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Infinite loop space structure(s) on the stable mapping class group

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Abstract

Tillmann introduced two infinite loop space structures on the plus construction of the classifying space of the stable mapping class group, each with different computational advantages (Invent. Math. 130 (1997) 257; Math. Ann. 317 (2000) 613). The first one uses disjoint union on a suitable cobordism category, whereas the second uses an operad which extends the pair of pants multiplication (i.e. the double loop space structure introduced by Miller, J. Differential Geom. 24 (1986) 1). She conjectured that these two infinite loop space structures were equivalent, and managed to prove that the first delooping are the same. In this paper, we resolve the conjecture by proving that the two structures are indeed equivalent, exhibiting an explicit geometric map.

Keywords: Mapping class groups; Infinite loop spaces; Rectification of diagrams

1. Introduction

Let $\Gamma_{g,n}$ be the mapping class group of a surface F of genus g with n boundary components. The classifying space $B\Gamma_{g,n}$ has the homotopy type of the moduli space of Riemann surface of type F when $n>0$. Attaching a torus with two boundary components to the surface induces a homomorphism $\Gamma_{g,1}\to\Gamma_{g+1,1}$. Let $\Gamma_\infty=\lim_{g\to\infty}\Gamma_{g,1}$ denote the stable mapping class group.

The space $\mathbb{Z}\times B\Gamma_\infty^+$, the group completion of $\coprod_{g\geqslant 0}B\Gamma_{g,1}$, has a natural double loop space structure induced by the pair of pants multiplication on $\coprod_{g\geqslant 0}B\Gamma_{g,1}$ ([9], see also [2]). In [15] Tillmann constructed an infinite loop space operad extending the pair of pants multiplication and acting on $\coprod_{g\geqslant 0}B\Gamma_{g,1}$, thus showing that the pair of pants multiplication actually induces an infinite loop space structure on $\mathbb{Z}\times B\Gamma_\infty^+$. This multiplication plays a role in conformal field theory [11],

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and has also proven useful for e.g. constructing homology operations [2,3]. Previously [13] she had exhibited $\mathbb{Z} \times B\Gamma_\infty^+$ as an infinite loop space in quite a different way, by constructing a cobordism category $\mathscr{S}$, symmetric monoidal under disjoint union of surfaces, such that $\Omega B\mathscr{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$. Note that the multiplication inducing the infinite loop space structure in this case is defined on $B\mathscr{S}$, and hence on a first deloop of $\mathbb{Z} \times B\Gamma_\infty^+$. This infinite loop space structure has also proven useful. Madsen-Tillmann [5] have constructed an infinite loop map from $\mathbb{Z} \times B\Gamma_\infty^+$, with the disjoint union infinite loop space structure, to $\Omega^\infty \mathbb{C}P_{-1}^\infty$. This map has lead recently to a proof of the Mumford conjecture [6].

Tillmann conjectured [13,3,15] that two infinite loop space structures were equivalent and managed to prove in [13] that their first deloopings are homotopy equivalent spaces. In this paper, we resolve the conjecture of Tillmann by proving that the two structures are indeed equivalent, exhibiting an explicit geometric map. Our map sends, up to homotopy, the pair of pants multiplication to the loop on disjoint union multiplication. We show that this map preserves the multiplication up to all higher homotopies, using the machinery of Dwyer and Kan [4], and hence produce an infinite loop map, which gives the equivalence.

To describe our result in more details, we have to introduce some notation. Let $\mathscr{M}$ denote Tillmann's operad as well as its associated monad. We will describe this operad in detail in Section 3. The operad has nth space $\mathscr{M}_n \simeq \coprod_{g\geqslant 0} B\Gamma_{g,n+1}$. So $\mathscr{M}(*) = \mathscr{M}_0 \simeq \coprod_{g\geqslant 0} B\Gamma_{g,1}$ and it is an $\mathscr{M}$-algebra. The cobordism category $\mathscr{S}$, described in Section 4, has objects the natural numbers. The morphism space $\mathscr{S}(n,m)$ is the classifying space of a category with objects disjoint union of surfaces with a total of n incoming and m outgoing boundaries, and with morphisms the appropriate mapping class groups (see Fig. 3). The category $\mathscr{S}$ is defined in such a way that $\mathscr{S}(n,1) = \mathscr{M}_n$. So an object of $\mathscr{M}(*)$ is a morphism from 0 to 1 in $\mathscr{S}$, and thus defines a 1-simplex in $B\mathscr{S}$. Hence there is a natural map

$$\phi : \mathscr{M}(*) \times \mathscr{M}(*) \to \Omega B\mathscr{S}$$

as two elements of $\mathscr{S}(0,1)$ define a loop in $B\mathscr{S}$. We use Barratt and Eccles' method to give the spectra of deloops explicitly, obtaining two sequences of simplicial spaces with space of p-simplices $E_p^i = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p((\mathscr{M}(*) \times \mathscr{M}(*))_+))$ and $F_p^i = \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathscr{S}))$, where Γ is the E_∞-operad with kth space $E\Sigma_k$, $\mathscr{G}$ is the group completion, and $\mathscr{M}^p$ (resp. Γ^p) means the monad functor iterated p times. (It is unfortunate that the letter Γ is used both for the mapping class groups and for the above operad. It should however always be clear from the context if it refers to the group $\Gamma_{g,k}$ or to the operad or monad Γ.) There is a map of operads $\mathscr{M} \to \Gamma$.

Theorem 1.1. *The adjoint of the map $\phi : \mathscr{M}(*) \times \mathscr{M}(*) \to \Omega B\mathscr{S}$ and the operad map $\mathscr{M} \to \Gamma$ induce maps*

$$f_p^i : E_p^i = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p((\mathscr{M}(*) \times \mathscr{M}(*))_+)) \to F_p^i = \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathscr{S}))$$

for $i \geqslant 1$ and $p \geqslant 0$, which can be rectified into an equivalence of spectra

$$(f')^i : (E')^i \xrightarrow{\simeq} (F')^i,$$

where E' and F' are spectra equivalent to E and F, respectively.

The maps f_p^i are almost simplicial maps in the sense that they satisfy all the simplicial identities except for $\delta_p f_p^i$ which is only homotopic to $f_{p-1}^i \delta_p$. The map f' is a rectification of f in the sense that the equivalence $E' \simeq E$ and $F' \simeq F$ is natural with respect to the maps f and f'.

In Section 2, we describe the method of rectification of diagrams which will be used in the proof. In Section 3, we give the construction of the operad $\mathscr{M}$, following [15], spelling out the details needed further on in the text and correcting a minor mistake. We also describe the spectrum of deloops of $\mathbb{Z} \times B\Gamma_\infty^+$ produced by $\mathscr{M}$. In Section 4, we give a description of the category $\mathscr{S}$, adapted to our needs, and produce an actual map inducing the equivalence $\Omega B\mathscr{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$. In the appendix, we related this map to Tillmann's original proof of this equivalence. In Section 5, we compare the two infinite loop space structures: we construct the map, rectify it, and show that it induces an equivalence of spectra.

We will be working most of the time in Top_*, the category of pointed topological spaces. Most of our spaces are realization of pointed simplicial spaces. We will use the notation $X_\bullet$ for a simplicial space and $X = |X_\bullet|$ for its realization.

2. Rectification of diagrams

Suppose we have a diagram of spaces and maps (of whatever shape, possibly infinite) which commutes only up to homotopy. If there are higher homotopies, it is possible to rectify it to a strictly commutative diagram, equivalent to the one we started with (in a sense to be made precise). We give here a method which is a special case of a theory treated by Dwyer and Kan [4]. The same construction was used by Segal [10]. The idea is to look at a commutative diagram as a functor from a discrete category $\mathscr{D}$ to $\mathsf{Top}_{(*)}$, the category of (pointed) topological space, and a homotopy commutative diagram as a functor from a category $\tilde{\mathscr{D}}$ to $\mathsf{Top}_{(*)}$, where the spaces of maps in $\tilde{\mathscr{D}}$ are "thicker" than in $\mathscr{D}$ (see Fig. 1 for the case of a square). As long as the morphism spaces in $\tilde{\mathscr{D}}$ are homotopy equivalent to the corresponding ones in $\mathscr{D}$, a rectification can be constructed.

We give here a precise description of the rectification in the unpointed case and prove a strong naturality statement (Proposition 2.1), as we need to know more about the rectification than what can be found in [4] or [10]. The pointed case is done similarly.

Let $\mathscr{D}$ be a discrete category and let $\tilde{\mathscr{D}}$ be a category enriched over Top with the same objects as $\mathscr{D}$ and such that there is a functor (path components functor)

$$p : \tilde{\mathscr{D}} \to \mathscr{D},$$

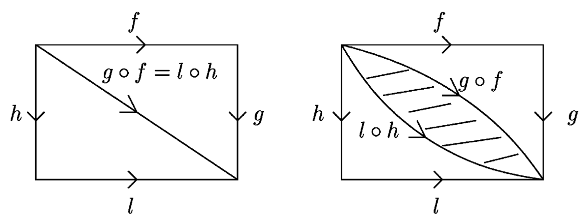


Fig. 1. Commutative and homotopy commutative square.

which is the identity on objects and induces a homotopy equivalence

$$\tilde{\mathscr{D}}(x,y) \simeq \mathscr{D}(x,y)$$

for each pair of objects x, y. So $\tilde{\mathscr{D}}$ has a contractible space of morphisms over each morphism in $\mathscr{D}$ and p is the projection. There is an induced functor

$$\mathsf{Top}^{\mathscr{D}} \xrightarrow{p^*} \mathsf{Top}^{\tilde{\mathscr{D}}},$$

from $\mathscr{D}$-*diagrams* to $\tilde{\mathscr{D}}$-*diagrams*. There is also a functor in the other direction:

$$\mathsf{Top}^{\mathscr{D}} \xleftarrow{p_*} \mathsf{Top}^{\tilde{\mathscr{D}}},$$

where p_*F is defined on an object x of $\mathscr{D}$ as the realization of a simplicial space whose nth space is

$$(p_*F)(x)_n = \coprod_{y_0,\dots,y_n \in Ob\tilde{\mathscr{D}}} F(y_0) \times \tilde{\mathscr{D}}(y_0,y_1) \times \cdots \times \tilde{\mathscr{D}}(y_{n-1},y_n) \times \mathscr{D}(p(y_n),x).$$

Two functors F and G are said to be *equivalent*, denoted $F \simeq G$, if there is a zig-zag of natural transformations $F \leftarrow F_1 \rightarrow \cdots \leftarrow F_k \rightarrow G$ which induces homotopy equivalences on objects.

Proposition 2.1. *There is an equivalence of functors*

$$p^* p_* F \simeq F$$

for any F in $\mathsf{Top}^{\tilde{\mathscr{D}}}$, *which is natural in F.*

As $\mathscr{D}$ and $\tilde{\mathscr{D}}$ have the same objects, this means in particular that $p_*F(x) \simeq F(x)$ for any object x. The functor p_*F is the *rectification* of F.

Proof. To prove the proposition, we will give an explicit sequence of natural transformations giving the equivalence and show that they are moreover natural with respect to F.

