} H_*(LM \times LM),$$

for  $R_{\frac{1}{2}}$  a retraction map defined just like the retraction map  $R$  used for  $\vee$ .

Let  $J : LM \times I \rightarrow LM$  be the reparametrizing map defined by  $J(\gamma, s) = \gamma \circ \theta_{\frac{1}{2} \rightarrow s}$  where  $\theta_{\frac{1}{2} \rightarrow s} : [0, 1] \rightarrow [0, 1]$  is the piecewise linear map that fixes 0 and 1 and takes  $\frac{1}{2}$  to  $s$ . Note that  $J$  restricts on the boundary to a map  $J : LM \times \partial I \rightarrow \mathcal{R}$  for

$$(2.7) \quad \mathcal{R} := \{\gamma \in LM \mid \gamma|_{[0, \frac{1}{2}]} \text{ or } \gamma|_{[\frac{1}{2}, 1]} \text{ is constant}\}$$

the subspace of  $LM$  of *half-constant loops*.

**Proposition 2.12.** *The loop coproduct  $\vee$  can equivalently be defined as the composition of the following sequence of maps:*

$$(2.8) \quad H_*(LM, M) \xrightarrow{\times I} H_{*+1}(LM \times I, LM \times \partial I \cup M \times \partial I) \xrightarrow{J} H_*(LM, \mathcal{R}) \\ \xrightarrow{(\text{ev}_{0, \frac{1}{2}})^* \tau_M \cap} H_*(U_\epsilon(\text{Fig}(8)), \mathcal{R}) \xrightarrow{R_{\frac{1}{2}}} H_*(\text{Fig}(8), \mathcal{R}) \xrightarrow{\text{cut}} H_*(LM \times LM, M \times LM \cup LM \times M).$$

See [43, Theorem 2.13] for a proof that this new definition is equivalent to the one of Section 2.2. Note that the last three maps in the statement indeed compose to a relative version of the coproduct  $\vee_{\frac{1}{2}}$ .

### 3. STRING TOPOLOGY VIA HOCHSCHILD COMPLEXES

In this section we define a product on the Tate-Hochschild complex of any connected dg Frobenius algebra  $A$ . The Tate-Hochschild complex is an amalgam of the Hochschild chains and cochains, that model, by results of Jones and Chen, the cohomology and homology of the free loop space of simply connected manifolds, respectively. We will see below and in Section 4 that, in this case, the product on the Tate-Hochschild complex relates to both the Chas-Sullivan product, when restricted to the Hochschild cochains, and the Goresky-Hingston coproduct, when restricted to the Hochschild chains.

**3.1. Differential graded algebras.** Let  $\mathbb{K}$  be a commutative ring with unit. Recall that a *dg  $\mathbb{K}$ -module*, or *chain complex*, is a graded  $\mathbb{K}$ -module  $V = \bigoplus_{j \in \mathbb{Z}} V^j$  equipped with a differential  $d_V : V \rightarrow V$ ; in this section, all differentials will have degree  $+1$ . The *dual* of  $(V, d_V)$  is the dg  $\mathbb{K}$ -module  $(V^\vee, d_{V^\vee})$  with  $(V^\vee)^{-j} = \text{Hom}_{\mathbb{K}}(V^j, \mathbb{K})$  and the differential defined by  $d_{V^\vee}(\alpha)(x) = -(-1)^{|\alpha|} \alpha(d_V(x))$  on homogeneous elements  $\alpha \in V^\vee$ , where  $|\alpha|$  denotes the degree of  $\alpha$ .

A *dg  $\mathbb{K}$ -algebra*  $A = (A, d, \mu)$ , or *dg-algebra* for short, is a dg  $\mathbb{K}$ -module  $(A, d)$  equipped with an associative product  $\mu : A \otimes A \rightarrow A$  of degree zero satisfying the Leibniz rule

$$\mu \circ (d \otimes \text{id} + \text{id} \otimes d) = d \circ \mu.$$

We write  $\mu(a \otimes b) = ab$ . The multiplication is (graded) commutative if  $ab = (-1)^{|a||b|}ba$ , and *unital* if there is a map  $u : \mathbb{K} \rightarrow A$  such that the image of  $1 \in \mathbb{K}$  is a unit for the multiplication of  $A$ .

The *cohomology*  $H^*(A)$  of a dg algebra  $A = (A, d, \mu)$  becomes a graded  $\mathbb{K}$ -algebra with product  $H^*(A) \otimes H^*(A) \rightarrow H^*(A)$  induced by  $\mu : A \otimes A \rightarrow A$ . A morphism of dg algebras  $f : A \rightarrow A'$  is a *quasi-isomorphism* if it induces an isomorphism of graded algebras  $H^*(f) : H^*(A) \xrightarrow{\cong} H^*(A')$ .



**Example 3.1.** The following examples are particularly relevant to our discussion:

- (1) The singular cochains on a topological space  $X$  equipped with the simplicial differential and cup product define a dg algebra  $(C^*(X; \mathbb{K}), d, \smile)$ . The cup product is associative and homotopy commutative.
- (2) When  $\mathbb{K} = \mathbb{Q}$  the dg algebra  $(C^*(X; \mathbb{Q}), d, \smile)$  is quasi-isomorphic to a *commutative* dg algebra  $(\mathcal{A}_{\text{pl}}(X), d, \wedge)$  of  $\mathbb{Q}$ -polynomial differential forms, as shown by Sullivan.

One of the main theorems discussed in this note, Theorem 3.17, involves the weaker notion of an  $A_\infty$ -algebra. Recall that an  $A_\infty$ -algebra is a graded  $\mathbb{K}$ -module  $A$  equipped with linear maps  $\{m_n : A^{\otimes n} \rightarrow A\}_{n \in \mathbb{Z}_{>0}}$ , where each  $m_n$  is of degree  $2 - n$ , satisfying the following relations:

- $m_1 \circ m_1 = 0$ , in other words,  $(A, m_1)$  is a dg  $\mathbb{K}$ -module,
- $m_1 \circ m_2 = m_2 \circ (m_1 \otimes \text{id}_A + \text{id}_A \otimes m_1)$ , in other words, the product  $m_2$  satisfies Leibniz rule with respect to  $m_1$ ,
- more generally, for each positive integer  $n$  we have

$$\sum (-1)^{p+qr} m_{p+1+r} \circ (\text{id}_A^{\otimes p} \otimes m_q \otimes \text{id}_A^{\otimes r}) = 0,$$

where the sum runs over all triples of positive integers  $(p, q, r)$  such that  $n = p + q + r$ .

In particular, the last equation implies that  $m_3 : A^{\otimes 3} \rightarrow A$  is a chain homotopy for the associativity of  $m_2$ . Hence, for any  $A_\infty$ -algebra  $A$ , the cohomology  $H^*(A, m_1)$  has an induced graded associative algebra structure.

**3.2. Differential graded Frobenius algebras.** The notion of a symmetric dg Frobenius algebra consists of a dg algebra equipped with a non-degenerate symmetric bilinear pairing compatible with the product structure. Our interest in symmetric dg Frobenius algebras is motivated by Poincaré duality.

**Definition 3.2.** A dg Frobenius  $\mathbb{K}$ -algebra of dimension  $n$  is a non-negatively graded unital dg  $\mathbb{K}$ -algebra  $(A, d, \mu)$  equipped with a pairing  $\langle -, - \rangle : A \otimes A \rightarrow \mathbb{K}$  such that

- (1)  $\langle -, - \rangle$  is of degree  $-n$ , i.e. non-zero only on  $A^i \otimes A^{n-i}$  for  $i = 0, \dots, n$
- (2)  $\langle -, - \rangle$  is non-degenerate, namely, the induced map

$$\rho : A \rightarrow A^\vee, \quad a \mapsto (b \mapsto \langle a, b \rangle)$$

is an isomorphism of degree  $-n$

- (3)  $\langle ab, c \rangle = \langle a, bc \rangle$  for any  $a, b, c \in A$
- (4)  $\langle d(a), b \rangle = -(-1)^{|a|} \langle a, d(b) \rangle$  for any  $a, b \in A$ .

Conditions (3) and (4) imply that  $\rho : A \rightarrow A^\vee$  is a map of dg  $A$ - $A$ -bimodules of degree  $-n$ , where the  $A$ - $A$ -bimodule structure on  $A^\vee$  is given by

$$(a \otimes b) \cdot \beta(c) = (-1)^{|\beta|(|a|+|b|)+|a|(|b|+|c|)} \beta(bca), \quad \text{for any } \beta \in A^\vee \text{ and } a, b, c \in A.$$

A dg Frobenius algebra  $A$  is said to be *symmetric* if  $\langle a, b \rangle = (-1)^{|a||b|} \langle b, a \rangle$  for any  $a, b \in A$ .

Note that the isomorphism  $\rho : A \rightarrow A^\vee$  gives rise to a degree  $n$  product on  $A^\vee$ :

$$A^\vee \otimes A^\vee \xrightarrow{\rho^{-1} \otimes \rho^{-1}} A \otimes A \xrightarrow{\mu} A \xrightarrow{\rho} A^\vee.$$

When  $A$  is a finitely generated free  $\mathbb{K}$ -module, e.g. when  $\mathbb{K}$  is a field, the linear dual of that product becomes a coproduct on  $A$ :

$$(3.1) \quad \Delta : A \xrightarrow{\rho} A^\vee \xrightarrow{\mu^\vee} (A \otimes A)^\vee \cong A^\vee \otimes A^\vee \xrightarrow{\rho^{-1} \otimes \rho^{-1}} A \otimes A$$

This coproduct is a map of  $A$ - $A$ -bimodules.

**Example 3.3** (Poincaré duality and relationship to the intersection product). Let  $M$  be a closed manifold of dimension  $n$ . The graded cohomology ring  $(H^*(M; \mathbb{K}), \smile)$  with coefficients in the commutative ring  $\mathbb{K}$  is an example of a symmetric dg Frobenius algebra of dimension  $n$  with trivial differential  $d = 0$  and pairing given by Poincaré duality. When  $\mathbb{K}$  is a field, the coproduct  $\Delta : H^*(M; \mathbb{K}) \rightarrow H^*(M; \mathbb{K}) \otimes H^*(M; \mathbb{K})$  is the dual of the intersection product on

homology. Indeed, the cup product is induced by the diagonal  $\Delta_M^* : H^*(M \times M) \rightarrow H^*(M)$ , so, following (3.1), the coproduct  $\Delta : H^*(M; \mathbb{K}) \rightarrow H^*(M; \mathbb{K}) \otimes H^*(M; \mathbb{K})$  is the composition

$$\begin{array}{ccc} H^k(M; \mathbb{K}) & \xrightarrow{\Delta} & \bigoplus_{i+j=n-k} H^{n-i}(M; \mathbb{K}) \otimes H^{n-j}(M; \mathbb{K}) \\ \downarrow [M] \cap \cong & & \cong \downarrow [M] \cap [M] \cap \\ H_{n-k}(M; \mathbb{K}) & \xrightarrow{(\Delta_M)^*} & H_{n-k}(M \times M; \mathbb{K}) \xleftarrow{\cong} \bigoplus_{i+j=n-k} H_i(M; \mathbb{K}) \otimes H_j(M; \mathbb{K}). \end{array}$$

And the intersection product can be defined as the Poincaré dual of the cup product, i.e. precisely the linear dual of the cohomology coproduct. (See e.g., [43, App B] for the relationship between that definition of the intersection product and the one given in Section 2.1.)

Applying the above composition of maps to  $1 \in H^0(M; \mathbb{K}) \cong \mathbb{K}$  we get a class  $\Delta(1) \in \bigoplus_{i+j=n} H_i(M; \mathbb{K}) \otimes H_j(M; \mathbb{K}) \cong H_n(M \times M; \mathbb{K})$  that is the uniquely such that  $[M \times M] \cap \Delta(1) = (\Delta_M)_*[M]$ . Hence  $\Delta(1)$  maps to the Thom class  $\tau_M \in H^n(M \times M, M \times M \setminus M)$  of Section 2.1 in relative cohomology, as the Thom class is determined by this same relation.

A dg algebra  $A$  is *simply connected* if it is non-negatively graded,  $A^0 = \mathbb{K}$ , and  $A^1 = 0$ . The following result of Lambrecht and Stanley shows that, when  $\mathbb{K}$  is a field and  $A$  is commutative and simply connected, a Frobenius structure on  $H^*(A)$  can be “lifted” to  $A$ .