For a functor $F : \tilde{\mathscr{D}} \to \mathsf{Top}$, define the functor $\overline{p^* p_* F}$ from $\tilde{\mathscr{D}}$ to Top simplicially by

$$(\overline{p^* p_* F})(y)_n = \coprod_{y_0,\dots,y_n \in Ob\tilde{\mathscr{D}}} F(y_0) \times \tilde{\mathscr{D}}(y_0,y_1) \times \cdots \times \tilde{\mathscr{D}}(y_{n-1},y_n) \times \tilde{\mathscr{D}}(y_n,y).$$

Then there are natural transformations

$$p^* p_* F \leftarrow \overline{p^* p_* F} \rightarrow F$$

inducing equivalences $p^* p_* F(y) \simeq \overline{p^* p_* F}(y) \simeq F(y)$, for all y in $\tilde{\mathscr{D}}$. The natural transformation $\overline{p^* p_* F} \to p^* p_* F$ is induced by the projection functor $\tilde{\mathscr{D}} \to \mathscr{D}$ which is a homotopy equivalence on the space of morphisms, so it clearly induces an equivalence. Now $\overline{p^* p_* F}$ is of the form of a two-sided bar construction $B(F,D,DX)$. May gives an explicit simplicial homotopy for the equivalence $B(F,D,DX) \simeq FX$ [7, Proposition 9.9]. It can be adapted to our case. Indeed, consider the inclusion $i : F(y) \hookrightarrow \overline{p^* p_* F}(y)$ defined by $a \mapsto (a, id_y) \in F(y) \times \tilde{\mathscr{D}}(y,y)$, and the obvious evaluation map $d : \overline{p^* p_* F}(y) \to F(y)$. Clearly, $d \circ i = id$. We are left to show that $i \circ d$ is homotopic to the identity. The simplicial homotopy is given explicitly on q-simplices by

$$h_i = s_q \dots s_{i+1} \circ \eta \circ \delta_{i+1} \dots \delta_q,$$

for $i = 0, \dots, q$, where $\eta : (\overline{p^* p_* F})_i \to (\overline{p^* p_* F})_{i+1}$ is defined by adding $id_x \in \tilde{\mathscr{D}}(x,x)$ on the right of the simplex.

The first natural transformation is clearly natural in F. The second is natural in F as the diagram

$$\begin{array}{ccc} F(y_0) \times \tilde{\mathcal{D}}(y_0, y) & \longrightarrow & F(y) \\ \downarrow & & \downarrow \\ F'(y_0) \times \tilde{\mathcal{D}}(y_0, y) & \longrightarrow & F'(y) \end{array}$$

commutes because a map of functors is itself a natural transformation. □

Note that the inclusion $i : F(y) \hookrightarrow \overline{p^* p_* F}(y)$ is not natural in y.

3. The mapping class groups operad

In this section, we describe Tillmann's operad $\mathscr{M}$, correcting a minor mistake from [15] in the construction. We also give explicitly the spectrum of deloops of $\mathbb{Z} \times B\Gamma_\infty^+$ produced by $\mathscr{M}$.

Let $F_{g,n+1}$ denote an oriented surface of genus g with $n+1$ boundary components. One of the boundary components is marked; we call the n other components *free*. Each free boundary component ∂_i comes equipped with a collar, a map from $[0,\varepsilon) \times S^1$ to a neighborhood of ∂_i; for the marked boundary component, there is a map from $(\varepsilon, 0] \times S^1$ to a neighborhood of the boundary. Let $\mathrm{Diff}^+(F_{g,n+1}; \partial)$ be the group of orientation preserving diffeomorphisms which fix the collars, and let

$$\Gamma_{g,n+1} = \pi_0(\mathrm{Diff}^+(F_{g,n+1}; \partial))$$

be its group of components, the associated *mapping class group*.

We want to construct a topological operad $\mathscr{M}$ with space of k-ary operations

$$\mathscr{M}_k \simeq \coprod_{g \geqslant 0} B\Gamma_{g,k+1}$$

and composition maps induced by gluing surfaces. To make gluing associative, one has to replace the groups $\Gamma_{g,k+1}$ by equivalent groupoids.

3.1. Construction of the operad

Pick a disc $D = F_{0,1}$, a pair of pants surfaces $P = F_{0,3}$ and a torus $T = F_{1,2}$ with two boundary components, all with fixed collars of the boundary components (see Fig. 2). Define a groupoid $\mathscr{E}_{g,n,1}$ with objects (F, σ), where F is a surface of type $F_{g,n+1}$ constructed from D, P and T by gluing the marked boundary of one surface to one of the free boundaries of another using the given parametrization, and σ is an ordering of the n free boundary components (see Fig. 2). Note that each boundary component of F comes equipped with a collar. The morphisms from (F, σ) to (F', σ') are

Fig. 2. Building blocks of $\mathscr{E}_{g,k,1}$ and element of $\mathscr{E}_{2,3,1}$.

the homotopy classes $\Gamma(F,F')=\pi_0\mathrm{Diff}^+(F,F';\partial)$ of orientation preserving diffeomorphisms preserving the collars and the ordering of the boundaries. The group Σ_n acts freely on $\mathscr{E}_{g,n,1}$ by permuting the labels and $B\mathscr{E}_{g,n,1}\simeq B\Gamma_{g,n+1}$.

Gluing of surfaces induces now an associative operation on the categories. Hence we have maps on the classifying spaces

$$\gamma : B\mathscr{E}_{g,k,1}\times B\mathscr{E}_{h_1,n_1,1}\times\cdots\times B\mathscr{E}_{h_k,n_k,1}\longrightarrow B\mathscr{E}_{g+h_1+\cdots+h_k,n_1+\cdots+n_k,1}$$

induced by gluing the k last surfaces to the first one according to the labels of its free boundaries. These maps are associative and Σ-equivariant. However, $\{\coprod_{g\geqslant 0}B\mathscr{E}_{g,n,1}\}_{n\in\mathbb{N}}$ does not precisely form an operad yet as there is no unit. We will apply a quotient construction on the categories $\mathscr{E}_{g,n,1}$ which will both provide a unit and make the product induced by the pair of pants associative and unital.

3.1.1. Quotient construction

To make the multiplication induced by the pair of pants associative, we need to identify subsurfaces of the form $\gamma(P;\_,P)$ to subsurfaces of the form $\gamma(P;P,\_)$. For the unit, we need to identify $\gamma(P;\_,D)$ and $\gamma(P;D,\_)$ to a circle. This circle will also be a unit for the operad. In [15], Tillmann does a quotient construction by picking morphisms $\phi_1:\gamma(P;D,\_)\to\gamma(P;\_,D)$ and $\phi_2:\gamma(P;\_,P)\to\gamma(P;P,\_)$ and uses composition of these morphisms to identify the surfaces. She then chooses an identification of $\gamma(P;\_,D)$ to the circle and repeats the process. This is not precisely correct as any choice of ϕ_1,ϕ_2 would not yield associative operad maps on the quotient categories. It will work only with the canonical choice which is the "identity". We prove here that this canonical choice exists and that it makes the quotient construction possible. We do both quotient constructions at once.

Define $\mathscr{S}_{0,1,1}$ to be the category with one object (thought of as the circle) and with $\mathbb{Z}$ as set of morphisms (thought of as the Dehn twists around that circle, which will make sense when we glue the circle to another surface). For $(g,n)\neq(0,1)$, define $\mathscr{S}_{g,n,1}$ to be the full subcategory of $\mathscr{E}_{g,n,1}$ with set of objects all surfaces which do not contain subsurfaces of the form $\gamma(P,\_,P)$, $\gamma(P,\_,D)$, and $\gamma(P,D,\_)$.

Claim 1. *For each object F in $\mathscr{E}_{g,n,1}$, there is a unique object $\bar{F}$ in $\mathscr{S}_{g,n,1}$ obtained from F by a sequence of the following operations: replacing a subsurface $\gamma(P,\_,P)$ by $\gamma(P,P,\_)$, and collapsing a subsurface of the form $\gamma(P,\_,D)$ or $\gamma(P,D,\_)$ to a circle. In particular, the gluing operation on objects of $\coprod\mathscr{S}_{g,n,1}$ defined by $F\Box_{\mathscr{S}}G:=\overline{F\Box G}$, using the gluing $\Box$ defined on $\mathscr{E}_{g,n,1}$, is associative.*

Claim 2. *For each F in $\mathscr{E}_{g,n,1}$, one can define a morphism*

$$\phi_F : F\to\bar{F}$$

in $\mathscr{E}_{g,n,1}$, *such that all diagrams of the form*

$$\begin{array}{ccccc} & & F\square\overline{G\square H} & \xrightarrow{\phi_{F\square\overline{G\square H}}} & \overline{F\square\overline{G\square H}} \\ & \nearrow^{\mathrm{id}\square\phi_{G\square H}} & & & \\ F\square G\square H & & & & \Big\downarrow = \\ & \searrow_{\phi_{F\square G}\square\mathrm{id}} & & & \\ & & \overline{F\square G}\square H & \xrightarrow[\phi_{\overline{F\square G}\square H}]{} & \overline{\overline{F\square G}\square H} \end{array}$$

commute.

Proof of Claim 1. Consider first an element (F,σ) of $\mathscr{E}_{0,n,1}$, i.e. a surface F of genus 0 together with a labeling σ of the free boundary components (σ is a permutation of the canonical labeling). The operations allowed do not change the number of free boundaries, nor does it permute the boundaries. In $\mathscr{S}_{0,n,1}$ there is only one surface with labeling σ, and this surface can clearly be obtained from F by a finite sequence of the prescribed moves. This surface is $\bar{F}$.

For a general surface F with labeling σ, the moves only affect the subsurfaces of F built out of P's and D's. Each of the maximal such subsurface has a unique image in the relevant $\mathscr{S}_{0,n_i,1}$. So $\bar{F}$ is the unique surface obtained by transforming each of those subsurfaces of F.

Finally, gluing as defined in the claim is associative by the uniqueness of the representative of $F\square G\square H$ in $\mathscr{S}_{g,n,1}$. □

Proof of Claim 2. Fix three non-intersecting curves on the pair of pants P, from 0 of the marked boundary to 0 of the first free boundary, from π of the this boundary to 0 of the second free boundary, and from π of this boundary to π of the marked boundary (where we think of S^1 as parametrized by $[0,2\pi[$). This divides the pair of pants into two discs. Fix also a curve on the disc D, from 0 to π of its boundary.

Now any surface built out of P's and D's comes equipped with a system of curves dividing the surface into two discs. These curves run from 0 of the marked boundary to 0 of the first (in the canonical ordering) free boundary, then from π of that boundary to 0 of the next, and so on until one goes back to π of the marked boundary. Choose a map $F\to\bar{F}$ which sends the curves of F to the corresponding ones in $\bar{F}$. As the curves divide the surfaces into discs, by the smooth "Alexander trick" [12] this map is unique up to isotopy. It is, up to isotopy, the identity on the discs. Now define ϕ_F to be the component of this map in $\mathrm{Diff}^+(F,\bar{F})$.

For a general surface F, define ϕ_F to be the map defined by the above on each maximal subsurface of F built out of P's and D's, and the identity on the tori. The diagram in the claim commutes by the Alexander trick. □

Now one can define an operad structure on the categories $\mathscr{S}_{g,n,1}$. The new structure maps $\bar{\gamma}$ are defined on objects by taking the unique representative of the image of γ on the surfaces:

$$\bar{\gamma}(F,G_1,\ldots,G_k)=\overline{\gamma(F,G_1,\ldots,G_k)}.$$

On morphisms,

$$\bar{\gamma}(f, g_1, \ldots, g_k) = \phi_{H'}\gamma(f, g_1, \ldots, g_k)\phi_H^{-1},$$

where H and H' are the images by γ of the sources and target surfaces of the maps f and g_i. The associativity of $\bar{\gamma}$ on objects follows from the associativity of gluing. On morphisms, it follows from the commutativity of the diagram in Claim 2.

Define the operad $\mathscr{M}$ by

$$\mathscr{M}_k = \coprod_{g \geqslant 0} B\mathscr{S}_{g,k,1}$$

with structure maps induced by $\bar{\gamma}$. Note that there is a map of operads

$$\mathscr{M} \xrightarrow{\pi} \Gamma,$$

where Γ is the infinite loop space operad with kth space $E\Sigma_k$, the classifying space of the translation category of the symmetric groups [1], and π is given on objects by the projection to the labels.

3.2. Infinite loop space structure

Let $\mathscr{G}$ denote the group completion functor from the category of monoids to the category of groups and let $(\mathscr{M}, \mu_{\mathscr{M}}, \eta_{\mathscr{M}})$ denote the monad associated to the operad $\mathscr{M}$ [7]. The pair of pants multiplication induces a monoid structure on $\mathscr{M}$-algebras. $\mathscr{M}(*)$ is an $\mathscr{M}$-algebra and

$$\mathscr{G}\mathscr{M}(*) \simeq \mathscr{G}\left(\coprod_{g \geqslant 0} B\Gamma_{g,1}\right) \simeq \mathbb{Z} \times B\Gamma_\infty^+.$$

Tillmann's main theorem in [15] can be restated as follows (cf. Theorem 4.1):

Theorem 3.1 (Tillmann [15]). *Let X be an $\mathscr{M}$-algebra. Then $\mathscr{G}X$ is weakly homotopy equivalent to an infinite loop space whose ith delooping is the realization of the simplicial space $\mathscr{G}\Gamma(S^i \wedge \mathscr{M}^\bullet(\mathscr{M}(*) \times X))$.*

We will denote by $E^i_\bullet$ the deloops of $\mathscr{G}\mathscr{M}(*)$. So the space of p-simplices of $E^i_\bullet$ is

$$E^i_p = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p(\mathscr{M}(*) \times \mathscr{M}(*))).$$

We describe next the simplicial structure on $E^i_\bullet$. Let $\mu_{\mathscr{M}}$, $\eta_{\mathscr{M}}$ and μ_Γ, η_Γ denote the product and unit maps of the monads $\mathscr{M}$ and Γ and let θ denote the diagonal $\mathscr{M}$-algebra structure map of $\mathscr{M}(*) \times \mathscr{M}(*)$. Let $\pi : \mathscr{M} \to \Gamma$ be the projection of operads.

For any operad $\mathscr{P}$, there is an assembly map $a : A \times P(X) \to P(A \times X)$, sending an element $(a, p, x_1, \ldots, x_k)$ to $(p, (a, x_1), \ldots, (a, x_k))$. As $\Gamma(*) = \{*\}$, the sequence of maps

$$\Gamma(A \times \mathscr{M}(X)) \xrightarrow{a} \Gamma(\mathscr{M}(A \times X)) \xrightarrow{\pi} \Gamma\Gamma(A \times X) \xrightarrow{\mu_\Gamma} \Gamma(A \times X)$$

induces a map on smash products

$$\lambda : \Gamma(A \wedge \mathscr{M}(X)) \to \Gamma(A \wedge X).$$

The simplicial structure on $E^i_\bullet$, $i \geqslant 0$, is defined as follows:

$$\delta_0 = \mathscr{G}(\lambda) : E^i_p \to E^i_{p-1},$$

$$\delta_i = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^{i-1}(\mu_{\mathscr{M}})) : E^i_p \to E^i_{p-1} \quad \text{for } 1 \leqslant i < p,$$

$$\delta_p = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^{p-1}(\theta)) : E^i_p \to E^i_{p-1},$$

$$s_i = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^i(\eta_{\mathscr{M}})) : E^i_p \to E^i_{p+1} \quad \text{for } 0 \leqslant i \leqslant p.$$

Let $\underline{F} = F_0 \overset{f_1}{\leftarrow} \cdots \overset{f_q}{\leftarrow} F_q$ be a p-simplex of $\mathscr{M}(*)$ and let D denote the 0-simplex represented by the disc.

Proposition 3.2. *Let $\phi : \mathscr{M}(*) \to |\mathscr{G}\Gamma(\mathscr{M}^\bullet(\mathscr{M}(*) \times \mathscr{M}(*)))|$ be the map which sends the point $[\underline{F}, t]$ to the 0-simplex $(1, D, [\underline{F}, t]) \in \mathscr{G}\Gamma(\mathscr{M}(*) \times \mathscr{M}(*))$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(*) & \xrightarrow{\ \phi\ } & |\mathcal{G}\Gamma(\mathcal{M}^\bullet(\mathcal{M}(*) \times \mathcal{M}(*)))| \\ \downarrow & \nearrow_{\simeq} & \\ \mathcal{G}\mathcal{M}(*) & & \end{array}$$

This can be proved by studying the map of fibrations which Tillmann uses to prove the equivalence $\mathscr{G}X \simeq \mathscr{G}\Gamma(\mathscr{M}^\bullet(\mathscr{M}(*) \times X))$.

4. The cobordism category

In this section, we first set up a variant of Tillmann's cobordism category $\mathscr{S}$. In [14], the morphism spaces are categories similar to the categories $\mathscr{E}_{g,n,1}$ defined in Section 3.1. Our version of $\mathscr{S}$ is obtained by applying the quotient construction of Section 3.1.1 to Tillmann's $\mathscr{S}$. This version is more amenable to the comparison of the two infinite loop space structures. We then construct an explicit equivalence $\mathbb{Z} \times B\Gamma^+_\infty \xrightarrow{\simeq} \Omega B\mathscr{S}$. We show in the appendix how this proof relates to Tillmann's proof.

The objects of $\mathscr{S}$ are the natural numbers $0, 1, 2, \ldots$. The morphism space $\mathscr{S}(n, m) = B\mathscr{S}_{g,n,m}$, where $\mathscr{S}_{g,n,m}$ is a category whose objects are surfaces built out of P, T and D as in the case of the operad $\mathscr{M}$ but allowing disjoint union of surfaces (with a component-wise quotient construction) and labeling both the inputs and the outputs (see Fig. 3). The morphisms of $\mathscr{S}_{g,n,m}$ are homotopy classes of diffeomorphisms preserving the orientation, the collars and the ordering of the boundary components. In particular $\mathscr{S}(k, 1) = \mathscr{M}_k$. Also, a morphism from n to m has exactly m components. The only morphism to 0 is the identity in $\mathscr{S}(0, 0)$. Note that $\mathscr{S}(n, n)$ contains the symmetric group, represented by disjoint copies of the circle with labels "on each side". Composition in $\mathscr{S}$ is induced

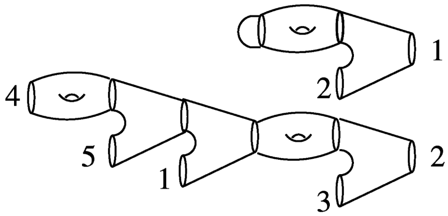


Fig. 3. Morphism from 5 to 2 in $\mathscr{S}$.

by gluing the surfaces according to the labels, which can be done using the structure maps of the operad $\mathscr{M}$ on each component. Disjoint union of surfaces induces a symmetric monoidal structure on $\mathscr{S}$. As $B\mathscr{S}$ is connected, it is an infinite loop space.

We use Barratt and Eccles' machinery [1] to produce the deloops of $\Omega B\mathscr{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$. Their main theorem can be restated as follows:

Theorem 4.1 (Barratt and Eccles [1]). *Let X be a Γ-algebra. Then $\mathscr{G}X$ is weakly equivalent to an infinite loop space whose ith deloop is the realization of the simplicial space $\mathscr{G}\Gamma(S^i \wedge \Gamma^\bullet(X))$.*

So in the case we are interested in, the space of p-simplices of the ith deloops is given by

$$F_p^i = \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathscr{S})) \quad \text{for } i \geqslant 1.$$

The simplicial structure of $F_\bullet^i$ is similar to the one of $E_\bullet^i$, which is given in detail in Section 3.2.

Note that $\mathscr{S}(0,1) = \mathscr{M}_0 = \mathscr{M}(*)$. Recall that the pair of pants multiplication induces a monoid structure on this space and that $\mathscr{G}\mathscr{M}(*) = \mathscr{G}(\mathscr{S}(0,1))$ denotes its group completion, which is homotopy equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$. In the following proposition, we use the fact that a morphism in $\mathscr{S}$ is a 1-simplex in $B\mathscr{S}$, and hence two morphisms from 0 to 1 define a loop in $B\mathscr{S}$.

Proposition 4.2. *Define $\psi : \mathscr{S}(0,1) \to \Omega B\mathscr{S}$ by $\psi(\underline{F}) = \underline{F}\binom{1}{0}D$ is the loop from 0 to 1 along the morphism defined by $\underline{F}$ followed by the morphism defined by the disc D taken backwards. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}(*) = \mathcal{S}(0,1) & \xrightarrow{\psi} & \Omega B\mathcal{S} \\
\downarrow & \nearrow{\scriptstyle\simeq} & \\
\mathcal{G}\mathcal{M}(*) = \mathcal{G}\mathcal{S}(0,1). & &
\end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{S}(0,1) & & \xrightarrow{\psi} & & & & \Omega B\mathcal{S} \\
\downarrow & \searrow{\scriptstyle f} & & & & & \downarrow{\scriptstyle =} \\
\mathcal{G}\mathcal{S}(0,1) & \xrightarrow{\simeq} & \Omega B(\mathcal{S}(0,1)) & \xrightarrow[g]{\simeq} & \Omega_{1,1}(B\mathcal{S}) & \xrightarrow[h]{\simeq} & \Omega B\mathcal{S}
\end{array}$$

where $\Omega_{1,1}(B\mathscr{S})$ denotes the space of paths from 1 to 1 in $B\mathscr{S}$. The map f is Quillen's group completion map, which sends an element $\underline{F}$ of the monoid $\mathscr{S}(0,1)$ to the loop it defines in its classifying space (as a monoid). To define g, consider the map $\mathscr{S}(0,1) \to \mathscr{S}(1,1)$ induced by gluing a pair of pants: $\underline{F} \mapsto \underline{F}\Box P$ (glue the surface to the left leg, i.e. compose $\underline{F} \coprod S^1$ with P in $\mathscr{S}$). This induces a functor from the monoid $\mathscr{S}(0,1)$ to the category $\mathscr{S}$. Indeed, one can check that the pairs of pants multiplication is mapped to composition in $\mathscr{S}$. Tillmann showed in [13] (Proposition 4.1) that g induces a homotopy equivalence on the classifying spaces. Finally, the map h is defined by precomposing with the path from 0 to 1 (1-simplex) given by the disc and postcomposing by the same path taken backwards.

The diagram commutes up to homotopy. Indeed, starting with $\underline{F}$ in $\mathscr{S}(0,1)$, the loop obtained by following the bottom of the diagram is going along D from 0 to 1, then $\underline{F}\Box P$ from 1 to 1 and lastly D again backwards from 1 to 0. This path is homotopy equivalent to $\psi(\underline{F})$ as $D \circ (\underline{F}\Box P) = \underline{F}$ in $\mathscr{S}$, which means that there is a 2-simplex in $B\mathscr{S}$ providing the required homotopy. □

Tillmann's proof that $\Omega B\mathscr{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$ is by showing that $\mathbb{Z} \times B\Gamma_\infty^+$ is equivalent to a homotopy fiber which is known to be $\Omega B\mathscr{S}$. We will show in the appendix that the map described above is a natural choice in this context to make the equivalence explicit. This leads to a more natural proof of the proposition.

5. Comparison of the two structures

In this section, we compare the two infinite loop space structures. We first construct in 5.1 the maps $f_p^i : E_p^i \to F_p^i$ of Theorem 1.1. In 5.2, we rectify these maps to simplicial maps between simplicial spaces $(E')_\bullet^i$ and $(F')_\bullet^i$ equivalent to $E_\bullet^i$ and $F_\bullet^i$. In 5.3, we show that this rectification provides a map of spectra which in 5.4 is shown to be an equivalence. Theorems 5.3, 5.8 and 5.9 combine to prove the main Theorem 1.1.

5.1. Construction of a map

We want to construct a map from $E_p^i = \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p(\mathscr{M}(*) \times \mathscr{M}(*)))$, the space of p-simplices of the ith deloop of $\mathscr{G}\mathscr{M}(*)$, to $F_p^i = \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathscr{S}))$, the p-simplices of the $(i-1)$th deloop of $B\mathscr{S}$, for $i \geqslant 1$. We will do this by combining a map from $\mathscr{M}(*) \times \mathscr{M}(*)$ to $\Omega B\mathscr{S}$, the operad map $\mathscr{M} \to \Gamma$ and an assembly map $S^1 \wedge \Gamma(X) \to \Gamma(S^1 \wedge X)$.