**Theorem 3.4.** [56, Theorem 1.1] *Let  $\mathbb{K}$  be any field and  $\mathcal{A}$  be a simply connected commutative dg  $\mathbb{K}$ -algebra equipped with a pairing  $\langle -, - \rangle_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{K}$  which induces a graded Frobenius algebra structure of dimension  $k$  on its cohomology  $H^*(\mathcal{A})$ . Then there exists a simply connected commutative symmetric dg Frobenius  $\mathbb{K}$ -algebra  $A$  and a zig-zag of quasi-isomorphisms of commutative dg algebras between  $A$  and  $\mathcal{A}$  inducing an isomorphism  $H^*(A) \cong H^*(\mathcal{A})$  of graded Frobenius algebras.*

**Example 3.5** (Frobenius models of manifolds). Let  $M$  be a simply-connected oriented closed manifold and assume  $\mathbb{K} = \mathbb{Q}$ . Then the polynomial forms  $\mathcal{A}_{\text{pl}}(M) \simeq C^*(M, \mathbb{Q})$  are a strictly commutative, simply connected model of the cochains. The above theorem then yields a commutative dg Frobenius algebra  $A_M \simeq C^*(M, \mathbb{Q})$ , that “lifts” the graded Frobenius structure of  $H^*(M; \mathbb{Q})$  to the cochain level.

**3.3. Hochschild chains and cochains.** We recall here the definition of the Hochschild chain and cochain complexes and their relevance in homological algebra and topology. We will work with the normalized version of the Hochschild complex, assuming that the algebra is unital. Let  $\bar{A}$  denotes the cokernel of the unit map  $\mathbb{K} \rightarrow A$ .

For any dg  $\mathbb{K}$ -module  $(V, d)$  we denote by  $(s^i V, s^i d)$  the  $i$ -th shifted module given by  $(s^i V)^j = V^{i+j}$  and  $s^i d(v) = (-1)^i d(s^i v)$  for any  $v \in V$ . The definition of the Hochschild complex will use the suspension  $s\bar{A}$ . For simplicity, we write  $\bar{a}$  for the element  $sa \in s\bar{A}$  where  $a \in \bar{A}$ .

**Definition 3.6.** Let  $A$  be an unital dg algebra. The *Hochschild chain complex* of  $A$  is the complex  $(C_*(A, A), \partial = \partial_v + \partial_h)$  where

$$C_*(A, A) = \bigoplus_{m \geq 0} (s\bar{A})^{\otimes m} \otimes A$$

and where  $\partial_v$ , the *vertical* differential, is given by

$$\begin{aligned} \partial_v(\bar{a}_1 \otimes \cdots \otimes \bar{a}_m \otimes a_{m+1}) &= - \sum_{i=1}^m (-1)^{\epsilon_{i-1}} \bar{a}_1 \otimes \cdots \otimes \bar{a}_{i-1} \otimes \overline{d(a_i)} \otimes \bar{a}_{i+1} \otimes \cdots \otimes a_{m+1} \\ &\quad + (-1)^{\epsilon_m} \bar{a}_1 \otimes \cdots \otimes \bar{a}_m \otimes d(a_{m+1}) \end{aligned}$$

and  $\partial_h$ , the *horizontal* differential, is given by

$$\begin{aligned} \partial_h(\overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}) &= \sum_{i=1}^{m-1} (-1)^{\epsilon_i} \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes a_{m+1} \\ &\quad - (-1)^{\epsilon_{m-1}} \overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}} \otimes a_m a_{m+1} \\ &\quad + (-1)^{(|a_2| + \cdots + |a_{m+1}| - m + 1)|a_1|} \overline{a_2} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} a_1. \end{aligned}$$

Here we denote  $\epsilon_i = |a_1| + \cdots + |a_i| - i$  and  $\epsilon_0 = 0$ .

We will denote by  $C_{-m,k}(A, A) = ((s\overline{A})^{\otimes m} \otimes A)^k$  the elements in  $(s\overline{A})^{\otimes m} \otimes A$  of total degree  $k$ . In particular,  $C_k(A, A) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} C_{-m,k}(A, A)$

The Hochschild homology of  $A$  is defined to be the homology of  $(C_*(A, A), \partial = \partial_v + \partial_h)$  and it is denoted by  $HH_*(A, A)$ . Hochschild homology is functorial with respect to maps of unital dg algebras. Furthermore, a quasi-isomorphism  $f : A \rightarrow A'$  between unital dg algebras that are flat as  $\mathbb{K}$ -modules induces an isomorphism

$$HH_*(f) : HH_*(A, A) \rightarrow HH_*(A', A').$$

**Remark 3.7** (The Hochschild complex in algebra and topology). The Hochschild chain complex originates in the context of homological algebra. When  $A$  is a dg algebra which is projective as a  $\mathbb{K}$ -module,  $C_*(A, A)$  is model for  $A \otimes_{A \otimes A^{op}}^{\mathbb{L}} A$ , the derived tensor product of  $A$  with itself in the category of  $A$ - $A$ -bimodules. Hence,  $HH_*(A, A) = \text{Tor}_*^{A \otimes A^{op}}(A, A)$ .

In topology, when  $\mathbb{K} = \mathbb{F}$  is a field, and  $A \simeq C^*(X; \mathbb{F})$  is a dg algebra cochain model for the singular cochains of a simply connected space  $X$ , then there is a quasi-isomorphism  $C_*(A, A) \simeq C^*(LX; \mathbb{F})$  between the Hochschild chains of  $A$  and the singular cochains of the free loop space of  $X$ . This relationship may be deduced over the reals using Chen iterated integrals (as introduced by Chen in [15], see also [33, 64]), or over any field using a cosimplicial model for the free loop space (as done by Jones in [47]). A dual version of the result, in terms of the coHochschild complex of the singular chains coalgebra, that works for coefficients in an arbitrary ring  $\mathbb{K}$  may be found in [71].

Goodwillie gave in [37] the following ‘‘Koszul dual’’ version of this model of the free loop space that does not assume simple connectivity. Let  $\mathbb{K}$  be any commutative ring and assume  $X$  is a path-connected and set instead  $A = C_*(\Omega X; \mathbb{K})$ , the singular chains on the space of (Moore) loops in  $X$ , equipped with the concatenation product. Then there is a quasi-isomorphism  $C_*(A, A) \simeq C_*(LX; \mathbb{K})$ .

**Definition 3.8.** Let  $A$  be a unital dg algebra. The *Hochschild cochain complex* of  $A$  is the complex  $(C^*(A, A), \delta = \delta^v + \delta^h)$  where

$$C^*(A, A) = \prod_{m \geq 0} \text{Hom}_{\mathbb{K}}((s\overline{A})^{\otimes m}, A)$$

and where  $\delta^v$  is given by

$$\delta^v(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) = d(f(\overline{a_1} \otimes \cdots \otimes \overline{a_m})) + \sum_{i=1}^m (-1)^{|f| + \epsilon_{i-1}} f(\overline{a_1} \otimes \cdots \otimes \overline{d(a_i)} \otimes \cdots \otimes \overline{a_m}),$$

and  $\delta^h$  by

$$\begin{aligned} \delta^h(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}}) &= - (-1)^{(|a_1| - 1)|f|} a_1 f(\overline{a_2} \otimes \cdots \otimes \overline{a_{m+1}}) \\ &\quad - \sum_{i=1}^m (-1)^{|f| + \epsilon_i} f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_i} \otimes \cdots \otimes \overline{a_{m+1}}) \\ &\quad + (-1)^{|f| + \epsilon_m} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) a_{m+1}, \end{aligned}$$

with  $\epsilon_i = |a_1| + \cdots + |a_i| - i$  and  $\epsilon_0 = 0$  as before.

Denoting by  $C^{m,k}(A, A) = \text{Hom}_{\mathbb{K}}^n((s\overline{A})^{\otimes m}, A)$  the submodule of  $\mathbb{K}$ -linear maps of degree  $k \in \mathbb{Z}$ , we have  $C^k(A, A) = \prod_{m \geq 0} C^{m,k}(A, A)$ .

The Hochschild cohomology  $\mathrm{HH}^*(A, A)$  of  $A$  is defined to be the cohomology of  $(C^*(A, A), \delta = \delta_v + \delta_h)$ . The Hochschild cochain complex is not at such natural in maps of dg algebras, but if  $f : A \rightarrow A'$  is a quasi-isomorphism of unital dg algebras that are flat as  $\mathbb{K}$ -modules, then there is an isomorphism  $\mathrm{HH}^*(A, A) \cong \mathrm{HH}^*(A', A')$ . We will see in the next section that the product structure of Hochschild cohomology is also invariant under quasi-isomorphisms.

**Remark 3.9.** When  $A$  is projective as a  $\mathbb{K}$ -module, the complex  $C^*(A, A)$  is a model for  $\mathbb{R}\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A)$ , the derived hom from  $A$  to itself in the category of  $A$ - $A$ -bimodules. Hence,  $\mathrm{HH}^*(A, A) = \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^*(A, A)$ . One may also model the Yoneda product  $\mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^*(A, A)$  via the chain level cup product  $\cup$  on  $C^*(A, A)$  of Definition 3.14. The graded algebra  $(\mathrm{HH}^*(A, A), \cup)$  may be also equipped with a Lie bracket of degree  $-1$  which is compatible with the cup product. The resulting algebraic structure is known as a Gerstenhaber algebra and was described in [32]. The Gerstenhaber algebra structure on  $\mathrm{HH}^*(A, A)$  may be lifted to an  $E_2$ -algebra structure at the cochain level on  $C^*(A, A)$ . This statement is known as the *Deligne conjecture* and was solved in [63].

**Remark 3.10** (Duality). For any dg algebra  $A$  the graded hom-tensor adjunction provides an isomorphism

$$C_{-m,*}(A, A)^\vee \cong C^{m,*}(A, A^\vee).$$

If  $A$  is a symmetric dg Frobenius algebra which is a finitely generated free  $\mathbb{K}$ -module then the isomorphism of  $A$ - $A$ -bimodules  $A \cong A^\vee$  induces an isomorphism of graded  $\mathbb{K}$ -modules

$$C_{-m,*}(A, A)^\vee \cong C^{m,*}(A, A^\vee) \cong C^{m,*}(A, A).$$

In particular, if  $A$  is a symmetric dg Frobenius algebra model over a field  $\mathbb{F}$  for a simply connected closed manifold  $M$ , e.g. as provided by Theorem 3.4, combining this duality with Remark 3.7 gives an isomorphism  $\mathrm{HH}^*(A, A) \cong H_*(LM; \mathbb{F})$ . In Section 4 we discuss how the Gerstenhaber algebra structure of  $\mathrm{HH}^*(A, A)$  corresponds to the Chas-Sullivan product of Section 2 and a loop bracket that in addition uses the circle action, see also [31].

**3.4. Tate-Hochschild complex.** In the presence of a Frobenius structure on an algebra  $A$  we may combine Hochschild chains and cochains of  $A$  into a single unbounded complex through a construction reminiscent of the Tate cohomology of a finite group.

**Definition 3.11.** [72] Let  $A$  a symmetric dg Frobenius  $\mathbb{K}$ -algebra of dimension  $n > 0$ . Write  $\Delta(1) = \sum_i e_i \otimes f_i \in A \otimes A$ . The *Tate-Hochschild complex*  $(\mathcal{D}^*(A, A), \delta)$  of  $A$  is the totalization of the double complex

$$\mathcal{D}^{*,*}(A, A) = \cdots \xrightarrow{\partial_h} s^{1-n}C_{-1,*}(A, A) \xrightarrow{\partial_h} s^{1-n}C_{0,*}(A, A) \xrightarrow{\gamma} C^{0,*}(A, A) \xrightarrow{\delta_h} C^{1,*}(A, A) \xrightarrow{\delta_h} \cdots$$

where  $\gamma : C_{0,*}(A, A) \cong A \rightarrow A \cong C^{0,*}(A, A)$  is given by

$$\gamma(a) = \sum_i (-1)^{|f_i||a|} e_i a f_i, \quad \text{for any } a \in A.$$

The fact that  $\partial_h \circ \gamma = 0 = \gamma \circ \delta^h$  follows from (4) Definition 3.2. Here totalization means the direct sum totalization in the Hochschild chains direction and the direct product totalization in the Hochschild cochains direction:

$$\begin{aligned} \mathcal{D}^k(A, A) &= \prod_{p \geq 0} \mathrm{Hom}_{\mathbb{K}}((s\bar{A})^{\otimes p}, A)^k \oplus \bigoplus_{p \geq 0} ((s\bar{A})^{\otimes p} \otimes A)^{k-n+1} \\ &= C^k(A, A) \oplus C_{k-n+1}(A, A). \end{aligned}$$

One can equivalently define the Tate-Hochschild complex  $\mathcal{D}^*(A, A)$  as the mapping cone of the chain map

$$(3.2) \quad \tilde{\gamma} : s^{1-n}C_*(A, A) \rightarrow C^*(A, A)$$

defined by  $\tilde{\gamma}(\alpha) = 0$  if  $\alpha \in C_{-m,*}(A, A)$  for  $m \neq 0$  and  $\tilde{\gamma}(\alpha) = \sum_i (-1)^{|f_i||\alpha|} e_i \alpha f_i$  if  $\alpha \in A = C_{0,*}(A, A)$ .

**Definition 3.12.** Let  $A$  be a dg Frobenius algebra with pairing  $\langle -, - \rangle_A : A \otimes A \rightarrow \mathbb{K}$ . Define a pairing

$$\langle -, - \rangle_{\mathcal{D}} : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \rightarrow \mathbb{K}$$

by

$$\langle f, \alpha \rangle_{\mathcal{D}} := \langle f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}), a_{m+1} \rangle_A$$

for any  $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in C_{-m,*}(A, A)$  and  $f \in C^{m,*}(A, A)$ , and 0 otherwise.