By construction, an element of $\mathscr{M}(*)$ is a morphism from 0 to 1 in the category $\mathscr{S}$, and hence a 1-simplex in its classifying space $B\mathscr{S}$. In particular, two such elements define a loop in $B\mathscr{S}$. The map obtained using this remark is naturally bisimplicial:

$$\tilde{\phi}_{p,q} : S_p^1 \times (\mathscr{M}_q(*) \times \mathscr{M}_q(*)) \to B_{p,q}\mathscr{S},$$

where S^1 is viewed as a simplicial space with two 0-simplices x_0, x_1 and two non-degenerate 1-simplices y_1, y_2 (see Fig. 4), and $B\mathscr{S}$ is viewed as a bisimplicial space with the second simplicial dimension coming from the simplicial structure of its morphism spaces.

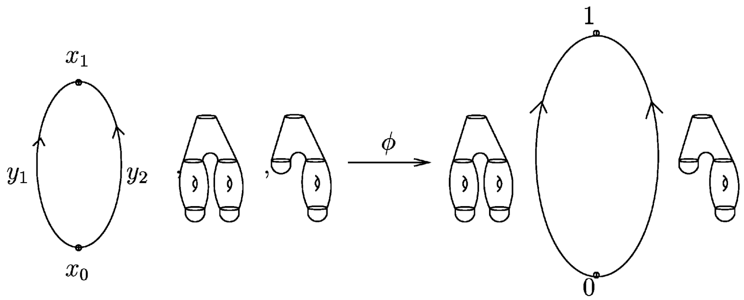


Fig. 4. Map from $S^1 \wedge (\mathscr{M}(*) \times \mathscr{M}(*))_+$ to $B\mathscr{S}$.

Define
$$\tilde{\phi}_{0,q}(x_i, \underline{F}, \underline{G}) = i \qquad \text{for } i = 0, 1,$$
$$\tilde{\phi}_{1,q}(y_1, \underline{F}, \underline{G}) = 0 \xrightarrow{F} 1,$$
$$\tilde{\phi}_{1,q}(y_2, \underline{F}, \underline{G}) = 0 \xrightarrow{G} 1.$$

This induces a map $\phi : S^1 \wedge (\mathscr{M}(*) \times \mathscr{M}(*))_+ \to B\mathscr{S}$. The base point of $\mathscr{M}(*) \times \mathscr{M}(*)$ is (D, D) and its image under $\tilde{\phi}$ is a canonically contractible loop, but it is not actually the trivial loop. The following proposition shows that we can work with $(\mathscr{M}(*) \times \mathscr{M}(*))_+$ instead of $\mathscr{M}(*) \times \mathscr{M}(*)$. This might sound like an unnatural solution to the problem, but it also seems to be the simplest solution.

Proposition 5.1. *Consider the diagonal $\mathscr{M}$-algebra structure on $(\mathscr{M}(*)\times\mathscr{M}(*))_+$ with the additional basepoint acting as an additional unit. Then the inclusion $\mathscr{M}(*)\times\mathscr{M}(*) \hookrightarrow (\mathscr{M}(*)\times\mathscr{M}(*))_+$ induces an equivalence of spectra*

$$\mathscr{G}\Gamma(S^i \wedge \mathscr{M}^\bullet(\mathscr{M}(*) \times \mathscr{M}(*))) \xrightarrow{\simeq} \mathscr{G}\Gamma(S^i \wedge \mathscr{M}^\bullet((\mathscr{M}(*) \times \mathscr{M}(*))_+)).$$

Proof. First consider the monoid $(\mathbb{N} \times \mathbb{N})_+$ where the added basepoint act as an additional unit. Then $\mathscr{G}((\mathbb{N} \times \mathbb{N})_+) = \mathbb{Z} \times \mathbb{Z}$, the two units being identified by the group completion. It follows that

$$\mathscr{G}((\mathscr{M}(*) \times \mathscr{M}(*))_+) \simeq \mathbb{Z} \times \mathbb{Z} \times B\Gamma_\infty^+ \times B\Gamma_\infty^+ \simeq \mathscr{G}(\mathscr{M}(*) \times \mathscr{M}(*)).$$

The inclusion $\mathscr{M}(*) \times \mathscr{M}(*) \hookrightarrow (\mathscr{M}(*) \times \mathscr{M}(*))_+$ clearly induces a map of spectra. By a fibration argument, using the equivalence above one shows that the map is an equivalence on the 0th spaces of the spectra. The result then follows from the fact that the spectra are connective. □

From now on, we will work with $\mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p((\mathscr{M}(*)\times\mathscr{M}(*))_+))$ instead of $\mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p(\mathscr{M}(*)\times \mathscr{M}(*)))$ and denote the former also by E_p^i for simplicity.

Recall from [1] that for the monad Γ there is an assembly map

$$a : A \wedge \Gamma(X) \to \Gamma(A \wedge X),$$

defined by $a(y,\underline{\sigma},x_1,\ldots,x_n)=(\underline{\sigma},[y,x_1],\ldots,[y,x_n])$. This also induces a map $\mathscr{A}\wedge\mathscr{G}\Gamma(X)\to\mathscr{G}\Gamma(A\wedge X)$. Combining a, ϕ and the operad map $\pi:\mathscr{M}\to\Gamma$, we get a map

$$\begin{array}{ccc}
\mathcal{G}\Gamma(S^i\wedge\mathcal{M}^p((\mathcal{M}(*)\times\mathcal{M}(*))_+)) & \xrightarrow{f^i_p} & \mathcal{G}\Gamma(S^{i-1}\wedge\Gamma^p(B\mathcal{S})) \\
\Big\downarrow{\scriptstyle \mathcal{G}\Gamma(S^i\wedge\pi^p)} & & \\
\mathcal{G}\Gamma(S^i\wedge\Gamma^p((\mathcal{M}(*)\times\mathcal{M}(*))_+)) & \nearrow{\scriptstyle \mathcal{G}\Gamma(S^{i-1}\wedge\Gamma^p(\phi))} & \\
\Big\downarrow{\scriptstyle \mathcal{G}\Gamma(S^{i-1}\wedge a)} & & \\
\mathcal{G}\Gamma(S^{i-1}\wedge\Gamma^p(S^1\wedge(\mathcal{M}(*)\times\mathcal{M}(*))_+)). & &
\end{array}$$

Proposition 5.2. *Let δ_j and s_j denote the boundary and degeneracy maps of both the simplicial spaces $E^i_\bullet$ and $F^i_\bullet$. Then the maps $f^i_p:E^i_p\to F^i_p$ for $i\geqslant 1$ and $p\geqslant 0$ satisfy*

$$\delta_j f^i_p = f^i_{p-1}\delta_j \quad \text{for } 0\leqslant j<p$$

and

$$s_j f^i_p = f^i_{p+1}s_j \quad \text{for } 0\leqslant j\leqslant p.$$

Proof. The boundary maps δ_j, for $0\leqslant j<p$, and the degeneracies for $E^i_\bullet$ and $F^i_\bullet$ are defined in terms of the operad structure maps of $\mathscr{M}$ and Γ. The map $f_\bullet$ commutes with all of those maps because f maps $\mathscr{M}$ to Γ via an operad map. □

The last commutation relation necessary to have a simplicial map, $\delta_p f_p = f_{p-1}\delta_p$, is satisfied only up to homotopy. The reason this happens, which is main difficulty in our construction, is that the map ϕ does not strictly commute with the action of $\mathscr{M}$ on $(\mathscr{M}(*)\times\mathscr{M}(*))_+$ and of Γ on $B\mathscr{S}$: the diagram

$$\begin{array}{ccc}
S^1\wedge\mathcal{M}((\mathcal{M}(*)\times\mathcal{M}(*))_+) & \longrightarrow & \Gamma(B\mathcal{S}) \\
\Big\downarrow & & \Big\downarrow \\
S^1\wedge(\mathcal{M}(*)\times\mathcal{M}(*))_+ & \longrightarrow & B\mathcal{S}
\end{array}$$

commutes only up to homotopy. Fig. 6 shows the two loops obtained using the two possible compositions in the diagram when the element in the first $\mathscr{M}$ is a pair of pants and the elements of $(\mathscr{M}(*)\times\mathscr{M}(*))_+$ are $(F_{1,1},F_{0,1})$ and $(F_{1,1},F_{1,1})$ (following the top of the diagram, we forget about the pair of pants and take the loop on the disjoint union of the surfaces, whereas following the bottom we glue the pair of pants to the surfaces and then form a loop in $B\mathscr{S}$). There is however a canonical homotopy given by the pair of pants (or more generally by the element of $\mathscr{M}$ used to multiply). As composition in $\mathscr{S}$ is by gluing, one can see in the picture that the two loops are boundaries of two 2-simplices of $B\mathscr{S}$ and hence are homotopy equivalent.

We show in the next section how to produce all the higher homotopies necessary to rectify our map into a simplicial map.

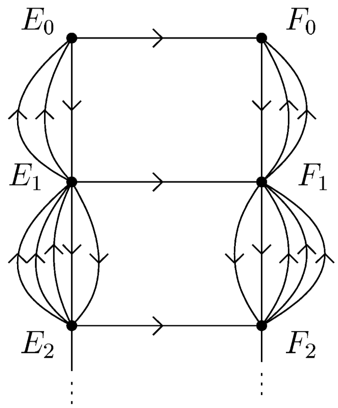


Fig. 5. Category $\mathscr{D}$.

5.2. Rectification of the map

Theorem 5.3. *There exist simplicial spaces $(E')^i_\bullet$ and $(F')^i_\bullet$ equivalent to the simplicial spaces $E^i_\bullet$ and $F^i_\bullet$ and a simplicial map $(f')^i_\bullet : (E')^i_\bullet \to (F')^i_\bullet$ such that the following diagram commutes*:

$$\begin{array}{ccc} E^i_p & \xrightarrow{f^i_p} & F^i_p \\ \simeq \updownarrow & & \updownarrow \simeq \\ (E')^i_p & \xrightarrow{(f')^i_p} & (F')^i_p \end{array}$$

To prove this theorem, we will use the method described in Section 2. We first need to construct the categories $\mathscr{D}$ and $\tilde{\mathscr{D}}$ relevant to our situation.

Let Δ^{op} denote the simplicial category: the objects of Δ^{op} are the natural numbers and there are maps $\delta_i : p \to p-1$ and $s_i : p \to p+1$ for each $i = 0,\dots,p$, satisfying the simplicial identities. So a Δ^{op}-diagram is a simplicial space. Any morphism in $\Delta^{\mathrm{op}}(p,q)$ can be expressed uniquely as a sequence $s_{j_t}\dots s_{j_1}\delta_{i_s}\dots\delta_{i_1}$ with $0 \leqslant i_s < \cdots < i_1 \leqslant p$ and $0 \leqslant j_1 < \cdots < j_t \leqslant q$ and $q-t+s = p$ [8].

Let $\mathscr{D}$ be the category whose $\mathscr{D}$-diagrams are precisely a couple of simplicial spaces with a simplicial map between them (see Fig. 5). So $\mathscr{D}$ has two copies of the natural numbers as set of objects, denoted E_p and F_p for $p \in \mathbb{N}$. The full subcategory of $\mathscr{D}$ containing all the E_p's is isomorphic to Δ^{op}. So $\mathscr{D}(E_p,E_q) = \Delta^{\mathrm{op}}(p,q)$. Similarly $\mathscr{D}(F_p,F_q) = \Delta^{\mathrm{op}}(p,q)$. Finally, there is a unique map $f_p \in \mathscr{D}(E_p,F_p)$ and it satisfies the simplicial identities $\delta_i f_p = f_{p-1}\delta_i$ and $s_i f_p = f_{p+1}s_i$ for $i = 0,\dots,p$. So any morphism in $\mathscr{D}$ from E_p to F_q can be written uniquely as a sequence $s_{j_t}\dots s_{j_1}\delta_{i_s}\dots\delta_{i_1} f_p$ where the indices are as above.

We now define the category $\tilde{\mathscr{D}}$ in such a way that the data given in 5.1 will induce a functor from $\tilde{\mathscr{D}}$ to Top_*. $\tilde{\mathscr{D}}$ has the same objects as $\mathscr{D}$ and we will again denote them by E_p and F_p for $p \in \mathbb{N}$. Also, $\tilde{\mathscr{D}}(E_p,E_q) = \mathscr{D}(E_p,E_q)$ and $\tilde{\mathscr{D}}(F_p,F_q) = \mathscr{D}(F_p,F_q)$. To describe $\tilde{\mathscr{D}}(E_p,F_q)$, we first need to define the *degeneracy degree* $d(g)$ of a morphism $g \in \mathscr{D}(E_p, F_q)$. If $g = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_1} f_p$ in the above notation, then $d(g)$ is the biggest k such that $i_k = p - k + 1$, and $d(g) = 0$ if no such k exists. In other words, d counts the number of "bad" maps, i.e. last boundary maps, occurring in g. Define

$$\tilde{\mathscr{D}}(E_p, F_q) = \coprod_{g \in \mathscr{D}(E_p, F_q)} \Delta_{d(g)},$$

where $\Delta_d = \{(t_0, \dots, t_d) \in \mathbb{R}^{d+1} | t_i \geqslant 0, \Sigma t_i = 1\}$ is the standard d-simplex. Note that for each $g \in \mathscr{D}(E_p, F_q)$, there is an inclusion

$$\tilde{g} : \Delta_{d(g)} \to \tilde{\mathscr{D}}(E_p, F_q)$$

whose image is the space of morphisms "sitting over g".

Recall that all the simplicial identities between the δ_i's and s_j's are satisfied in $\tilde{\mathscr{D}}$. Also, note that there is a unique map (0-simplex) between E_p and F_p sitting over the simplicial map $f_p \in \mathscr{D}(E_p, F_p)$. We denote this map again by f_p and we set the relation $s_i f_p = f_{p+1} s_i$ for $0 \leqslant i \leqslant p$ and $\delta_i f_p = f_{p-1} \delta_i$ for $0 \leqslant i < p$. Because the last relation, when $i = p$, does not hold in $\tilde{\mathscr{D}}$, there are exactly $d(g) + 1$ maps formed of compositions of δ_i's, s_j's and an f_k projecting down to g in $\mathscr{D}$. We define those compositions to be the vertices of $\Delta_{d(g)}$. More precisely, for $0 \leqslant k \leqslant d(g)$, define

$$g_k := s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{k+1}} f_{p-k} \delta_{i_k} \dots \delta_{i_1} := \tilde{g}(0, \dots, 1, \dots, 0),$$

where 1 is in the kth position counting backwards.

Now composition in $\tilde{\mathscr{D}}$ is determined by the vertices of the simplices. A map in $\tilde{\mathscr{D}}(E_p, F_q)$ can only be pre-composed by a map in $\tilde{\mathscr{D}}(E_r, E_p)$ or post-composed by a map in $\tilde{\mathscr{D}}(F_q, F_s)$. Using the identities given above, we know how those compositions are defined on the vertices of the simplices of $\tilde{\mathscr{D}}(E_p, F_q)$. We then extend the composition simplicially.

Theorem 5.4. *Let $\tilde{\mathscr{D}}$ be the category defined above. For each $i \geqslant 1$, there is a functor*

$$L_i : \tilde{\mathscr{D}} \to \mathsf{Top}_*$$

such that $L_i(E_\bullet) = E^i_\bullet$, $L_i(F_\bullet) = F^i_\bullet$, where $E_\bullet$ and $F_\bullet$ denote the two subcategories of $\tilde{\mathscr{D}}$ isomorphic to Δ^{op}, and $L_i(f_p) = f^i_p$.

Proof. As $E^i_\bullet$ and $F^i_\bullet$ are simplicial spaces, the restriction of L_i to each copy of Δ^{op} in $\tilde{\mathscr{D}}$ is a well-defined functor. As the map f^i_p satisfies the identities satisfied by f_p with the boundary and degeneracy maps (Proposition 5.2), L_i is also well defined on the vertices of the simplices of the morphism spaces $\tilde{\mathscr{D}}(E_p, F_q)$. We have to show that we can extend the definition of L_i to the whole simplices.

Let $d = d(g)$ and g_k be the kth vertex of the d-simplex over g as described above. To simplify notations, let

$$X = (\mathscr{M}(*) \times \mathscr{M}(*))_+.$$

Because only the first d last boundaries appear in g, the map g_k defined from the space E^i_p to F^i_q is determined by a map

$$h_k : S^1 \wedge \mathscr{M}^d(X) \xrightarrow{\delta_{i_k} \dots \delta_{i_1}} S^1 \wedge \mathscr{M}^{d-k}(X) \to \Gamma^{d-k}(B\mathscr{S}) \xrightarrow{\delta_{i_d} \cdots \delta_{i_{k+1}}} B\mathscr{S}.$$

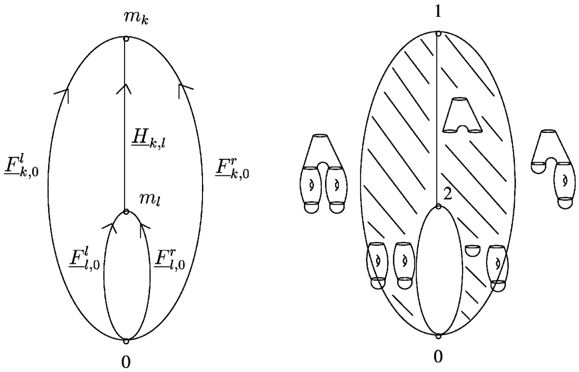


Fig. 6. Homotopy $H_{k,l}$.

More precisely, $g_k = \beta \circ \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^{p-d}(h_k)) \circ \alpha$, where α is a composition of $\pi : M \to \Gamma$ and the assembly map and β is a composition of boundaries and degeneracies. The point is that both α and β are independent of k. We show first that the maps h_k are all canonically homotopic, constructing a map

$$\Delta_d \to Map_*(S^1 \wedge \mathscr{M}^d(X), B\mathscr{S})$$

taking the value h_k at the kth vertex of Δ_d.

The main idea is the following. We think of an element $\mathbf{F} \in \mathscr{M}^d(X)$ as being divided into $d+1$ levels $d, \dots, 0$: level d is an element of $\mathscr{M}_{m_{d-1}}$ for some $m_{d-1} \in \mathbb{N}$ coming from the M the most on the left, level $d-1$ is composed of m_{d-1} "surfaces with elements of their mapping class groups" (i.e. an element of $\mathscr{M}_{j_1} \times \cdots \times \mathscr{M}_{j_{m_{d-1}}}$ for some j_i's but determined up to actions of the symmetric groups) coming from the second $\mathscr{M}$, and so on up to level 0 which is composed of m_0 elements of $X = (\mathscr{M}(*) \times \mathscr{M}(*))_+$. Denote by $\underline{F_i}$ the disjoint union of the elements of level i for $i \geqslant 1$ (we will explain how to take such a disjoint union) and $(\underline{F^l_0}, \underline{F^r_0})$ is the disjoint union, component-wise, of level 0 (left and right component). This defines two $d+1$-simplices in $B\mathscr{S}$:

$$m_d \overset{\underline{F_d}}{\longleftarrow} m_{d-1} \dots \overset{\underline{F_1}}{\longleftarrow} m_0 \overset{\underline{F^l_0}}{\longleftarrow} 0 \quad \text{and} \quad m_d \overset{\underline{F_d}}{\longleftarrow} m_{d-1} \cdots \overset{\underline{F_1}}{\longleftarrow} m_0 \overset{\underline{F^r_0}}{\longleftarrow} 0,$$

where $m_d = 1$. The map h_k essentially glues levels $k, \dots, 1$ to level 0 and then takes disjoint union, producing the couple of morphisms $\underline{F}^l_{k,0} = \underline{F}_k \circ \cdots \circ \underline{F}^l_0$ and $\underline{F}^r_{k,0} = \underline{F}_k \circ \cdots \circ \underline{F}^r_0$ from 0 to m_k in $\mathscr{S}$. Note that for each $k > l$, the surface $\underline{H}_{k,l} = \underline{F}_k \circ \cdots \circ \underline{F}_{l+1}$ induces a natural homotopy between h_k and h_l. Indeed, $\underline{H}_{k,l}$ is a morphism from m_l to m_k in $\mathscr{S}$ such that $\underline{H}_{k,l} \circ \underline{F}^l_{l,0} = \underline{F}^l_{k,0}$ and $\underline{H}_{k,l} \circ \underline{F}^r_{l,0} = \underline{F}^r_{k,0}$. This produces two 2-simplices in $B\mathscr{S}$ and hence a homotopy (see Fig. 6). The higher homotopies exist essentially because the $\underline{H}_{k,l}$'s form the edges of the following d-simplex in $\mathscr{S}$:

$$m_d \overset{\underline{F_d}}{\longleftarrow} m_{d-1} \cdots \overset{\underline{F_1}}{\longleftarrow} m_0.$$

Up to a permutation question to be clarified, this d-simplex parametrizes the d-simplex of maps we want to produce.

The general situation is indeed slightly more complicated as the disjoint union taken by h_k is determined by a point in a p-simplex of Γ_{m_k}. As a result, h_k will actually take $p+1$ copies of this disjoint union (one for each vertex of the simplex) with various labels. The two $(d+1)$-simplices given above have to be replaced by two product simplices $(\Delta_d)_+ \wedge \Delta_{p+1}$ containing $p+1$ copies of the above d-simplex of homotopies. We describe this precisely below.

Recall that $B\mathscr{S}$ is a bisimplicial set with "vertical" simplicial level coming from the classifying space construction and "horizontal" simplicial level coming from the simplicial structure of the morphism spaces. The map $S^1 \wedge X \to B\mathscr{S}$ sends the simplicial level of S^1 to the vertical simplicial level of $B\mathscr{S}$ and the simplicial level of X to the horizontal simplicial level. However, when considering the maps $h_k : S^1 \wedge \mathscr{M}^d(X) \to B\mathscr{S}$, these simplicial levels get mixed up as $\mathscr{M}^{d-k}$ acts on $B\mathscr{S}$ (through the map $\mathscr{M} \to \Gamma$), hence the simplicial level of $\mathscr{M}$ acts on the vertical simplicial level of $B\mathscr{S}$.