The above pairing is compatible with the Tate-Hochschild differential, i.e. it satisfies

$$\langle \delta x, y \rangle_{\mathcal{D}} = (-1)^{|x|} \langle x, \delta y \rangle_{\mathcal{D}}.$$

Consequently, we obtain an induced pairing  $H^*(\mathcal{D}^*(A, A)) \otimes H^*(\mathcal{D}^*(A, A)) \rightarrow \mathbb{K}$ .

**Remark 3.13** (The Tate complex in algebra and topology). Let  $\mathbb{K}$  be a field and  $A$  a symmetric dg Frobenius  $\mathbb{K}$ -algebra  $A$ . Then  $H^*(\mathcal{D}^*(A, A))$  is isomorphic to the graded  $\mathbb{K}$ -vector space of morphisms from  $A$  to itself in the *singularity category*

$$\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}}) = \mathcal{D}^b(A \otimes A^{\text{op}}) / \text{Perf}(A \otimes A^{\text{op}}),$$

i.e. the Verdier quotient of the bounded derived category of finitely generated dg  $A$ - $A$ -bimodules by the full subcategory of perfect dg  $A$ - $A$ -bimodules. This statement was originally proven in Proposition 6.9 of [82] when  $A$  is a (non-graded) symmetric Frobenius algebra and extended in Proposition 3.11 of [72] to the case when  $A$  is a symmetric dg Frobenius algebra.

The singularity category was used in [70] to study singularities of algebraic varieties.

In topology, when  $A$  is a commutative symmetric dg Frobenius model for  $C^*(M, \mathbb{K})$  for a simply connected manifold  $M$ , Remarks 3.7 and 3.10, we can think of  $\mathcal{D}^*(A, A)$  as a way of connecting the singular chains and cochains on  $LM$  into a single unbounded complex via the Euler characteristic of  $M$ . Indeed, the map  $\gamma : A \rightarrow A$  in that case takes the product with the element  $\sum_i e_i f_i$ , that identifies with the Euler class of  $M$ . In other words, the map  $\gamma$  is determined by taking a representative of the Poincaré dual of the fundamental class  $[M]$  to the Euler characteristic  $\chi(M)$  thought of as a top dimensional cochain on  $M$  by using a representative of the volume form. On cohomology this is just multiplication by  $\chi(M)$  thought of as a map  $\mathbb{K} \cong H^0(A) \rightarrow H^n(A) \cong \mathbb{K}$ . A symplectic version of the Tate-Hochschild construction has been described and studied in [17, 23] by combining symplectic homology and cohomology via a “V-shaped” Hamiltonian.

**3.5. Two operations on Hochschild complexes.** We recall the classical cup product on the Hochschild cochains of a dg algebra, and define afterwards a form of dual operation on the Hochschild chains.

**Definition 3.14.** Let  $A$  be a dg  $\mathbb{K}$ -algebra. The *cup product*

$$\cup : C^{m,*}(A, A) \otimes C^{n,*}(A, A) \rightarrow C^{m+n,*}(A, A)$$

is defined on any  $f \in C^{m,*}(A, A)$ ,  $g \in C^{n,*}(A, A)$  by the formula

$$f \cup g(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n}}) = (-1)^{|g|\epsilon_m} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) g(\overline{a_{m+1}} \otimes \cdots \otimes \overline{a_{m+n}}),$$

where  $\epsilon_m = \sum_{i=1}^m |a_i| - m$ .

The cup product gives rise to an associative product of degree 0 on  $C^*(A, A)$  that satisfies the graded Leibniz identity with respect to the Hochschild cochains differential  $\delta$ . Therefore  $(C^*(A, A), \delta, \cup)$  is a dg algebra and, consequently, the induced product on  $\text{HH}^*(A, A)$  defines a graded associative algebra structure. This computes the endomorphism graded algebra  $\text{Ext}_{A \otimes A^{\text{op}}}^*(A, A)$  with the categorical Yoneda product.

We now describe a product on the Hochschild chains of a symmetric dg Frobenius algebra that behaves as a “dual” to this cup product, following [72, Section 2.3]. This product has also appear in a slight variation in e.g. [1, Section 6] and [54, Example 2.12].

A dg algebra  $A$  is *connected* if it is non-negatively graded and  $A^0 = \mathbb{K}$ . When  $A$  is a Frobenius of dimension  $n$ , finitely generated free as a  $\mathbb{K}$ -module, this implies that also  $A^n \cong \mathbb{K}$ .

**Definition 3.15.** Suppose  $A$  is a connected symmetric dg Frobenius  $\mathbb{K}$  algebra of dimension  $n > 0$ . The *algebraic Goresky-Hingston product*

$$*: C_*(A, A) \otimes C_*(A, A) \rightarrow C_*(A, A)$$

is defined on any  $\alpha = \bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1}$  and  $\beta = \bar{b}_1 \otimes \cdots \otimes \bar{b}_q \otimes b_{q+1}$  by the formula

$$\alpha * \beta = \sum_i (-1)^{\eta_i} \bar{b}_1 \otimes \cdots \otimes \bar{b}_{q+1} e_i \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1} f_i,$$

where  $\eta_i = |\alpha| |f_i| + |b_{q+1}| + (|\alpha| + n - 1)(|\beta| + n - 1)$ . The product  $*$  induces a degree zero product on the  $(1 - n)$ -shifted graded  $\mathbb{K}$ -module  $s^{1-n} C_*(A, A)$ .

Note that  $*$  does *not* satisfy the Leibniz rule with respect to the Hochschild chains differential  $\partial$ . In fact, the product  $*$  may be understood as a secondary operation, or a chain homotopy, between two operations. If  $p > 0$  and  $q > 0$  we do have

$$(3.3) \quad \partial(\alpha * \beta) - \partial(\alpha) * \beta - (-1)^{|\alpha|+k-1} \alpha * \partial(\beta) = 0.$$

However, if  $p = 0$ , so that  $\alpha = a_1 \in C_{0,*}(A, A) = A$ , we may compute

$$\partial(\alpha * \beta) - \partial(\alpha) * \beta - (-1)^{|\alpha|+n-1} \alpha * \partial(\beta) = \sum_i (-1)^{\eta_i+|\beta|-1-|b_{q+1}|} \bar{b}_1 \otimes \cdots \otimes \bar{b}_q \otimes b_{q+1} e_i a_1 f_i.$$

The case  $q = 0$  is analogous.

Note that, for degree reasons,  $e_i a_1 f_i$  is only non-zero if  $a_1 \in A^0 \cong \mathbb{K}$  and, in such case,  $e_i a_1 f_i \in A^n \cong \mathbb{K}$ . It follows that  $*$  induces a well-defined chain map on the complement of  $C_{0,0}(A, A) = A^0 \cong \mathbb{K} \subset C_*(A, A)$ , which we call the reduced Hochschild complex.

**Definition 3.16.** The *reduced Hochschild chain complex*  $\bar{C}_*(A, A)$  of a connected dg algebra  $A$  is the subcomplex  $\bar{C}_{*,*}(A, A) \subset C_{*,*}(A, A)$  given by  $\bar{C}_{0,0}(A, A) = 0$  and  $\bar{C}_{i,j}(A, A) = C_{i,j}(A, A)$  for all pairs of integers  $(i, j) \neq (0, 0)$ . We denote by  $\overline{\text{HH}}_*(A, A)$  its homology.

The algebraic Goresky-Hingston product  $*$  gives rise to an associative product of degree 0

$$*: s^{1-n} \bar{C}_*(A, A) \otimes s^{1-n} \bar{C}_*(A, A) \rightarrow s^{1-n} \bar{C}_*(A, A)$$

that satisfies the graded Leibniz identity with respect to the reduced Hochschild chains differential. The elements that lead to obstructions for the Leibniz rule on  $C_*(A, A)$  to be satisfied are now removed in the sub-complex  $\bar{C}_*(A, A)$ . Consequently, the induced product on  $s^{1-n} \overline{\text{HH}}_*(A, A)$  defines a graded associative algebra structure.

**3.6. Cyclic  $A_\infty$ -algebra on the Tate-Hochschild complex.** The following natural questions now arise:

- (Q<sub>1</sub>) In what sense are the products  $\cup$  and  $*$  dual to each other?
- (Q<sub>2</sub>) What is the compatibility between  $\cup$  and  $*$  and what is the general algebraic structure they are part of?
- (Q<sub>3</sub>) Do  $\cup$  and  $*$  satisfy a form of homotopy invariance?
- (Q<sub>4</sub>) Is there a homological interpretation for the product  $*$  similar to the interpretation of  $\cup$  as the endomorphism algebra in the derived category of  $A$ - $A$ -bimodules?
- (Q<sub>5</sub>) What is the precise relationship between the geometrically defined Chas-Sullivan and Goresky-Hingston operations and  $\cup$  and  $*$ ?

Question (Q<sub>5</sub>) will be discussed in Section 4, following [67]. The following two statements address the remaining questions (Q<sub>1</sub>)–(Q<sub>4</sub>), saying in particular that  $\cup$  and  $*$  naturally combine to a single product on the Tate-Hochschild complex.

**Theorem 3.17.** [72, Theorem 6.3, Proposition 6.5] *Let  $\mathbb{K}$  be a field and  $A$  be a connected symmetric dg Frobenius  $\mathbb{K}$ -algebra of dimension  $n$ . There exists a (strictly unital)  $A_\infty$ -algebra structure  $\{m_1, m_2, m_3, \dots\}$  on  $\mathcal{D}^*(A, A) = s^{1-n} C_*(A, A) \oplus C^*(A, A)$  such that*

- (1)  $m_1 = \delta$  is the Tate-Hochschild complex differential,  $m_2$  extends both  $*$  and  $\cup$  (i.e.  $m_2|_{s^{1-n} C_*(A, A)} = *$ ,  $m_2|_{C^*(A, A)} = \cup$ ), and  $m_i = 0$  for  $i > 3$ .

(2) The  $A_\infty$ -algebra is cyclically compatible with the pairing  $\langle -, - \rangle_{\mathcal{D}}$ :

$$\langle m_p(\alpha_0 \otimes \cdots \otimes \alpha_{p-1}), \alpha_p \rangle_{\mathcal{D}} = (-1)^{|\alpha_0|(|\alpha_1| + \cdots + |\alpha_p|)} \langle m_p(\alpha_1 \otimes \cdots \otimes \alpha_p), \alpha_0 \rangle_{\mathcal{D}}.$$

(3) The induced homology product is (graded) commutative, and there is an isomorphism of graded algebras

$$H^*(\mathcal{D}^*(A, A)) \cong \mathrm{HH}_{sg}^*(A, A),$$

where the latter is the endomorphism algebra from  $A$  to itself in the singularity category of  $A$ - $A$ -bimodules.

(4) Connes' operator  $B : C_*(A, A) \rightarrow C_{*-1}(A, A)$  extends to an operator  $B_{\mathcal{D}} : \mathcal{D}^*(A, A) \rightarrow \mathcal{D}^{*-1}(A, A)$  satisfying  $B_{\mathcal{D}} \circ \delta + \delta \circ B_{\mathcal{D}} = 0$ ,  $B_{\mathcal{D}} \circ B_{\mathcal{D}} = 0$ , and making  $H^*(\mathcal{D}^*(A, A))$  into a  $BV$ -algebra.

Statement (3) in Theorem 3.17 provides a homological algebra interpretation for the graded associative algebra structure on  $H^*(\mathcal{D}^*(A, A))$ , thus giving an answer to  $(Q_4)$ , while an answer to questions  $(Q_1)$  and  $(Q_2)$  is given by (1) and (2).

**Remark 3.18** (Manin triples). Using constructions and language originated in the theory of quantum groups, we can say a little more about  $(Q_1)$  and  $(Q_2)$ . Denote the associative product on  $H^*(\mathcal{D}^*(A, A))$  by

$$\star : H^*(\mathcal{D}^*(A, A)) \otimes H^*(\mathcal{D}^*(A, A)) \rightarrow H^*(\mathcal{D}^*(A, A)).$$

Observe that there is an isomorphism

$$H^*(\mathcal{D}^*(A, A)) \cong H^*(\ker(\tilde{\gamma})) \oplus H^*(\mathrm{coker}(\tilde{\gamma})),$$

where  $\tilde{\gamma} : s^{1-n}C_*(A, A) \rightarrow C^*(A, A)$  is the map  $\gamma$  considered as map of chain complexes that is mostly zero. In this language, the above result imply the existence of a commutative product  $\star$  on the direct sum  $H^*(\ker(\gamma)) \oplus H^*(\mathrm{coker}(\gamma))$ , together with a pairing  $\langle -, - \rangle_{\mathcal{D}}$ , satisfying the following properties:

(i) The pairing  $\langle -, - \rangle_{\mathcal{D}}$  of Definition 3.12 is non-degenerate with respect to the ‘‘monomial length’’ chain level filtration on  $\mathcal{D}^{*,*}(A, A) = s^{1-n}C_{*,*}(A, A) \oplus C^{*,*}(A, A)$ . More precisely, it induces an isomorphism of graded vector spaces

$$C_{-m,*}(A, A) \xrightarrow{\cong} C^{m,*}(A, A)^\vee.$$

(ii) For any  $x, y, z \in H^*(\ker(\gamma)) \oplus H^*(\mathrm{coker}(\gamma))$  we have  $\langle x \star y, z \rangle_{\mathcal{D}} = \langle x, y \star z \rangle_{\mathcal{D}}$ .