Consider a p-simplex $\mathbf{F} \in \mathscr{M}^d(X)_p$ with level d a p-simplex of $\mathscr{M}_{m_{d-1}}$ (up to the right action of $\Sigma_{m_{d-1}}$), level $d-1$ composed of m_{d-1} p-simplices (that is a p-simplex of $\mathscr{M}_{j_1} \times \cdots \times \mathscr{M}_{j_{m_{d-1}}}$ for some j_i's but determined up to actions of the symmetric groups from the left and from the right), and so on up to level 0 which is composed of m_0 p-simplices of $X = (\mathscr{M}(*) \times \mathscr{M}(*))_+$. Choose a representative of $\mathbf{F}$, i.e. choose an ordering of the components on each level with the appropriate labeling. The image of h_k is of course independent of such a choice. (When taking disjoint union, h_k will relabel the disjoint union which uses the chosen order, and this relabeling is equivariant.) We want to describe the image of h_k at a point of the p-simplex $\mathbf{F}$ and a point t of S^1. Suppose that t is on the left half of S^1 (i.e. $t \in y_1$). Recall that

$$h_k : S^1 \wedge \mathscr{M}^d(X) \overset{\delta_{i_k}\cdots\delta_{i_1}}{\longrightarrow} S^1 \wedge \mathscr{M}^{d-k}(X) \to \Gamma^{d-k}(B\mathscr{S}) \overset{\delta_{i_d}\cdots\delta_{i_{k+1}}}{\longrightarrow} B\mathscr{S}.$$

So h_k starts by gluing levels $k, \ldots, 1$ to level 0, then maps it to $B\mathscr{S}$ (after mapping the remaining $\mathscr{M}$'s to Γ's) obtaining elements $[\underline{G}^k_1, t], \ldots, [\underline{G}^k_{m_k}, t] \in B\mathscr{S}$, where $\underline{G}^k_i$ is a morphism of $\mathscr{S}$ from 0 to 1 (left component of the image $\mathscr{M}^k X \to X$) and t, coming from S^1, is the same for each element of $B\mathscr{S}$. The map h_k then let Γ act on these elements. By associativity of the operad action, this action is $\Gamma^{d-k}(X) \overset{\mu_\Gamma}{\to} \Gamma(X) \to X$. The element of Γ_{m_k} acting on $B\mathscr{S}$ is a point in a p-simplex

$$\sigma^k_0 \leftarrow \sigma^k_1 \leftarrow \cdots \leftarrow \sigma^k_p$$

where $\sigma^k_i \in \Sigma_{m_k}$. This simplex is the image of the glued levels $d, \ldots, d-k$ of $\mathbf{F}$ under the map $\mathscr{M} \to \Gamma$. (Note that the vertices of the simplex depend the choice made earlier on but not the permutation $\sigma^k_i(\sigma^k_{i+1})^{-1}$ defining the morphisms. Indeed, Σ_{m_k} acts on such a simplex by right multiplication on the vertices and our choice comes to choosing a point in the orbit of that action.) This p-simplex has to act on points of 1-simplices of $B\mathscr{S}$. Generically, we have to degenerate the p-simplex and the 1-simplices to $(p+1)$-simplices. As all the 1-simplices are degenerated in the same way, the image of $\Gamma(B\mathscr{S}) \to B\mathscr{S}$ will be a point in a $(p+1)$-simplex of $B\mathscr{S}$ of the form

$$m_k \overset{\sigma^k_0(\sigma^k_1)^{-1}}{\longleftarrow} \cdots \overset{\sigma^k_{q-1}(\sigma^k_q)^{-1}}{\longleftarrow} m_{d-k} \overset{\sigma^k_q \underline{F}^l_{k,0}}{\longleftarrow} 0 \leftarrow \cdots \leftarrow 0,$$

where the morphisms $\sigma^k_i(\sigma^k_{i+1})^{-1}$ are disjoint union of circles with labels σ^k_i on the left and σ^k_{i+1} on the right, and $\sigma^k_q \underline{F}_{k,0}$ is a disjoint union of the $\underline{G}^k_i$'s (in our chosen order) relabeled by the permutation σ^k_q. Note that q depends on t, but as the right part of the simplex is degenerate, all the $p+1$-simplices we have to consider are degenerate faces of the $p+1$-simplex

$$m_k \overset{\sigma^k_0(\sigma^k_1)^{-1}}{\longleftarrow} \cdots m_k \overset{\sigma^k_{p-1}(\sigma^k_p)^{-1}}{\longleftarrow} m_k \overset{\sigma^k_p \underline{F}^l_{k,0}}{\longleftarrow} 0.$$

If t is in the right half of the circle, the image is obtained by replacing $\underline{F}^l_{k,0}$ by $\underline{F}^r_{k,0}$, the disjoint union of the right components.

So h_k defines two maps (left and right component)

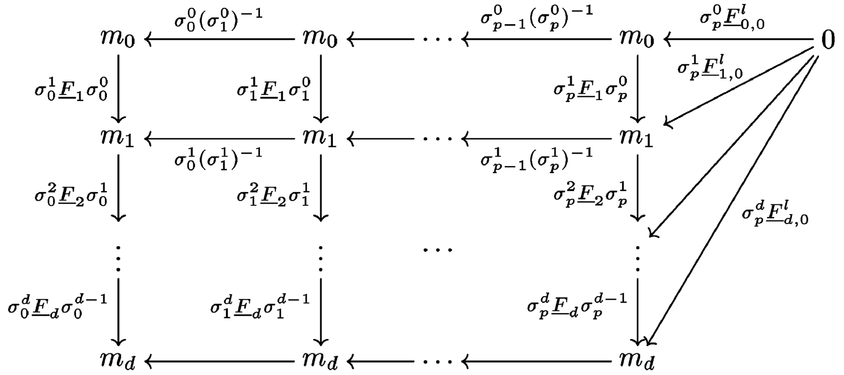
$$h_k^l, h_k^r : \Delta_1 \wedge \mathscr{M}^d(X)_p \to B_{p+1,p}\mathscr{S}$$

agreeing on $\{0\} \times \mathscr{M}^d(X)_p$ (mapped to 0) and on $\{1\} \times \mathscr{M}^d(X)_p$ (mapped to $m_k \overset{\sigma_0^k(\sigma_1^k)^{-1}}{\longleftarrow} \cdots \overset{\sigma_{p-1}^k(\sigma_p^k)^{-1}}{\longleftarrow} m_k$). Hence we recover $h_k : S^1 \wedge \mathscr{M}^d(X) \to B\mathscr{S}$.

We want to show that all the maps h_k fit together as the vertices of a map $\Delta_d \to Map_*(S^1 \wedge \mathscr{M}^d(X), B\mathscr{S})$. Define first a map

$$\widetilde{h}^l : (\Delta_d)_+ \wedge (\Delta_1 \wedge \mathscr{M}^d(X)) \to B\mathscr{S}$$

with image on the p-simplex $\underline{F}$ of $\mathscr{M}^d(X)$, the $(\Delta_d)_+ \wedge \Delta_{p+1}$ simplex of $B\mathscr{S}$



where the kth row is the image of h_k^l and the vertical morphism $\sigma_j^i \underline{F}_i \sigma_j^{i-1}$ is the disjoint union of the surfaces of level i in $\mathbf{F}$ with left labels σ_j^i and right labels σ_j^{i-1}. Note that $m_d = 1$ so all the morphisms on the bottom line are the identity. So this diagram gives various possible decompositions of a single morphism from 0 to 1 (the left component of the image). One can also see in this diagram the $p+1$ copies (with different labels) of the d-simplex of homotopies mentioned earlier.

Define a map $\tilde{h}^r$ similarly, replacing $\underline{F}_{k,0}^l$ by $\underline{F}_{k,0}^r$. As the left and right maps agree on $\Delta_d \times \{1\} \times \Delta_p$ we obtain a map

$$\tilde{h} : (\Delta_d)_+ \wedge S^1 \wedge \mathscr{M}^d X \to B\mathscr{S}$$

that is a map $\Delta_d \to Maps_*(S^1 \wedge \mathscr{M}^d X, B\mathscr{S})$.

Now recall that $g_k = \beta \circ \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^{p-d}(h_k)) \circ \alpha$, where α and β are independent of k. We thus obtain a map

$$\tilde{g} : \Delta_d \to Map_*(E_p^i, F_q^i)$$

by continuity of the functor $A = \mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^{p-d}(\_\_))$, which extends the definition already given on the vertices of Δ_d.

We are left to check that this defines a functor $\tilde{\mathscr{D}} \to \mathsf{Top}_*$. Recall that composition in $\tilde{\mathscr{D}}$ is determined (simplicially) by composition of the vertices and we know by Proposition 5.2 that we have a functor when restricting to the vertices. So all we need to check is that all compositions involving a map in a d-simplex defined above are induced by simplicial maps $\Delta_d \to \Delta_{d+1}$ or $\Delta_d \to \Delta_{d-1}$. Consider a d-simplex of maps Δ_d from E_p^i to F_q^i. The only interesting compositions are when the maps h_k are affected (otherwise the composition is just induced by the identity on Δ_d). This happens when precomposing or postcomposing by a (sequence of) face or degeneracy map which acts on $\mathscr{M}^d(X)$ or $\Gamma^d(X)$. As the action is by monad multiplication or action, or by insertion of a unit, it induces face maps $\Delta_d \times \Delta_{p+1} \to \Delta_{d-1} \times \Delta_{p+1}$ or $\Delta_d \times \Delta_{p+1} \to \Delta_{d+1} \times \Delta_{p+1}$ on the simplex image (either a new row is added or two rows are glued), which implies the result. □

Proof of Theorem 5.3. The projection $p:\tilde{\mathscr{D}} \to \mathscr{D}$ induces homotopy equivalences $\tilde{\mathscr{D}}(A,B) \simeq \mathscr{D}(A,B)$. Hence the functor L_i defined in Theorem 5.4 has a rectification $L_i' = p_*(L_i):\mathscr{D} \to \mathsf{Top}_*$. Denote by $(E')^i_p$, $(F')^i_p$ and $(f')^i_p$ the images of E_p, F_p and f^i_p via L_i'. By definition of the category $\mathscr{D}$, $(E')^i_\bullet$ and $(F')^i_\bullet$ are simplicial spaces and $(f')^i_\bullet : (E')^i_\bullet \to (F')^i_\bullet$ is a simplicial map.

We know that $p^*L_i' \simeq L_i : \tilde{\mathscr{D}} \to \mathsf{Top}_*$. Now note that $p^*(L_i')|_{\Delta^{\rm op}} = (L_i')|_{\Delta^{\rm op}} : \Delta^{\rm op} \to \mathsf{Top}_*$, for each copy of $\Delta^{\rm op}$ in $\tilde{\mathscr{D}}$ and the corresponding one in $\mathscr{D}$, as p is the identity on those subcategories. So $L_i'|_{\Delta^{\rm op}} \simeq L_i|_{\Delta^{\rm op}}$, which precisely says that the simplicial spaces $(E')^i_\bullet$ and $(F')^i_\bullet$ are equivalent to $E^i_\bullet$ and $F^i_\bullet$ respectively. Lastly, as $p^*L_i'(f_p) = (f')^i_p$, the diagram given in the corollary commutes by naturality of the equivalence of functors. □

5.3. Map of spectra

The spaces E^i and F^i for $i \geqslant 1$ form two Ω-spectra. This means that there are equivalences $\varepsilon^i : E^i \xrightarrow{\simeq} \Omega E^{i+1}$ and $\varepsilon^i : F^i \xrightarrow{\simeq} \Omega F^{i+1}$. Both adjoints $\bar{\varepsilon}^i$ can be expressed simplicially:

$$\bar{\varepsilon}^i_p : \Sigma\mathscr{G}\Gamma(S^i \wedge \mathscr{M}^p((\mathscr{M}(*) \times \mathscr{M}(*))_+)) \to \mathscr{G}\Gamma(S^{i+1} \wedge \mathscr{M}^p((\mathscr{M}(*) \times \mathscr{M}(*))_+))$$

$$\bar{\varepsilon}^i_p : \Sigma\mathscr{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathscr{S})) \to \mathscr{G}\Gamma(S^i \wedge \Gamma^p(B\mathscr{S})),$$

where both maps are given by the assembly map $A \wedge \mathscr{G}\Gamma(X) \to \mathscr{G}\Gamma(A \wedge X)$ described earlier. The following proposition is a direct consequence of the definitions of f^i_p and of the spectrum structure maps.

Proposition 5.5. *For $p \geqslant 0$ and $i \geqslant 1$, the following diagram commutes*:

$$\begin{array}{ccc}
\Sigma\mathcal{G}\Gamma(S^i \wedge \mathcal{M}^p((\mathcal{M}(*) \times \mathcal{M}(*))_+)) & \xrightarrow{\Sigma f^i_p} & \Sigma\mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathcal{S})) \\
\downarrow{\scriptstyle \bar{\epsilon}^i_p} & & \downarrow{\scriptstyle \bar{\epsilon}^i_p} \\
\mathcal{G}\Gamma(S^{i+1} \wedge \mathcal{M}^p((\mathcal{M}(*) \times \mathcal{M}(*))_+)) & \xrightarrow{f^{i+1}_p} & \mathcal{G}\Gamma(S^i \wedge \Gamma^p(B\mathcal{S}))
\end{array}$$

We want to show that the simplicial spaces $(E')^i$ and $(F')^i$ form two spectra equivalent to the original ones and that the maps $(f')^i$ form a map of spectra. For this, we need two lemmas.

Lemma 5.6. *For each $i \geqslant 1$, the two sequences of maps $\bar{\varepsilon}^i_p$ induce a natural transformation of functors*

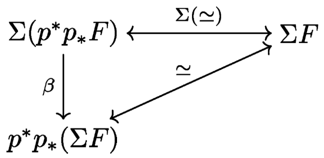
$$\bar{\varepsilon}^i : \Sigma L_i \to L_{i+1}.$$

Proof. ΣL_i and L_{i+1} are functors from $\tilde{\mathscr{D}}$ to Top_*. We already know that the $\bar{\varepsilon}_p^i$'s form a couple of simplicial maps and commute with the maps f_p (Proposition 5.5). So we only need to check that the $\bar{\varepsilon}_p^i$ commute with all the homotopies. This follows from the fact that the map $\bar{\varepsilon}_p^i$ is induced by an assembly map $\Sigma\mathscr{G}\Gamma X \to \mathscr{G}\Gamma\Sigma X$, which is natural in X [1]. □

Lemma 5.7. *For any functor* $F : \mathscr{D} \to \mathsf{Top}_*$, *there is a natural transformation*

$$\beta : \Sigma(p^* p_* F) \to p^* p_*(\Sigma F)$$

such that the following diagram is commutative:

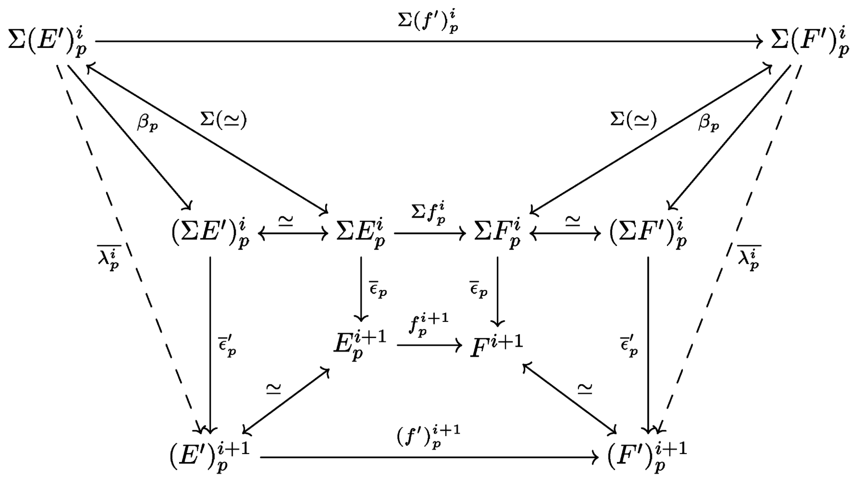


In particular, β *is an equivalence.*

Proof. There is a natural map $\beta_n : (\Sigma(p_*F)(x))_n \to (p_*(\Sigma F)(x))_n$ on each simplicial level as the second space is a quotient of the first. The resulting map β collapses a contractible subspace of the first simplicial space. One then checks that the diagram commutes. □

Theorem 5.8. *The spaces* $(E')^i$ *and* $(F')^i$ *for* $i \geqslant 1$ *form two spectra which are equivalent to the spectra* E^i *and* F^i, *and the maps* $(f')^i : (E')^i \to (F')^i$ *form a map of spectra.*

Proof. Recall that p_* is a functor $\mathsf{Top}_*^{\tilde{\mathscr{D}}} \to \mathsf{Top}_*^{\mathscr{D}}$. By Lemma 5.6, we have a natural transformation $\Sigma L_i \xrightarrow{\bar{\varepsilon}} L_{i+1}$. Denote by $(\Sigma L_i)' \xrightarrow{\bar{\varepsilon}'} L'_{i+1}$ its image under p_*. We need to construct maps $\lambda^i : (E')^i \xrightarrow{\simeq} \Omega(E')^{i+1}$ and $\lambda^i : (F')^i \xrightarrow{\simeq} \Omega(F')^{i+1}$. We define their adjoint $\bar{\lambda}$ simplicially in the following diagram:



This diagram commutes by Proposition 5.5 for the commutation of the square in the center, Proposition 2.1 (naturality in F of the equivalence $p^* p_* F \simeq F$) for the commutation of the left and right squares, because the equivalence is a natural transformation of functors for the top and bottom squares, and by Lemma 5.7 for the two triangles.

Now the adjoint of $\bar{\lambda}$ are equivalences as

$$\begin{array}{ccc} \Sigma(E')^i & \xrightarrow{\Sigma(\simeq)} & \Sigma E^i \\ {\scriptstyle \bar{\lambda}^i}\downarrow & & \downarrow{\scriptstyle \bar{\epsilon}^i} \\ (E')^{i+1} & \xrightarrow{\simeq} & E^{i+1} \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} (E')^i & \xrightarrow{\simeq} & E^i \\ {\scriptstyle \lambda^i}\downarrow & & \downarrow{\scriptstyle \epsilon^i} \\ \Omega(E')^{i+1} & \xrightarrow{\Omega(\simeq)} & \Omega E^{i+1} \end{array}$$

the commutation of the left diagram implies the commutation of the right one, and the maps ε^i are equivalences. Moreover, this last diagram shows that the equivalences $E^i \simeq (E')^i$ and $F^i \simeq (F')^i$ are equivalences of spectra.

Finally, the commutation of the larger square in the big diagram implies that the maps $(f')^i$ form a map of spectra. □

5.4. Equivalence

So far, we have defined the spectra $(E')^i$ and $(F')^i$ only for $i \geqslant 1$, i.e. starting with the first deloop of $\mathbb{Z} \times B\Gamma_\infty^+$. Define $(E')^0 := \Omega(E')^1$, $(F')^0 := \Omega(F')^1$ and

$$(f')^0 := \Omega(f')^1 : (E')^0 \to (F')^0.$$

Note that $\Omega(E')^1 \simeq \mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega(F')^1$.

Theorem 5.9. *The map of spectra $\{(f')^i\}_{i \geqslant 0} : \{(E')^i\} \to \{(F')^i\}$ is an equivalence.*

Lemma 5.10. *There are maps $\phi_p : \mathscr{M}(*) \to \mathscr{G}\Gamma(\mathscr{M}^p((\mathscr{M}(*) \times \mathscr{M}(*))_+))$ and $\psi : \mathscr{M}(*) \to \Omega B\mathscr{S}$ such that the following diagram commutes*:

$$\begin{array}{ccc} (E')^1_p & \xrightarrow{(f')^1_p} & (F')^1_p \\ {\scriptstyle \simeq}\updownarrow & & \updownarrow{\scriptstyle \simeq} \\ E^1_p = \mathcal{G}\Gamma(S^1 \wedge \mathcal{M}^p((\mathcal{M}(*) \times \mathcal{M}(*))_+)) & \xrightarrow{f^1_p} & \mathcal{G}\Gamma(\Gamma^p(B\mathcal{S})) = F^1_p \\ \uparrow & & \downarrow \\ \Sigma\mathcal{G}\Gamma(\mathcal{M}^p((\mathcal{M}(*) \times \mathcal{M}(*))_+)) & & B\mathcal{S} \\ & {\scriptstyle \Sigma(\phi_p)}\nwarrow \quad \Sigma\mathcal{M}(*) \quad \nearrow{\scriptstyle \bar{\psi}} & \end{array}$$

Proof. Let $\underline{F}$ be an element of $\mathscr{M}(*)$. Define $\phi_p(\underline{F}) := (1, \ldots, 1, (\underline{F}, D))$, where D is the disc, and let ψ be the map defined in Proposition 4.2 which sends $\underline{F}$ to the loop in $B\mathscr{S}$ going from 0 to 1 along $\underline{F}$ and back to 0 along D. The bottom part of the diagram is easily seen to commute. The top part commutes by naturality of the equivalence. □

Proof of Theorem 5.9. The spectra $(E')^i$ and $(F')^i$, for $i \geqslant 0$ are connective. Indeed, they are equivalent to the spectra E^i and F^i. As the functor Γ preserves connectedness, E^i and F^{i+1} are connected for $i \geqslant 1$. Moreover, $F^1 \simeq B\mathscr{S}$ is also connected. Hence it is enough to show that $(f')^0$ is an equivalence.

Thinking of $\mathscr{M}(*)$ as a constant simplicial space, Lemma 5.10 yields a commutative diagram of simplicial spaces (with no map from $E^1_\bullet$ to $F^1_\bullet$). Taking adjoints and using Propositions 3.2 and 4.2 we get a homotopy commutative diagram

$$\begin{array}{ccccc} (E')^0 & & \xrightarrow{\quad (f')^0 \quad} & & (F')^0 \\ \updownarrow\simeq & & & & \updownarrow\simeq \\ |\mathcal{G}\Gamma(\mathcal{M}^\bullet((\mathcal{M}(*)\times\mathcal{M}(*))_+))| & \xleftarrow{\ \phi\ } & \mathcal{M}(*) & \xrightarrow{\ \psi\ } & \Omega B\mathcal{S} \\ & \nwarrow\simeq & \downarrow & \nearrow\simeq & \\ & & \mathcal{G}\mathcal{M}(*) & & \end{array}$$

(the left triangle commutes only up to homotopy). Hence $(f')^0$ is a homotopy equivalence. □

Acknowledgements

I would like to thank my supervisor Ulrike Tillmann for her support and encouragements, and Bill Dwyer for a very illuminating discussion about higher homotopies.

Appendix.

The proof that $\Omega B\mathscr{S} \simeq \mathbb{Z}\times B\Gamma_\infty^+$ which appeared in [13,14] relies on a generalized group completion theorem. Tillmann constructs a homology fibration with fiber $\mathbb{Z}\times B\Gamma_\infty$ and homotopy fiber of the homotopy type of $\Omega B\mathscr{S}$. The canonical map from the fiber to the canonical homotopy fiber is thus a homology equivalence. We use an explicit identification of the homotopy fiber with $\Omega B\mathscr{S}$ (before stabilization) to show that the map given in Proposition 4.2 induces the homology equivalence.

Consider the simplicial space with space of n-simplices

$$(E_{\mathscr{S}}\mathscr{S}_1)_n = \coprod_{m_0,\ldots,m_n\in Ob\mathscr{S}} \mathscr{S}(m_0,m_1)\times\cdots\times\mathscr{S}(m_{n-1},m_n)\times\mathscr{S}(m_n,1)$$

and boundary maps induced by composition in $\mathscr{S}$. Consider also the telescope

$$\mathscr{S}_\infty(n) = Tel(\mathscr{S}(n,1)\xrightarrow{T}\mathscr{S}(n,1)\xrightarrow{T}\ldots),$$

where $\mathscr{S}(n,1)\xrightarrow{T}\mathscr{S}(n,1)$ is induced by gluing the torus T. Note that $\mathscr{S}_\infty(0) \simeq \mathbb{Z}\times B\Gamma_\infty$. As composition induces maps $\mathscr{S}(n,m)\times\mathscr{S}_\infty(m) \to \mathscr{S}_\infty(n)$, we can also define a simplicial space $E_{\mathscr{S}}\mathscr{S}_\infty = Tel(E_{\mathscr{S}}\mathscr{S}_1\xrightarrow{T}\ldots)$. The map

$$\pi : E_{\mathscr{S}}\mathscr{S}_\infty \to B\mathscr{S},$$

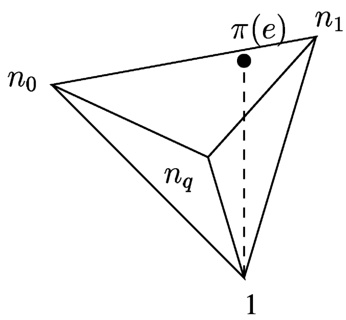
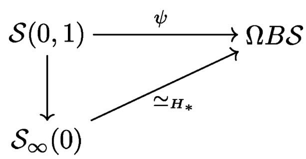


Fig. 7. Path δ_e from $\pi(e)$ to 1.