(iii) Both  $(H^*(\mathrm{coker}(\gamma)), \cup)$  and  $(H^*(\ker(\gamma)), \star)$  are isotropic sub-algebras of

$$(H^*(\ker(\gamma)) \oplus H^*(\mathrm{coker}(\gamma)), \star)$$

with respect to the pairing  $\langle -, - \rangle_{\mathcal{D}}$ .

The algebraic structure just described is reminiscent of a *Manin triple*, a notion originally introduced in the context of quantum groups. A Manin triple consists of a triple of Lie algebras  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  over a field  $\mathbb{K}$  such that  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  as vector spaces and  $\mathfrak{g}$  is equipped with a symmetric bilinear pairing  $\langle -, - \rangle_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{K}$  satisfying  $\langle [x, y], z \rangle_{\mathfrak{g}} = \langle x, [y, z] \rangle_{\mathfrak{g}}$ , inducing an isomorphism  $\mathfrak{g}_+ \cong \mathfrak{g}_-^\vee$ , and for which  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are isotropic Lie sub-algebras. If  $\mathfrak{h}$  is a finite dimensional Lie algebra then there is a 1-1 correspondence between Manin triples with  $\mathfrak{g}_+ = \mathfrak{h}$  and Lie bialgebra structures on  $\mathfrak{h}$ . In particular, if  $\mathfrak{g}$  is a Lie bialgebra then one can describe a canonical Lie bialgebra structure on  $\mathfrak{g} \oplus \mathfrak{g}^\vee$  called the *Drinfeld double* of  $\mathfrak{g}$ . Drinfeld showed this construction yields a quasi-triangular Lie bialgebra. A complete reference for these notions and results is [12].

Define analogously a *graded commutative Manin triple* to be a triple of graded commutative  $\mathbb{K}$ -algebras  $(V, V_+, V_-)$  over a field  $\mathbb{K}$  such that

- (i)  $V = V_+ \oplus V_-$  as a vector space and  $V$  is equipped with a symmetric bilinear pairing  $\langle -, - \rangle_V : V \otimes V \rightarrow \mathbb{K}$  inducing an isomorphism  $V_+ \cong V_-^\vee$ ,
- (ii) for any  $a, b, c \in V$ , we have  $\langle ab, c \rangle_V = \langle a, bc \rangle_V$ , and
- (iii) both  $V_+$  and  $V_-$  are isotropic sub-algebras of  $V$ .

As in the Lie case, one can use the duality given by the pairing to reformulate the defining equations of this structure in terms of a type of bialgebra structure on  $V$ . More precisely, if  $W$  is a finite dimensional graded commutative algebra, there is a 1-1 correspondence between graded commutative Manin triples with  $V_+ = W$  and graded commutative cocommutative *infinitesimal bialgebra* structures on  $W$ , as introduced by Joni and Rota in [48]. The data of a graded infinitesimal bialgebra structure on  $W$  consists of a product  $\cdot : W \otimes W \rightarrow W$  of degree 0 and coproduct  $\Delta : W \rightarrow W \otimes W$  of degree  $k$  such that  $\Delta$  is a derivation of the product, namely

$$\Delta(a \cdot b) = \Delta(a) \cdot b + (-1)^{|a|k} a \cdot \Delta(b),$$

where we define  $(a' \otimes a'') \cdot b := a' \otimes (a'' \cdot b)$  and  $a \cdot (b' \otimes b'') := (a \cdot b') \otimes b''$ . See [5] for more about infinitesimal biaglegbras. See [68] for (a non-graded version of) the correspondence between commutative cocommutative infinitesimal bialgebras with Manin triples of commutative algebras and, more generally, between Poisson bialgebras and Manin triples of Poisson algebras.

The following result provides an answer to question (Q<sub>3</sub>).

**Theorem 3.19.** [73, Theorem 1.1] *Let  $\mathbb{K}$  be a field and  $(A, \langle -, - \rangle_A)$  and  $(B, \langle -, - \rangle_B)$  be two simply connected symmetric dg Frobenius  $\mathbb{K}$ -algebras of dimension  $n$ . Suppose that there is a zig-zag of quasi-isomorphisms of dg algebras*

$$A \xleftarrow{\cong} \bullet \xrightarrow{\cong} \dots \xleftarrow{\cong} \bullet \xrightarrow{\cong} B.$$

Then there is an isomorphism of algebras

$$(H^*(\mathcal{D}^*(A, A)), \star) \cong (H^*(\mathcal{D}^*(B, B)), \star)$$

restricting to an isomorphism of subalgebras

$$(s^{1-n}\overline{\text{HH}}_*(A, A), \star) \cong (s^{1-n}\overline{\text{HH}}_*(B, B), \star).$$

The proof the above theorem relies on the homological interpretation of the Tate-Hochschild cohomology algebra as the endomorphism algebra in the singularity category of  $A$ - $A$ -bimodules (see Remark 3.13). The isomorphism class of the latter, just like for the Hochschild cohomology algebra with cup product, is an invariant of the quasi-isomorphism type of the underlying dg algebra. A careful analysis of the relationship between Tate-Hochschild cohomology and singular Hochschild cohomology allows to conclude that the isomorphism  $(H^*(\mathcal{D}^*(A, A)), \star) \cong (H^*(\mathcal{D}^*(B, B)), \star)$  restricts to an isomorphism  $(s^{1-n}\overline{\text{HH}}_*(A, A), \star) \cong (s^{1-n}\overline{\text{HH}}_*(B, B), \star)$  in the simply connected case. We refer to [73] for further details.

As a direct consequence of Theorem 3.19 is the following.

**Corollary 3.20.** [73, Corollary 1.2]

- (1) *Let  $M$  be a simply connected oriented closed manifold of dimension  $n$  and  $A$  a Poincaré duality model for the cdga of rational polynomial forms  $\mathcal{A}_{\text{pl}}(M, \mathbb{Q})$ , as provided by Theorem 3.4. The isomorphism class of the graded algebra structure on  $s^{1-n}\overline{\text{H}}^*(LM; \mathbb{Q})$  induced by the product  $\star : s^{1-n}\overline{\text{HH}}_*(A, A)^{\otimes 2} \rightarrow s^{1-n}\overline{\text{HH}}_*(A, A)$  through the isomorphism  $\overline{\text{H}}^*(LM; \mathbb{Q}) \cong \overline{\text{HH}}_*(A, A)$  is independent of the choice of Poincaré duality model  $A \simeq \mathcal{A}_{\text{pl}}(M, \mathbb{Q})$ .*
- (2) *If  $M$  and  $M'$  are homotopy equivalent simply connected oriented closed manifolds of dimension  $n$ , then the algebra structures on  $s^{1-n}\overline{\text{H}}^*(LM; \mathbb{Q})$  and  $s^{1-n}\overline{\text{H}}^*(LM'; \mathbb{Q})$  are isomorphic.*

**3.7. Final remarks.** One would like to understand the complete algebraic chain level structure of the Tate-Hochschild complex of a symmetric dg Frobenius algebra. The type of cyclic  $A_\infty$ -algebra described in Theorem 3.17 is a finite type version of a notion discussed in [46] under the name of *Pre Calabi-Yau algebra*. It is explain there how the associator  $m_3$  of a Pre-Calabi Yau algebra gives rise to a *double Poisson bracket*. A precise formula for the map  $m_3$  on the Tate-Hochschild complex may be found in Remark 6.4 of [72].

This is only the tip of the iceberg of a very rich algebraic structure on the Tate-Hochschild complex. Part (4) of Theorem 3.17 tells us that  $B_{\mathcal{D}}$  and the product  $\star$  define a *BV-algebra*



structure on  $H^*(\mathcal{D}^*(A, A)) = H^*(\text{coker}(\gamma)) \oplus H^*(\text{ker}(\gamma))$ . By definition, a BV-algebra consists of a triple  $(V, \star, B)$  where  $(V, \star)$  is a graded commutative algebra,  $B : V \rightarrow V$  is a degree  $-1$  operator satisfying  $B \circ B = 0$ , and the operation

$$\{x, y\} := B(x \star y) - B(x) \star y - (-1)^{|x|} x \star B(y)$$

is a Lie bracket of degree  $-1$  which is a derivation of  $\star$  on each variable, i.e.  $\{-, -\}$  is Poisson compatible with  $\star$ .

The BV-algebra structure on Tate-Hochschild cohomology extends the BV-algebra structure of the Hochschild cohomology of a symmetric dg Frobenius algebra to the Tate-Hochschild cohomology. In [49] we lift the BV-algebra structure of Tate-Hochschild cohomology to the chain level, building upon the framework of [50, 51], and solve a cyclic Deligne conjecture for the Tate-Hochschild complex. The Lie bracket associated to the BV-algebra structure gives rise to a compatible (Lie) Manin triple structure on  $(H^*(\mathcal{D}^*(A, A)), H^*(\text{coker}(\gamma)), H^*(\text{ker}(\gamma)))$  extending the classical Gerstenhaber algebra structure on Hochschild cohomology. The Lie algebra structure on  $H^*(\mathcal{D}^*(A, A))$  was also lifted to a cyclic  $L_\infty$ -algebra structure on  $\mathcal{D}^*(A, A)$  in [72]. After dualizing and completing the tensor product appropriately, we obtain on  $H^*(\mathcal{D}^*(A, A))$  a graded commutative cocommutative infinitesimal bialgebra equipped with a Gerstenhaber bracket and a Gerstenhaber cobracket that are Lie bialgebra compatible. Furthermore, the Gerstenhaber bracket and the cocommutative coproduct, as well as the Gerstenhaber cobracket and the commutative product, satisfy additional second order compatibility equations. This algebraic structure, which may be called a *Gerstenhaber bialgebra*, is a graded version of a *Poisson bialgebra*, defined and studied in [68].

Gerstenhaber bialgebras are reminiscent of similar structures appearing in the theory of quantum groups, where associated to a Lie bialgebra  $\mathfrak{g}$ , such as the structure induced on the tangent Lie algebra of a Poisson-Lie group, one may consider the commutative cocommutative Hopf algebra  $S(\mathfrak{g})$ , the symmetric algebra on the vector space  $\mathfrak{g}$ , with the Poisson bracket and Poisson cobracket induced by the Lie bialgebra structure on  $\mathfrak{g}$ . Then one proceeds to deform the product to obtain the non-commutative cocommutative universal enveloping algebra  $U(\mathfrak{g})$  and then deforms the coproduct in the Poisson cobracket direction to obtain a non-commutative non-cocommutative Hopf algebra  $U_h(\mathfrak{g})$ . Motivated by the above discussion and by the question of constructing examples of non-commutative non-cocommutative infinitesimal bialgebras one can replace the notion of Hopf algebra by infinitesimal bialgebra. More precisely, one could ask if given a Poisson bialgebra  $A$  there exists a deformation to a (possibly non-commutative non-cocommutative) infinitesimal bialgebra  $A[[\hbar]]$  in the direction of the Poisson bracket and cobracket. One may also study analogous questions in the graded setting for Gerstenhaber bialgebras.

Lie bialgebras also appear in  $S^1$ -equivariant string topology. In fact, the Chas-Sullivan loop product and the Goresky-Hingston loop coproduct induce a Lie bialgebra structure once we pass to the reduced  $S^1$ -equivariant homology of the free loop space of a manifold. This structure generalizes previous constructions of Goldman and Turaev from surfaces to manifolds of arbitrary dimension [75], [36], [78]. In the algebraic context, this construction is modeled by a dg Lie bialgebra structure on the reduced cyclic chain complex of a dg Frobenius algebra ([16], [67], [18]), a construction foreshadowed by Ginzburg's necklace Lie bialgebra [34]. Turaev described the quantization of the Lie bialgebra structure on the zeroth  $S^1$ -equivariant homology of the free loop space of a surface in terms of skein invariants of links in 3-manifolds. This quantization has also been studied from an algebraic perspective: in [74] a quantization of Ginzburg's necklace Lie bialgebra of a quiver is constructed and this is generalized in [16] where a quantization of the Lie bialgebra on the cyclic homology of a Frobenius algebra is constructed. We expect that the functorial theory of quantization of Lie bialgebras described by Etingof and Kazhdan in [28] may be adapted to quantize infinitesimal bialgebras in the direction of a compatible bracket and cobracket. This theory should give rise to explicit and interesting examples of non-commutative non-cocommutative infinitesimal bialgebras associated to dg Frobenius algebras by quantizing the infinitesimal bialgebra structure of  $H^*(\mathcal{D}^*(A, A))$  in the direction of the Gerstenhaber bracket and cobracket.

## 4. STRING TOPOLOGY AND CONFIGURATION SPACES

In this section we compare the geometrically defined string topology operations of Section 2 with the ones defined algebraically using a dg Frobenius model as in Section 3, under the assumption that the coefficients  $\mathbb{K} = \mathbb{R}$  are the real numbers. The main ingredient is an algebraic model for the Fulton-McPherson compactification of  $M \times M \setminus M$ , the configuration space of two points in  $M$ .

Let  $M$  be a simply connected oriented closed manifold. By a theorem of Lambrechts and Stanley (stated here as Theorem 3.4), applied to the case  $\mathbb{K} = \mathbb{R}$ , there exists a commutative symmetric dg Frobenius algebra  $A$  quasi-isomorphic to real cochains  $C^*(M; \mathbb{R})$ . As discussed in Remark 3.7, we have isomorphisms

$$(4.1) \quad HH_*(A, A) \cong HH_*(C^*(M; \mathbb{R}), C^*(M; \mathbb{R})) \cong H^*(LM; \mathbb{R}).$$

**Definition 4.1.** Define the *relative Hochschild complex* by

$$\underline{C}_*(A, A) = \bigoplus_{m \geq 1} (s\bar{A})^{\otimes m} \otimes A$$

Because  $A$  is commutative,  $\underline{C}_*(A, A)$  is a sub chain complex of  $C_*(A, A)$ .

The chain complex  $\underline{C}_*(A, A)$  may also be regarded as the kernel of the natural chain map  $C_*(A, A) \rightarrow A$ , which models the map  $\text{cst}: M \rightarrow LM$  (see Example 4.15). Hence (4.1) restricts to an isomorphism

$$(4.2) \quad \underline{HH}_*(A, A) \cong H^*(LM, M; \mathbb{R}).$$

The algebraic Goresky-Hingston product given in Definition 3.15 restricts to a product on this relative version of the Hochschild chain complex (see also e.g., [1, Sec 6]). The purpose of this section is to sketch a proof of the following result:

**Theorem 4.2.** [67, Theorem 1.3] *Let  $M$  be a simply-connected oriented closed manifold with commutative dg Frobenius algebra model  $A \simeq C^*(M; \mathbb{R})$ . Then the isomorphism (4.2)*

$$\underline{HH}_*(A, A) \cong H^*(LM, M; \mathbb{R}),$$

*intertwines the algebraic with the topological Goresky-Hingston product of Definitions 2.2 (dualised) and 3.15.*

We will follow the line of argument of [67]. A similar argument to the one presented here gives the equivalence between the algebraic and topological Chas-Sullivan products of Definitions 2.1 and 3.14 (dualized), giving an alternative proof of [30, Theorem 11]. Here we focus on the Goresky-Hingston product.

We will use the definition of the coproduct given in Section 2.5. Before embarking into the proof of the theorem in Section 4.4, we will take a closer look at the crucial step in the definition of the coproduct, namely the intersection map, defining a general notion of *intersection products* (see Sections 4.1 and 4.2). Section 4.3 then analyses invariance properties of such intersection products.

**Remark 4.3** (Dependence on the manifold  $M$ ). Note that we can take  $A$  *any* commutative dg Frobenius model of  $C^*(M; \mathbb{R})$  in the statement. As the right hand side in the theorem is model-independent, it follows that the algebraic Goresky-Hingston product on  $H_*(A, A)$  does not depend on the particular model  $A$ . This partially recovers Corollary 3.20.

We saw in Section 2.3 through a lens space example that the coproduct on  $H_*(LM)$  is in general not a homotopy invariant of  $M$ , at least with integral coefficients, see Theorem 2.10. In the proof of Theorem 4.2, the topology of  $M$  will enter through the homotopy type of the complement of the diagonal  $M \times M \setminus M$ . This last space identifies with the configuration space of 2 points, a space known to depend in general on more than the homotopy type from the same lens space example, see [58]. We will use a recent result by Campos-Willwacher and Idrissi [11, 45] to obtain an algebraic model for this space in the case of simply-connected manifolds (together with some compatibility datum).

To simplify presentation and notation, we will show the corresponding statement for the operation

$$H_{*+n-1}(LM; \mathbb{R}) \rightarrow H_*(LM, M; \mathbb{R}) \xrightarrow{\vee} H_*(LM, M; \mathbb{R})^{\otimes 2},$$

that is the pre-composition with the canonical map  $H_*(LM; \mathbb{R}) \rightarrow H_*(LM, M; \mathbb{R})$ .

**4.1. Intersection products.** Recall from Section 2.5 that the loop coproduct can be defined as a relative version of the trivial coproduct  $\vee_{\frac{1}{2}}$ , intersecting with the figure eights space  $\text{Fig}(8) \subset LM$ . The crucial step in this definition of the coproduct is the composition

$$(4.3) \quad R_{\frac{1}{2}} \circ ((\text{ev}_{0, \frac{1}{2}})^* \tau_M \cap): H_*(LM, \mathcal{R}) \rightarrow H_{*-n}(\text{Fig}(8), \mathcal{R}),$$

see (2.8). Here  $\mathcal{R}$  is the subspace of half-constant loops,  $\text{ev}_{0, \frac{1}{2}} = (\text{ev}_0, \text{ev}_{\frac{1}{2}}): LM \rightarrow M \times M$  is the evaluation at 0 and  $\frac{1}{2}$ , the cochain  $\tau_M \in C^n(M \times M, M \times M \setminus M)$  is a representative of the Thom class of the normal bundle of the diagonal  $M \rightarrow M \times M$ , and  $R_{\frac{1}{2}}$  is a retraction map. In Sections 4.1–4.3, homology can be taken with integral coefficients.

Note that  $\text{Fig}(8)$  is the pullback of  $\text{ev}_{0, \frac{1}{2}}$  along the diagonal

$$\begin{array}{ccc} \text{Fig}(8) & \longrightarrow & LM \\ \text{ev}_0 \downarrow & & \downarrow \text{ev}_{0, \frac{1}{2}} \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

and one can show that, just like the evaluation map  $\text{ev}_0$ , the map  $\text{ev}_{0, \frac{1}{2}}$  is a fibration. The map (4.3) is the lift along  $\text{ev}_{0, \frac{1}{2}}$  of the intersection product  $H_*(M \times M) \xrightarrow{\bullet} H_{*-n}(M)$ , taken relative to  $\mathcal{R}$ . We will think of it as a “relative intersection product” and will now abstract what is needed to define it.

**4.1.1. Relative intersection products.** The definition of the relative intersection product (4.3) immediately generalizes to the following situation. Suppose  $p_{\mathcal{E}}: \mathcal{E} \rightarrow M \times M$  is a fibration, and  $\mathcal{R}$  is a space equipped with maps  $p_{\mathcal{R}}: \mathcal{R} \rightarrow M$  and  $f: \mathcal{R} \rightarrow \mathcal{E}$  such that the diagram

$$(4.4) \quad \begin{array}{ccc} \mathcal{R} & \xrightarrow{f} & \mathcal{E} \\ p_{\mathcal{R}} \downarrow & & \downarrow p_{\mathcal{E}} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

commutes. From this data, we can define the following zig-zag of chain maps:

$$C_*(\mathcal{E}) \longrightarrow C_*(\mathcal{E}, \mathcal{E}|_{M \times M \setminus M}) \xleftarrow{\sim} C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{U_M \setminus M}) \xrightarrow{\cap p_{\mathcal{E}}^* \tau_M} C_{*-n}(\mathcal{E}|_{U_M}) \xleftarrow{\sim} C_{*-n}(\mathcal{E}|_M),$$

where  $TM \cong U_M \subset M \times M$  is a tubular neighborhood of the diagonal as in Section 2.1. Both wrong-way maps are quasi-isomorphisms: the first one by excision and the second one since we are pulling back a fibration along the homotopy equivalence  $M \xrightarrow{\sim} U_M$ . Thus we get a map in homology

$$(4.5) \quad H_*(\mathcal{E}) \xrightarrow{\text{int}_M} H_{*-n}(\mathcal{E}|_M),$$

which we call the (*absolute*) *intersection product* associated to the fibration  $p_{\mathcal{E}}$ . To refine this operation to a relative version, we note that the following diagram commutes.

$$(4.6) \quad \begin{array}{ccccccc} C_*(\mathcal{E}) & \longrightarrow & C_*(\mathcal{E}, \mathcal{E}|_{M \times M \setminus M}) & \xleftarrow{\sim} & C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{U_M \setminus M}) & \xrightarrow{\cap p_{\mathcal{E}}^* \tau} & C_{*-n}(\mathcal{E}|_{U_M}) & \xleftarrow{\sim} & C_{*-n}(\mathcal{E}|_M) \\ f \uparrow & & f \uparrow & & f \uparrow & & f \uparrow & & f \uparrow \\ C_*(\mathcal{R}) & \xlongequal{\quad} & C_*(\mathcal{R}) & \xlongequal{\quad} & C_*(\mathcal{R}) & \xrightarrow{\cap f^* p_{\mathcal{E}}^* \tau} & C_{*-n}(\mathcal{R}) & \xlongequal{\quad} & C_{*-n}(\mathcal{R}) \end{array}$$

Taking vertical mapping cones, this again defines a zig-zag of complexes such that the wrong-way maps are quasi-isomorphisms and thus we obtain a map in homology

$$(4.7) \quad H_*(\mathcal{E}, \mathcal{R}) \xrightarrow{\text{int}_M} H_{*-n}(\mathcal{E}|_M, \mathcal{R})$$

which we call the *relative intersection product* associated to the diagram (4.4).

**Proposition 4.4.** *For  $\mathcal{E} = LM$  with  $p_{\mathcal{E}} = \text{ev}_{0, \frac{1}{2}} = (\text{ev}_0, \text{ev}_{\frac{1}{2}}): LM \rightarrow M \times M$  and  $\mathcal{R} \hookrightarrow LM$  the space of half-constant loops, the operation*

$$\text{int}_M: H_*(LM, \mathcal{R}) \longrightarrow H_{*-n}(\text{Fig}(8), \mathcal{R})$$

*coincides with the corresponding map in the definition (2.8) of the loop coproduct.*

*Proof.* The first three commuting squares in (4.6) are simply spelling out the details in (2.8) (as in (2.3)), with the only difference that a homotopy inverse to excision was chosen in (2.8). The last step follows from the fact that the retraction map  $R_{\frac{1}{2}}$  in (2.8) is a homotopy inverse to the inclusion  $\text{Fig}(8) \hookrightarrow LM|_{U_M}$  (this is essentially [43, Lemma 2.11]), thus inducing an inverse to the map  $H_*(\text{Fig}(8), \mathcal{R}) \rightarrow H_*(LM|_{U_M}, \mathcal{R})$  in relative homology.  $\square$

Similarly, we obtain the loop product as an example of the (non-relative) intersection product:

**Proposition 4.5.** *For  $\mathcal{E} = LM \times LM$  with  $p_{\mathcal{E}} = (\text{ev}_0, \text{ev}_0): LM \times LM \rightarrow M \times M$  and  $\mathcal{R} = \emptyset$ , the operation*

$$\text{int}_M: H_*(LM \times LM) \rightarrow H_{*-n}(\text{Fig}(8))$$

*coincides with the corresponding map in the definition (2.5) of the loop product.*

The following properties of the relative intersection product follow directly from the definitions.

**Proposition 4.6.** *The relative intersection product (4.7) is natural in diagrams (4.4) over a fixed manifold  $M$  and refines the absolute intersection product (4.5) in the sense that*

$$\begin{array}{ccc} H_*(\mathcal{E}, \mathcal{R}) & \xrightarrow{\text{int}_M} & H_{*-n}(\mathcal{E}|_M, \mathcal{R}) \\ \uparrow & & \uparrow \\ H_*(\mathcal{E}) & \xrightarrow{\text{int}_M} & H_{*-n}(\mathcal{E}|_M) \end{array}$$

*commutes. The absolute intersection product is natural in fibrations  $p_{\mathcal{E}}$ , and identifies with the classical intersection product of Section 2.1 in the case  $\mathcal{E} = M \times M$  with  $p_{\mathcal{E}} = \text{id}$ :*

$$H_*(M \times M) \xrightarrow{\text{int}_M = \bullet} H_{*-n}(M).$$

**4.2. Intersection contexts.** The definition of the relative intersection product uses the following data from the manifold: the diagram

$$(4.8) \quad \begin{array}{ccc} U_M \setminus M^c & \longrightarrow & M \times M \setminus M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sim} & U_M \hookrightarrow M \times M \end{array}$$

and the class  $\tau_M \in H^n(M \times M, M \times M \setminus M) \cong H^n(U_M, U_M \setminus M)$ . This is also the data used to define the classical intersection product. We note that the spaces  $U_M$ ,  $U_M \setminus M$  and  $M \times M \setminus M$  only appear in the intermediate steps of the definition.

We now describe a slight generalization of a relative intersection product

$$H_*(\mathcal{E}, \mathcal{R}) \rightarrow H_{*-n}(\mathcal{E}|_M, \mathcal{R}).$$

Such a construction may be defined from the data of a diagram (4.4) as before together with the “manifold data” recorded by any homotopy pushout diagram of the shape

$$(4.9) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sim} & C \longrightarrow M \times M \end{array}$$

equipped with a class  $\tau \in H^n(C, A)$ . Indeed, if we denote  $\mathcal{E}|_A, \mathcal{E}|_B, \mathcal{E}|_C$  the pull-back of  $\mathcal{E}$  along the maps  $A, B, C \rightarrow M \times M$ , to construct the relative intersection using the corresponding zig-zag (4.6), all we need is that the maps

$$C_*(\mathcal{E}, \mathcal{E}|_B) \longleftarrow C_*(\mathcal{E}|_C, \mathcal{E}|_A)$$

and

$$C_*(\mathcal{E}|_C) \longleftarrow C_*(\mathcal{E}|_M)$$

are quasi-isomorphisms. For the second one, this follows as before from our assumption that  $M \rightarrow C$  is a homotopy equivalence, given that  $p_{\mathcal{E}}$  is a fibration. For the first, it follows from the assumption that (4.9) is a homotopy pushout, using Mather’s second cube theorem [60, Theorem 25] applied to the pullback of the square along the fibration  $p_{\mathcal{E}}$ , as a replacement of excision.

**Definition 4.7.** We call a homotopy pushout diagram of the shape (4.9) an *intersection context*, and a cohomology class  $\tau \in H^n(C, A)$  an *n-orientation*.

We define two oriented intersection contexts to be equivalent if there is a zig-zag of diagrams that is a pointwise homotopy-equivalence, compatible with the orientations. A diagram chase gives the following.

**Proposition 4.8.** *Two equivalent oriented intersection contexts associate the same relative intersection map*

$$\text{int}_M: H_*(\mathcal{E}, \mathcal{R}) \rightarrow H_{*-n}(\mathcal{E}|_M, \mathcal{R})$$

to a tuple  $(\mathcal{E}, \mathcal{R}, p_{\mathcal{E}}, p_{\mathcal{R}}, f)$  as in diagram (4.4).

The intersection context we will be using in our proof of Theorem 4.2 is the following. Let  $FM_2$  denote the Fulton-McPherson compactification of the configurations space of two points. It is obtained as the real oriented blowup of  $M \times M$  along the diagonal. That is  $FM_2$  is a manifold with boundary whose interior is  $M \times M \setminus M$  and with boundary the unit tangent bundle  $UTM$  of  $M$ . In particular, it fits into the following commuting square

$$(4.10) \quad \begin{array}{ccc} UTM & \longrightarrow & FM_2 \\ \downarrow & & \downarrow \\ M & \xlongequal{\quad} & M \longrightarrow M \times M. \end{array}$$

**Proposition 4.9.** *Together with the class  $\tau_M \in H^n(M, UTM) \cong H^n(U_M, U_M \setminus M)$ , Diagram (4.10) defines an oriented intersection context equivalent to (4.8).*

*Proof.* There is a zig-zag of equivalences between the two diagrams coming from the pair of zig-zag  $UTM \rightarrow \overline{U_M} \setminus M \leftarrow U_M \setminus M$  and  $FM_2 = FM_2 \leftarrow M \times M \setminus M$ , for  $\overline{U_M} \setminus M$  an epsilon neighborhood of  $UTM$  in  $FM_2$ .  $\square$

**4.3. Invariance of intersection products.** Suppose  $f: M \rightarrow N$  is a smooth map, and that  $M$  comes equipped with an intersection context, for example one of the form (4.10). Composing with  $f$ , we obtain an intersection context for  $N$  from that of  $M$ . We denote the corresponding relative intersection product by  $f_* \text{int}_M$ . By construction we have the following naturality property:

**Lemma 4.10.** *For*

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ N & \longrightarrow & N \times N \end{array}$$

as in (4.4), the square

$$\begin{array}{ccc} H_*(f^*\mathcal{E}, f^*\mathcal{R}) & \xrightarrow{\text{int}_M} & H_{*-n}(f^*\mathcal{E}|_M, f^*\mathcal{R}) \\ \downarrow & & \downarrow \\ H_*(\mathcal{E}, \mathcal{R}) & \xrightarrow{f_* \text{int}_M} & H_{*-n}(\mathcal{E}|_N, \mathcal{R}) \end{array}$$

commutes, where  $f^*\mathcal{E}$  and  $f^*\mathcal{R}$  are the homotopy pullback of  $\mathcal{E}$  and  $\mathcal{R}$  along  $f \times f: M \times M \rightarrow N \times N$  and  $f: M \rightarrow N$ . Note that the vertical maps are isomorphisms if  $f$  is a homotopy equivalence.

We are interested in the case  $\mathcal{E} = LN \rightarrow N \times N$  with  $\mathcal{R}_N \rightarrow N$  the space of half-constant loops, as defined in 2.5. In that case,  $f$  also induces compatible natural maps  $LM \rightarrow LN$  and  $\mathcal{R}_M \rightarrow \mathcal{R}_N$  giving a commuting diagram

$$\begin{array}{ccc} H_*(LM, \mathcal{R}_M) & \xrightarrow{\text{int}_M} & H_{*-n}(\text{Fig}(8)_M, \mathcal{R}_M) \\ \downarrow & & \downarrow \\ H_*(f^*LN, f^*\mathcal{R}_N) & \xrightarrow{\text{int}_M} & H_{*-n}(f^*LN|_M, f^*\mathcal{R}_N) \\ \downarrow & & \downarrow \\ H_*(LN, \mathcal{R}_N) & \xrightarrow{f_* \text{int}_M} & H_{*-n}(\text{Fig}(8)_N, \mathcal{R}_N), \end{array}$$

where again the vertical arrows are all isomorphisms if  $f$  is a homotopy equivalence. Hence comparing the loop coproduct for two manifolds  $M$  and  $N$  is equivalent to comparing the relative intersection products  $f_* \text{int}_M$  and  $\text{int}_N$  on the pair  $(LN, \mathcal{R}_N)$ . In general, these are not equal. Otherwise, since the loop coproduct may be described in terms of the above intersection products (as in Proposition 2.12), this would yield a proof for homotopy invariance of the loop coproduct, contradicting Theorem 2.10.

In contrast, the loop product is known to satisfy homotopy invariance (see Theorem 2.9), and the (failed) line of argument suggested above for the coproduct does go through for the product. The essential difference is that the loop product only uses the non-relative intersection product (see Proposition 4.5). Its homotopy invariance follows from the following result.

**Theorem 4.11.** *Let  $f: M \rightarrow N$  be an orientation-preserving homotopy equivalence of manifolds, each equipped with its intersection context of the form (4.10). Then for any fibration  $\mathcal{E} \rightarrow N \times N$  the intersection product*

$$\text{int}_N: H_*(\mathcal{E}) \longrightarrow H_{*-n}(\mathcal{E}|_N)$$

*coincides with the transferred intersection product  $f_* \text{int}_M$ .*

*Sketch proof.* The above theorem is proved in the papers [25, 26, 40, 30] in the context of string topology, i.e. in the special case when  $\mathcal{E} = LN \rightarrow N \times N$ , as the crucial ingredient in the homotopy invariance of the loop product, and the proofs generalise to our context. The proof of Gruher-Salvatore in [40] is closest to our language, so we follow that paper. Translating to our notation, Theorem 8 in that paper defines a product preserving map  $\theta_f$  in homology from  $(\mathcal{E}, \text{int}_N)$  to  $(f^*\mathcal{E}, \text{int}_M)$ . This map can be composed by the product-preserving map  $(f^*\mathcal{E}, \text{int}_M) \rightarrow (\mathcal{E}, f_* \text{int}_M)$  given by the non-relative version of Lemma 4.10. As both maps preserve the product, it is enough to show that they compose to the identity on  $\mathcal{E}$ . This statement corresponds to the last display in the proof of Proposition 23 in [40]. This last computation is only stated in the case of the loop space in that paper, but it comes from

an analysis of the maps using Thom isomorphisms that only use what the maps do on the underlying manifolds.  $\square$

An alternative approach to the above statement is to use parametrized homotopy theory as in [61], identifying the intersection product considered here with the evaluation map of the Costenoble-Waner duality for  $M$ .

**Remark 4.12.** As the example of lens spaces shows (Theorem 2.10), the above theorem does not generalize to the relative intersection product. The above argument fails in that the composition  $\iota \circ \theta_f$  may fail to be equal to the identity in relative homology. This is equivalent to the lack of a Thom isomorphism type map in the computation to be an isomorphism in relative homology, relating to the issue discussed in [44, Sec 3.8].

#### 4.4. Equivalence between algebraic and geometric models for the loop coproduct.

We will now give a sketch of the proof of Theorem 4.2. We first describe real models (in the sense of rational homotopy theory) for each of the steps in the definition of the loop coproduct and compare the final result with the description in 3.15. More precisely, up to crossing with an interval, we can write the geometric coproduct (2.8) as the composition of the following three maps:

$$(4.11) \quad C_{*+n}(LM \times I, LM \times \partial I) \xrightarrow{J} C_{*+n}(LM, \mathcal{R}) \xrightarrow{\text{int}} C_*(\text{Fig}(8), \mathcal{R}) \xrightarrow{\text{cut}} C_*(LM, M)^{\otimes 2},$$

where the middle map is the intersection product discussed in Sections 4.1 and 4.2. We will give models for each of these three maps. Most of what we do in this section can be done with rational coefficients; real coefficients will only be needed at the very end of the section, when picking a particular model of the configuration space  $FM_2$ . For simplicity, we will ignore sign issues in this section.

A major ingredient will be the Eilenberg-Moore theorem, that we will use to give rational models of homotopy pull-backs. We will apply it to the functorial rational model of polynomial forms  $\mathcal{A}_{\text{pl}}$ , with  $\mathcal{A}_{\text{pl}}(X) \simeq C^*(X; \mathbb{Q})$ :

**Theorem 4.13** (Eilenberg-Moore; see for instance Theorem 7.14 in [62]). *Suppose that*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

*is a homotopy pullback of spaces, such that  $Z$  is simply-connected and either  $X$  or  $Y$  are connected. Then the natural map*

$$\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)} \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(W)$$

*induced by  $f^*: \mathcal{A}_{\text{pl}}(X) \rightarrow \mathcal{A}_{\text{pl}}(W)$  and  $g^*: \mathcal{A}_{\text{pl}}(Y) \rightarrow \mathcal{A}_{\text{pl}}(W)$  is a quasi-isomorphism. Here  $\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y)$  denotes the derived tensor product.*

In the following we use the bar construction model for the derived tensor product:

$$\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y) = \bigoplus_{p \geq 0} \mathcal{A}_{\text{pl}}(X) \otimes \overline{s\mathcal{A}_{\text{pl}}(Z)}^{\otimes p} \otimes \mathcal{A}_{\text{pl}}(Y),$$

with differential analogous to that of the Hochschild complex (see Definition 3.6). Note that with this definition  $\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y)$  is a quasi-free (i.e. free after forgetting the differential)  $\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y)$ -module. Moreover, there is an  $\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y)$ -module map

$$\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y)$$

given by inclusion of the ( $p = 0$ )-summand.

The map

$$\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(W)$$

in the theorem is then given by projecting onto the  $\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y)$  summand on which the map is  $f^* \cup g^*$ . We obtain the following commutative diagram

$$(4.12) \quad \begin{array}{ccc} \mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y) & \xrightarrow{\sim} & \mathcal{A}_{\text{pl}}(W) \\ \uparrow & \nearrow & \\ \mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y) & & \end{array}$$

of  $\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y)$ -modules. In other words, both  $\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y)$  and  $\mathcal{A}_{\text{pl}}(W)$  come with a distinguished element, called the *pointing* and the equivalence respects that distinguished element. That is, we have the following

**Corollary 4.14.** *The map  $\mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)}^L \mathcal{A}_{\text{pl}}(Y) \xrightarrow{\sim} \mathcal{A}_{\text{pl}}(W)$  of Theorem 4.13 is a quasi-isomorphism of pointed  $\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y)$ -modules.*

**Example 4.15** (The Hochschild complex as a model for  $LM$ ). The loop space  $LM$  can be defined as a pullback

$$\begin{array}{ccc} LM & \xrightarrow{\text{ev}_0} & M \\ \downarrow & & \downarrow \Delta_M \\ PM & \xrightarrow{\text{ev}_0 \times \text{ev}_1} & M \times M. \end{array}$$

We then obtain the following zig-zag of pointed  $\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)$ -modules

$$\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M) \xleftarrow{\sim} \mathcal{A}_{\text{pl}}(PM) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M) \xrightarrow{\sim} \mathcal{A}_{\text{pl}}(LM),$$

where the first arrow is the quasi-isomorphism induced by  $\mathcal{A}_{\text{pl}}(PM) \simeq \mathcal{A}_{\text{pl}}(M)$  and the second one comes from Theorem 4.13. The above zig-zag thus exhibits  $\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M)$  as a model for  $C^*(LM; \mathbb{Q})$ . Additionally, we obtain a model for the map  $\text{ev}_0 : LM \rightarrow M$  as follows:

$$\begin{array}{ccccc} \mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M) & \xleftarrow{\sim} & \mathcal{A}_{\text{pl}}(PM) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M) & \xrightarrow{\sim} & \mathcal{A}_{\text{pl}}(LM) \\ \uparrow 1 \otimes \text{id} & & \uparrow 1 \otimes \text{id} & & \uparrow \text{ev}_0 \\ \mathcal{A}_{\text{pl}}(M) & \xlongequal{\quad} & \mathcal{A}_{\text{pl}}(M) & \xlongequal{\quad} & \mathcal{A}_{\text{pl}}(M). \end{array}$$

Finally, let

$$B\mathcal{A}_{\text{pl}}(M) = \bigoplus_{p \geq 0} \mathcal{A}_{\text{pl}}(M) \otimes \overline{s\mathcal{A}_{\text{pl}}(M)^{\otimes p}} \otimes \mathcal{A}_{\text{pl}}(M)$$

denote the two-sided bar construction computing  $\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M)$ . There is a quasi-isomorphism of pointed  $\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)$ -modules

$$B\mathcal{A}_{\text{pl}}(M) \xrightarrow{\sim} \mathcal{A}_{\text{pl}}(M).$$

Since  $B\mathcal{A}_{\text{pl}}(M)$  is a quasi-free  $\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)$ -module we obtain quasi-isomorphisms

$$\begin{array}{ccc} B\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)} \mathcal{A}_{\text{pl}}(M) & \xleftarrow{\sim} & B\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M) \\ \uparrow 1 \otimes \text{id} & \nearrow 1 \otimes \text{id} & \downarrow \sim \\ \mathcal{A}_{\text{pl}}(M) & \xrightarrow{1 \otimes \text{id}} & \mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)}^L \mathcal{A}_{\text{pl}}(M). \end{array}$$

The left hand side is now exactly the definition of the Hochschild complex:

$$C_*(\mathcal{A}_{\text{pl}}(M), \mathcal{A}_{\text{pl}}(M)) \equiv B\mathcal{A}_{\text{pl}}(M) \otimes_{\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M)} \mathcal{A}_{\text{pl}}(M).$$

This shows that the Hochschild complex, as a pointed  $\mathcal{A}_{\text{pl}}(M)$ -module, is a model for  $\text{ev}_0 : LM \rightarrow M$ , giving a proof of the isomorphism (4.1) in the rational case. (See also [29, Proposition 1].)

Note that the above computations also shows that the map  $\mathcal{A}_{\text{pl}}(M) \otimes \mathcal{A}_{\text{pl}}(M) \rightarrow B\mathcal{A}_{\text{pl}}(M)$  is a model for the fibration  $PM \rightarrow M \times M$ .



Let  $A \simeq \mathcal{A}_{\text{pl}}(M)$  be any commutative dg algebra model of  $M$ . We can now replace  $\mathcal{A}_{\text{pl}}(M)$  by  $A$  in the models for  $LM$  and  $PM$  that we have obtained. As above, let

$$BA = \bigoplus_{p \geq 0} A \otimes (s\bar{A})^{\otimes p} \otimes A$$

be the two sided bar-resolution, considered as a pointed  $A \otimes A$ -module. We then have models

$$A \otimes A \longrightarrow BA \quad \text{and} \quad A \longrightarrow C_*(A, A)$$

for the fibrations  $\text{ev}_0 \times \text{ev}_1 : PM \rightarrow M$  and  $\text{ev}_0 : LM \rightarrow M$ . The latter map admits a section  $\text{cst} : M \rightarrow LM$ , which, under the identification  $LM \cong PM \times_{M \times M} M$ , is given by the diagonal embedding. Analysing the zig-zags in Example 4.15 we find that it is modelled by

$$C_*(A, A) = BA \otimes_{A \otimes A} A \longrightarrow A \otimes_{A \otimes A} A \xrightarrow{m} A$$

where  $m : A \otimes A \rightarrow A$  is the multiplication map of  $A$ .

4.4.1. *Reparametrization map  $J$ .* In this section, we will give a model of the reparametrization map

$$J : sC_*(LM) \cong C_*(LM \times I, LM \times \partial I) \longrightarrow C_*(LM, \mathcal{R}).$$

We have so far seen that the Hochschild complex  $C_*(A, A)$  can be used to model the loop space together with the evaluation and inclusion maps  $LM \hookrightarrow M$ . This model however does not come with a convenient description of the map  $\text{ev}_{0, \frac{1}{2}} = (\text{ev}_0, \text{ev}_{\frac{1}{2}}) : LM \rightarrow M \times M$ . We start the section by giving a model of  $LM$  that is more convenient to describe that map.

**Lemma 4.16.** *The fibration  $\text{ev}_{0, \frac{1}{2}} : LM \rightarrow M \times M$  admits the following pointed  $A \otimes A$ -module model:*

$$A \otimes A \longrightarrow A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$$

where  $A^{\otimes 2}$  is an  $A^{\otimes 4}$  module via the map  $(x, y, z, w) \rightarrow (xz, yw)$ . As a vector space

$$A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} = \bigoplus_{p, q \geq 0} (s\bar{A})^{\otimes p} \otimes A \otimes (s\bar{A})^{\otimes q} \otimes A$$

and the map is the inclusion into the summand with  $p, q = 0$ .

*Proof.* The map  $\text{ev}_{0, \frac{1}{2}} = (\text{ev}_0, \text{ev}_{\frac{1}{2}}) : LM \rightarrow M \times M$  is the product over  $M \times M$  of two copies of the path fibration:

$$\begin{array}{ccc} LM & \longrightarrow & PM \times PM \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{\Delta_M \times \Delta_M} & (M \times M) \times (M \times M). \end{array}$$

As in the Example 4.15 we use  $A \otimes A \rightarrow BA$  as a model for  $PM \rightarrow M \times M$ . Applying Theorem 4.13, we get a model for  $\text{ev}_{0, \frac{1}{2}} : LM \rightarrow M \times M$  as

$$\mathcal{A}_{\text{pl}}(LM) \xleftarrow{\sim} (A \otimes A) \otimes_{A^{\otimes 4}}^L (BA \otimes BA) \simeq A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$$

where one checks that  $a \otimes b \in A \otimes A$  is mapped to the right-hand side as claimed in the statement.  $\square$

**Lemma 4.17.** *The fibration  $\text{Fig}(8) \rightarrow M$  admits the following pointed  $A$ -module model*

$$A \rightarrow C_*(A, A) \otimes_A C_*(A, A) \cong A \otimes_{A^{\otimes 4}} (BA \otimes BA)$$

with the cut map and inclusions  $LM \times LM \xleftarrow{\text{cut}} \text{Fig}(8) \hookrightarrow LM$  given by the quotient maps

$$C_*(A, A) \otimes C_*(A, A) \longrightarrow C_*(A, A) \otimes_A C_*(A, A) \cong A \otimes_{A^{\otimes 4}} (BA \otimes BA) \longleftarrow BA \otimes_{A^{\otimes 2}} BA.$$

*Proof.* The space  $\text{Fig}(8)$  can be seen to be the pullback of  $PM \times PM \rightarrow M^2 \times M^2 = M^4$  along the diagonal  $M \rightarrow M^4$ , which gives us a model for  $\text{Fig}(8) \rightarrow M$  as  $A \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$ . Consider the two factorizations of the diagonal as  $M \rightarrow M^2 \rightarrow M^4$ , where the second map  $M^2 \rightarrow M^4$  is either  $(x, y) \mapsto (x, x, y, y)$  or  $(x, y) \mapsto (x, y, x, y)$ . The first version exhibits  $\text{Fig}(8)$  as the pullback of  $LM \times LM \rightarrow M \times M$  along the diagonal and gives the description of the cut map. The second version exhibits  $\text{Fig}(8)$  as the pullback of  $\text{ev}_{0, \frac{1}{2}}: LM \rightarrow M \times M$  giving the description of the inclusion  $\text{Fig}(8) \rightarrow LM$ .  $\square$

The above description of the figure eight space, allows us now to give a model for the map  $\mathcal{R} \rightarrow LM$ . Let

$$C_*(A, A) \oplus_A C_*(A, A) = \text{cone}(C_*(A, A) \oplus C_*(A, A) \rightarrow A)$$

where the map is the composition  $C_*(A, A) \oplus C_*(A, A) \rightarrow A \oplus A \rightarrow A$ , with the second map being the difference. Here, as we are working with cochain complexes, by ‘‘cone’’ we mean the following construction:

$$\text{cone}(A \xrightarrow{f} B) = (A \oplus sB, d_A + d_B + f).$$

**Lemma 4.18.** *The map  $\mathcal{R} \rightarrow \text{Fig}(8)$  is modelled by the map*

$$\begin{aligned} C_*(A, A) \otimes_A C_*(A, A) &\longrightarrow C_*(A, A) \oplus_A C_*(A, A) \\ \bar{\mathbf{a}} \otimes \bar{\mathbf{b}} \otimes c &\longmapsto \epsilon(\bar{\mathbf{a}})(\bar{\mathbf{b}} \otimes c) \oplus \epsilon(\bar{\mathbf{b}})(\bar{\mathbf{a}} \otimes c), \end{aligned}$$

where  $\epsilon(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p) = 0$  if  $p \geq 1$  and 1 if  $p = 0$ .

*Proof.* Consider the commuting diagram

$$\begin{array}{ccc} LM \times LM & \longleftarrow & LM \times M \\ \uparrow & & \uparrow \\ M \times LM & \longleftarrow & M \times M \end{array}$$

of spaces over  $M \times M$ . By pulling back along the diagonal we obtain

$$\begin{array}{ccc} \text{Fig}(8) & \longleftarrow & LM \\ \uparrow & & \uparrow \\ LM & \longleftarrow & M \end{array}$$

and  $\mathcal{R}$  is the pushout of the lower right triangle; the diagram thus encodes the inclusion map  $\mathcal{R} \rightarrow \text{Fig}(8)$ . Hence we can get a model for that commuting square in algebra, by pulling-back in the same way the previous square and using the naturally part of Theorem 4.13. This becomes

$$\begin{array}{ccc} C_*(A, A) \otimes_A C_*(A, A) & \xrightarrow{\text{id} \otimes \epsilon} & C_*(A, A) \\ \downarrow \epsilon \otimes \text{id} & & \downarrow \epsilon \\ C_*(A, A) & \xrightarrow{\epsilon} & A, \end{array}$$

from which one can read off the map given in the statement.  $\square$

We now assemble the models of  $LM$ ,  $\text{Fig}(8)$  and  $\mathcal{R}$  just obtained to give a model of reparametrization map:

**Proposition 4.19.** *In our models, the reparametrization map  $J^*: \mathcal{A}_{pl}(LM, \mathcal{R}) \rightarrow s\mathcal{A}_{pl}(LM)$*

$$\text{cone}(A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \longrightarrow C_*(A, A) \oplus_A C_*(A, A)) \xrightarrow{J^*} sC_*(A, A)$$

takes  $\alpha = (\bar{a}_1 \otimes \cdots \otimes \bar{a}_p) \otimes c \otimes (\bar{b}_1 \otimes \cdots \otimes \bar{b}_q) \otimes d$  of the subcomplex  $A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$  of the source to

$$B(\alpha) := \pm(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes \bar{c} \otimes \bar{b}_1 \otimes \cdots \otimes \bar{b}_q) \otimes d$$

in the target and maps  $\beta \oplus \gamma \in C_*(A, A) \oplus_A C_*(A, A)$  to  $\beta - \gamma$ .

One can give a proof of the above proposition using Chen's iterated integrals, see [67, Section 4.2]. We give here an alternative proof.

*Proof.* We split the reparametrization into two maps

$$(LM \times I, LM \times \partial I) \xrightarrow{\cong} (LM, LM \sqcup LM) \rightarrow (LM, \mathcal{R})$$

where  $LM \sqcup LM \rightarrow LM$  maps the two copies of  $LM$  to the left (resp. right) half-constant loops. Now there is an equivalence of pairs (an equivalence of the corresponding cones, to be precise)

$$(LM \times I, LM \times \partial I) \xrightarrow{\cong} (\text{pt}, LM \sqcup \text{pt})$$

via the map that sends one of the  $LM$  factors to  $\{\text{pt}\}$ . We can thus think of the reparametrization map as the zig-zag

$$(\text{pt}, LM \sqcup \text{pt}) \xleftarrow{\cong} (LM, LM \sqcup LM) \rightarrow (LM, \mathcal{R}).$$

In our rational model, this becomes a map

$$sC_*(A, A) \xrightarrow{\cong} \text{cone} \left( \begin{array}{c} A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \\ \downarrow \\ C_*(A, A) \oplus C_*(A, A) \end{array} \right) \leftarrow \text{cone} \left( \begin{array}{c} A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \\ \downarrow \\ C_*(A, A) \oplus_A C_*(A, A) \end{array} \right)$$

where the first map is the inclusion  $C_*(A, A) \rightarrow C_*(A, A) \oplus C_*(A, A)$  in the first summand, and the second map is the natural projection. It remains to give a left-inverse to the first map. One can check that sending  $(\alpha, \beta \oplus \gamma) \rightarrow B(\alpha) + \beta - \gamma$  defines such a chain model for such a homotopy inverse. The result follows.  $\square$

**4.4.2. Cut map.** We give now a model for the cut map used in the definition of the coproduct. Its target is  $C^*(LM \times LM, M \times LM \cup LM \times M) \simeq C^*(LM, M)^{\otimes 2}$ . Recall that the relative Hochschild chain complex  $\underline{C}_*(A, A)$  is the kernel of the (surjective) map  $C_*(A, A) \rightarrow A$  and hence a model for  $C^*(LM, M)$ .

**Proposition 4.20.** *The cut map  $(\text{Fig}(8), R) \rightarrow (LM \times LM, M \times LM \cup LM \times M)$  can be modelled as the map*

$$(4.13) \quad \text{cone} \left( \begin{array}{c} C_*(A, A) \otimes_A C_*(A, A) \\ \downarrow \\ C_*(A, A) \oplus_A C_*(A, A) \end{array} \right) \xleftarrow{\text{cut}} \text{cone} \left( \begin{array}{c} C_*(A, A) \\ \downarrow \\ A \end{array} \right)^{\otimes 2} \xleftarrow{\cong} \underline{C}_*(A, A)^{\otimes 2}.$$

defined by

$$\text{cut}((\bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1}) \otimes (\bar{b}_1 \otimes \cdots \otimes \bar{b}_p \otimes b_{p+1})) = \pm (\bar{a}_1 \otimes \cdots \otimes \bar{a}_p) \otimes (\bar{b}_1 \otimes \cdots \otimes \bar{b}_p) \otimes a_{p+1} b_{p+1}$$

sitting in the subcomplex  $C_*(A, A) \otimes_A C_*(A, A)$  of the target.

*Proof.* We have already seen in Lemma 4.17 that the cut map  $\text{Fig}(8) \rightarrow LM \times LM$  can be described as the quotient map

$$C_*(A, A) \otimes_A C_*(A, A) \leftarrow C_*(A, A) \otimes C_*(A, A).$$

To see that this map descends to a relative map, we use the same diagrams of spaces as in the proof of Lemma 4.18.  $\square$

4.4.3. *Model for the relative intersection.* We are left to find a model for the relative intersection step of (4.11). We will use the decomposition of this map given by the relative intersection product using the oriented intersection context (4.10):

$$(4.14) \quad \begin{array}{ccccccc} C_*(LM) & \longrightarrow & C_*(LM, LM|_{FM_2}) & \xleftarrow{\sim} & C_*(LM|_M, LM|_{UTM}) & \xrightarrow{\cap p_{\mathcal{E}}^* \tau} & C_{*-n}(LM|_M) \\ f \uparrow & & f \uparrow & & f \uparrow & & f \uparrow \\ C_*(\mathcal{R}) & \xlongequal{\quad} & C_*(\mathcal{R}) & \xlongequal{\quad} & C_*(\mathcal{R}) & \xrightarrow{\cap f^* p_{\mathcal{E}}^* \tau} & C_{*-n}(\mathcal{R}). \end{array}$$

The middle map is the “excision” map

$$C_*(\text{Fig}(8), LM|_{UTM}) \cong C_*(LM|_M, LM|_{UTM}) \xrightarrow{\sim} C_*(LM, LM|_{FM_2}),$$

and we need a cochain model for a homotopy inverse of that map. We start by giving a model of the spaces involved, starting from appropriate models of  $UTM$  and  $FM_2$ .

Suppose now that  $A = A_M$  is a Poincaré duality model for  $M$ , as given by Theorem 3.4. Then  $A$  has a coproduct map  $\Delta: s^{-n}A \rightarrow A \otimes A$  (dual to the intersection product of  $M$ , see Example 3.3). Lambrechts-Stanley conjectured in [57] explicit commutative dg algebra models for configuration spaces. This conjecture was shown to hold over the reals by Idrissi and Campos-Willwacher, see [45], [11, Appendix A].

For  $FM_2$ , this model is the quotient of the truncated polynomial algebra

$$\mathcal{F}_A = \left( \frac{A \otimes A[\omega_{1,2}]}{(\omega_{1,2}^2 = 0, (a \otimes 1)\omega_{1,2} = (1 \otimes a)\omega_{1,2})}, d\omega_{1,2} = \Delta(1) \right),$$

where  $\omega_{1,2}$  is a degree  $n-1$  class. The spherical fibration  $UTM$  more classically admits a model

$$\mathcal{U}_A = \left( \frac{A[\vartheta]}{(\vartheta^2 = 0)}, d\vartheta = e \right),$$

where  $\vartheta$  has degree  $n-1$ , representing the fiber, and  $e = (m \circ \Delta)(1) \in A$  is the Euler class of  $M$ .

These algebras fit into the commutative diagram

$$(4.15) \quad \begin{array}{ccc} \mathcal{U}_A & \longleftarrow & \mathcal{F}_A \\ \uparrow & & \uparrow \\ A & \xleftarrow{m} & A \otimes A, \end{array}$$

where the vertical maps are the natural inclusions and the top map takes  $\vartheta$  to  $\omega_{1,2}$ .

**Theorem 4.21.** *Let  $A$  be a Poincaré duality model for a simply-connected manifold  $M$ . Then the following hold:*

- (1) *The diagram (4.15) is a real model for (4.10), i.e. there exists a zig-zag of quasi-isomorphisms of squares of commutative dg  $\mathbb{R}$ -algebras connecting (4.15) to the diagram obtained from (4.10) by applying  $\mathcal{A}_{pl}(-)$ .*
- (2) *The map  $\phi: \text{cone}(A \rightarrow \mathcal{U}_A) \rightarrow \text{cone}(A \otimes A \rightarrow \mathcal{F}_A)$  taking  $(x, y + \vartheta z) \in A \oplus s\mathcal{U}_A$  to  $(\Delta(z), (z \otimes 1)\omega_{1,2}) \in A \otimes A \oplus s\mathcal{F}_A$ , is a model for the homotopy inverse of the map of pairs  $\iota: (UTM, M) \xrightarrow{\sim} (FM_2, M \times M)$ , and is a map of  $A \otimes A$ -modules.*
- (3) *A representative of the Thom class  $\tau \in \text{cone}(A \rightarrow \mathcal{U}_A)$  is given by*

$$\tau = (e, \vartheta),$$

where  $e = m \circ \Delta(1) \in A$  is the Euler class as above.

*Proof sketch.* Part (1) follows from the works [11] and [45]: the model of  $FM_2$  given here is that of Lambrechts-Stanley, and it is a commutative dg algebra model of  $FM_2$  over the reals by these two papers. Analysing the models, we see that the maps in Diagram (4.10) are modeled as stated, as the multiplication of  $A$  models the diagonal, and the class  $\omega_{1,2}$  corresponds to the class of the sphere in  $UTM$ . Going through the proof in [11] or [45] that  $\mathcal{F}_A$  is quasi-isomorphic to

$\mathcal{A}_{\text{pl}}(FM_2, \mathbb{R})$ , one can strengthen the statements to obtain a zig-zag of squares of commutative dg algebras, as claimed. See also [67, Proposition 8.3].

For part (2), we know that the map of pairs is a homotopy equivalence, and that it is modeled by the map  $\widehat{m}$  coming from the diagram. So it is enough to check that  $\phi$  a 1-sided homotopy inverse to  $\widehat{m}$ . The composite  $\widehat{m} \circ \phi$  takes  $(x, y + \vartheta z)$  to  $(ze, z\vartheta)$  in  $\text{cone}(A \rightarrow \mathcal{U}_A)$ . This map is homotopic to the identity by the homotopy  $h(x, y + \vartheta z) \mapsto (y, 0)$ . One checks that  $\phi$  is a map of  $A \otimes A$ -modules.

Part (3) follows from the analysis of the models in (1). Alternatively, note that  $\text{cone}(A \rightarrow \mathcal{U}_A) \sim s^{n-1}A$  and thus there is only one candidate up to a scalar for the Thom class. The scalar is determined by the condition that the image of the Thom class under the isomorphism  $H^n(M, UTM) \cong H^n(M \times M, M \times M \setminus M) \rightarrow H^n(M \times M)$  is the diagonal class. By (2) this image is  $\Delta(1) \in A \times A$ , which is the diagonal class.  $\square$

4.4.4. *Proof of Theorem 4.2.* We now assemble the results of the previous sections to give a sketch proof of Theorem 4.2. Let

$$\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}, \overline{b}_1 \otimes \cdots \otimes \overline{b}_q \otimes b_{q+1} \in \underline{C}_*(A, A)$$

be two Hochschild chains. By Proposition 4.20, applying the cut map to their tensor product we get

$$\pm(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p) \otimes (\overline{b}_1 \otimes \cdots \otimes \overline{b}_q) \otimes a_{p+1}b_{q+1} \in C_*(A, A) \otimes_A C_*(A, A).$$

Next we apply the relative intersection product as given by our algebraic model of Diagram (4.14): writing  $\mathcal{L}_A := A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$  and  $\mathcal{R}_A := C_*(A, A) \oplus_A C_*(A, A)$  for our models of  $LM$  and  $\mathcal{R}$  of Section 4.4.1, and applying Theorem 4.13 to the pullbacks of  $LM$  along the maps of Diagram (4.10), as modeled by (4.15), the intersection product is modeled by

$$\begin{array}{ccccccc} \mathcal{L}_A & \longleftarrow & \mathcal{L}_A \otimes_{A^{\otimes 2}} \text{cone}(A^{\otimes 2} \rightarrow \mathcal{F}_A) & \xrightarrow{\sim} & \mathcal{L}_A \otimes_{A^{\otimes 2}} \text{cone}(A \rightarrow \mathcal{U}_A) & \xleftarrow{\cup \tau_M} & \mathcal{L}_A \otimes_{A^{\otimes 2}} A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_A & \xlongequal{\quad} & \mathcal{R}_A & \xlongequal{\quad} & \mathcal{R}_A & \xleftarrow{\cup \tau_M} & \mathcal{R}_A, \end{array}$$

where the first map has degree  $n$ , with source

$$\mathcal{L}_A \otimes_{A^{\otimes 2}} A = (A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}) \otimes_{A^{\otimes 2}} A \cong C_*(A, A) \otimes_A C_*(A, A).$$

Recall from Theorem 4.21(3) that the Thom class is given by  $\tau_M = (e, \vartheta)$  in our model, so applying the first map to our element gives

$$\pm(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p) \otimes (\overline{b}_1 \otimes \cdots \otimes \overline{b}_q) \otimes (a_{p+1}b_{q+1}e, a_{p+1}b_{q+1}\vartheta)$$

in  $C_*(A, A) \otimes_A C_*(A, A) \otimes_A \text{cone}(A \rightarrow \mathcal{U}_A) \cong \mathcal{L}_A \otimes_{A^{\otimes 2}} \text{cone}(A \rightarrow \mathcal{U}_A)$ . Now we apply the explicit inverse of  $\text{cone}(A \otimes A \rightarrow \mathcal{F}_A) \rightarrow \text{cone}(A \rightarrow \mathcal{U}_A)$  given in Theorem 4.21 which yields

$$\pm(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p) \otimes (\overline{b}_1 \otimes \cdots \otimes \overline{b}_q) \otimes (\Delta(a_{p+1}b_{q+1}), (a_{p+1}b_{q+1} \otimes 1)\omega_{1,2})$$

in  $\mathcal{L}_A \otimes_{A^{\otimes 2}} \text{cone}(A \otimes A \rightarrow \mathcal{F}_A)$ . Next applying  $\text{cone}(A^{\otimes 2} \rightarrow \mathcal{F}_A) \rightarrow A^{\otimes 2}$ , we obtain

$$\pm(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p) \otimes (\overline{b}_1 \otimes \cdots \otimes \overline{b}_q) \otimes \Delta(a_{p+1}b_{q+1})$$

in  $\mathcal{L}_A \otimes_{A^{\otimes 2}} A^{\otimes 2} \cong A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$ . Finally, the reparametrization map  $J$  is given by Proposition 4.19 after applying the last identification and yields the formula for the coproduct as

$$\sum \pm(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes \overline{a}_{p+1}e_i \otimes \overline{b}_1 \otimes \cdots \otimes \overline{b}_q) \otimes b_{q+1}f_i \in sC_*(A, A)$$

matching the formula for the algebraic Goresky-Hingston product of Definition 3.15 (up to switching the factors, which does not make a difference on cohomology by the graded commutativity of the product, see Theorem 3.17).

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