induced by collapsing $\mathscr{S}_\infty(n)$ to $\{n\}$, is a homology fibration. Let $hF_\infty := PB\mathscr{S} \times_{B\mathscr{S}} E_{\mathscr{S}}\mathscr{S}_\infty$ denote the homotopy fiber. As $E_{\mathscr{S}}\mathscr{S}_\infty$ is contractible, hF_∞ is of the homotopy type of $\Omega B\mathscr{S}$. Hence, we have

$$\mathbb{Z} \times B\Gamma_\infty \simeq \mathscr{S}_\infty(0) \overset{\simeq_{H_*}}{\rightarrow} hF_\infty \simeq \Omega B\mathscr{S}.$$

Theorem A.1. *The map $\psi : \mathscr{S}(0,1) \to \Omega B\mathscr{S}$ defined by $\psi(\underline{F}) = \underline{F}\binom{1}{0}D$ induces the homology equivalence $Z \times B\Gamma_\infty \simeq_{H_*} \Omega B\mathscr{S}$, i.e. there is a commutative diagram*



To prove the theorem, we first study the non-stable case. Consider the map $\pi_{(1)} : E_{\mathscr{S}}\mathscr{S}_1 \to B\mathscr{S}$, and let $hF_1 := PB\mathscr{S} \times_{B\mathscr{S}} E_{\mathscr{S}}\mathscr{S}_1$ denote its homotopy fiber. As $E_{\mathscr{S}}\mathscr{S}_1$ is also contractible, hF_1 is homotopy equivalent to $\Omega B\mathscr{S}$ (but not equivalent to the fiber in this case). We want to construct an explicit homotopy equivalence

$$\rho : hF_1 \to \Omega_{01}B\mathscr{S},$$

where $\Omega_{01}B\mathscr{S}$ is the space of loops in $B\mathscr{S}$ starting at 0 and ending at 1. To a q-simplex $(\underline{F}_1, \ldots, \underline{F}_q, \underline{G}) \in \mathscr{S}(n_0, n_1) \times \cdots \times \mathscr{S}(n_{q-1}, n_q) \times \mathscr{S}(n_q, 1)$ of $E_{\mathscr{S}}\mathscr{S}_1$ corresponds a $q+1$-simplex of $B\mathscr{S}$

$$\sigma = n_0 \overset{\underline{F}_1}{\rightarrow} n_1 \rightarrow \cdots \overset{\underline{F}_q}{\rightarrow} n_q \overset{\underline{G}}{\rightarrow} 1$$

having 1 as last vertex. The face opposite to 1 in σ is $\pi_{(1)}(\underline{F}_1, \ldots, \underline{F}_q, \underline{G})$. For $e \in E_{\mathscr{S}}\mathscr{S}_1$, define the path δ_e from $\pi(e)$ to 1, to be the straight line in σ between $\pi(e)$ and 1 (see Fig. 7). Now for $(p, e) \in hF_1$ ($e \in E_{\mathscr{S}}\mathscr{S}_1$ and p is a path in $B\mathscr{S}$ from 0 to $\pi(e)$), define $\rho(p, e)$ to be the product of paths $p.\delta_e$.

Lemma A.2. *The map $\rho : hF_1 \to \Omega_{01}B\mathscr{S}$ is a homotopy equivalence.*

Proof. Define the map $\xi : \Omega_{01}B\mathscr{S} \to hF_1$ by $\xi(\lambda) = (\lambda, Id_1)$, where $Id_1 \in \mathscr{S}(1,1)$ is the identity at 1. Then $\rho \circ \xi$ is the identity on $\Omega_{01}B\mathscr{S}$. On the other hand, $\xi \circ \rho$ is homotopic to the identity on hF_1. Indeed, for e in the q-simplex $(\underline{F}_1, \ldots, \underline{F}_q, \underline{G})$, we have $(\xi \circ \rho)(p, e) = (p.\delta_e, Id_1)$. Consider the $q+1$-simplex of $E_{\mathscr{S}}\mathscr{S}_1$ defined by $(\underline{F}_1, \ldots, \underline{F}_q, \underline{G}, Id_1)$. As in the case of δ_e, we can define a straight line γ_e in the $q+1$-simplex from e to Id_1 (e lies in the face opposite to Id_1). Now $\pi_{(1)}(\gamma_e) = \delta_e$. This induces the required homotopy in hF_1 by truncating the path $p.\delta_e$ at $\pi_{(1)}(\gamma_e(t))$ (explicitly $H(t, p, e) = (p.\delta|_{\pi_{(1)}(\gamma_e(t))}, \gamma_e(t))$). □

Proof of Theorem A.1. Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{S}(0,1) & \xrightarrow{T} & \mathcal{S}(0,1) & \xrightarrow{T} & \mathcal{S}(0,1) & \longrightarrow & \cdots \\
{\scriptstyle \rho\circ j}\downarrow & & {\scriptstyle \rho\circ j}\downarrow & & {\scriptstyle \rho\circ j}\downarrow & & \\
\Omega_{01}B\mathcal{S} & \xrightarrow{\lambda_T} & \Omega_{01}B\mathcal{S} & \xrightarrow{\lambda_T} & \Omega_{01}B\mathcal{S} & \longrightarrow & \cdots
\end{array}$$

where $j : \mathscr{S}(0,1) \to hF_1$ is the canonical map from the fiber to the homotopy fiber, ρ and T are defined above and $\lambda_T : \Omega_{01}B\mathscr{S} \to \Omega_{01}B\mathscr{S}$ is the multiplication with the loop from 1 to 1 defined by the torus T. As the squares commute only up to homotopy, we need to rectify this diagram to get a map of telescopes.

Let $\mathscr{D}$ be the discrete category with two copies of $\mathbb{N}$ as set of objects and morphisms as shown in the following diagram:

$$\begin{array}{ccccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots
\end{array}$$

where the squares commute strictly. Now consider the standard k-simplex $\Delta_k \subset \mathbb{R}^{k+1}$ and consider the path space

$$P_k = \{\delta : I \to \Delta_k \mid \delta(0) = (1, 0, \ldots, 0) \text{ and } \delta(1) = (0, \ldots, 0, 1)\}.$$

So $P_k \simeq \Omega\Delta_k$ is a contractible space. Let $\tilde{\mathscr{D}}$ be the category enriched over Top having the same objects as $\mathscr{D}$ and morphism spaces defined as follows: for $n \in \mathbb{N}$ and $k \geqslant 0$,

$$\tilde{\mathscr{D}}(A_n, A_{n+k}) = \mathscr{D}(A_n, A_{n+k}) = \{*\},$$

$$\tilde{\mathscr{D}}(B_n, B_{n+k}) = \mathscr{D}(B_n, B_{n+k}) = \{*\},$$

$$\tilde{\mathscr{D}}(A_n, B_{n+k}) = P_{k+1},$$

the other morphism spaces being empty. Labeling the $k+2$ vertices of Δ_{k+1} with $A_n, B_n, \ldots, B_{n+k}$ induces a face inclusion $i : \Delta_{k+1} \hookrightarrow \Delta_{n+k+l+1}$. The composition of a morphism $f : A_n \to B_{n+k}$ with the unique morphism $g : B_{n+k} \to B_{n+k+l}$ is defined to be the product of paths $(i \circ f).p$ in $\Delta_{n+k+l+1}$, where p is the path from B_{n+k} to B_{n+k+l} following the edges $B_{n+k} - B_{n+k+1} - \cdots - B_{n+k+l}$. To compose the morphism $f : A_n \to A_{n+k}$ with a morphism $g : A_{n+k} \to B_{n+k+l}$, one uses the inclusion $j : \Delta_{l+1} \hookrightarrow \Delta_{k+l+1}$ sending Δ_{l+1} to the face having vertices labeled $A_n, B_{n+k}, B_{n+k+1}, \ldots, B_{n+k+l}$. Define $g \circ f$ to be $j \circ g$.

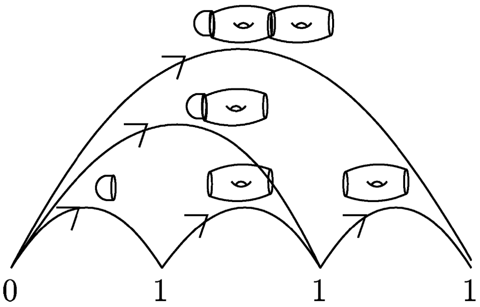


Fig. 8. Images of the three possible compositions from A_n to B_{n+2} at the disc.

The categories $\mathscr{D}$ and $\tilde{\mathscr{D}}$ satisfy the hypothesis of Section 2. We need to show that our data gives a functor $J:\tilde{\mathscr{D}} \to \mathsf{Top}$. Define J on object by $J(A_n) := \mathscr{S}(0,1)$ and $J(B_n) := \Omega_{01}B\mathscr{S}$. The diagram given at the beginning of the proof defines J on the morphisms of the type $A_n \to A_{n+k}$, $B_n \to B_{n+k}$ and $A_n \to B_n$ as well as their composition. Such compositions between A_n and B_{n+k} are paths following the edges in the relevant simplex. Fig. 8 shows in the case of $A_n \to B_{n+2}$ that the images are actually paths following the edges in a simplex of $B\mathscr{S}$. For any morphism $\delta : A_n \to B_{n+k}$, define $J(\delta) : \mathscr{S}(0,1) \to \Omega_{01}B\mathscr{S}$ by setting $J(\delta)(\underline{F})$ to be the corresponding path in the $k+1$-simplex of $B\mathscr{S}$

$$0 \xrightarrow{F} 1 \xrightarrow{T} 1 \xrightarrow{T} \cdots \xrightarrow{T} 1.$$

J is functorial essentially because we defined composition in $\tilde{\mathscr{D}}$ precisely to make it functorial.

Let $J' = p_*J : \mathscr{D} \to \mathsf{Top}$ be the rectification of J. The rectification produces two telescopes $J'(A_*)$ and $J'(B_*)$ which are equivalent to the one we started with by naturality of the equivalence $p^*p_*J \simeq J$. Moreover, we now have a map of telescopes $f : Tel(J'(A_*)) \to Tel(J'(B_*))$. From [13], we know that the map $\mathscr{S}_\infty(0) \to hF_\infty = PB\mathscr{S} \times_{B\mathscr{S}} E_{\mathscr{S}}\mathscr{S}_\infty$ is a homology equivalence. Putting all this information together, we have

$$\begin{array}{ccccc} Tel(J'(A_*)) & \xleftrightarrow{\simeq} & \mathcal{S}_\infty(0) & \xrightarrow{\simeq_{H_*}} & hF_\infty \\ \downarrow & & & & \\ Tel(J'(B_*)) & \xleftrightarrow{\simeq} & Tel(\Omega_{01}B\mathcal{S}) & & \end{array}$$

Now on each "level" of the telescope, by naturality of the equivalence and by Lemma A.2, we have a commutative diagram

$$\begin{array}{ccccc} J'(A_n) & \longleftrightarrow & \mathcal{S}(0,1) & \xrightarrow{j} & hF_1 \\ \downarrow & & \downarrow{\scriptstyle \rho\circ j} & \swarrow{\scriptstyle \rho,\ \simeq} & \\ J'(B_n) & \longleftrightarrow & \Omega_{01}B\mathcal{S} & & \end{array}$$

It follows that the map $f : Tel(J'(A_*)) \to Tel(J'(B_*))$ is a homology equivalence. Hence we have a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{S}(0,1) & \longrightarrow & \mathcal{S}_\infty(0) \\ & \swarrow^{\psi} & \downarrow{\scriptstyle \rho\circ j} & & \downarrow{\scriptstyle \simeq_{H_*}} \\ \Omega B\mathcal{S} & \xleftarrow{\simeq} & \Omega_{01}B\mathcal{S} & \xrightarrow{\simeq} & Tel(\Omega_{01}B\mathcal{S}) \end{array}$$

where the map $\Omega_{01}B\mathscr{S} \to \Omega B\mathscr{S}$ is the multiplication with the path from 0 to 1 along the disc (taken backwards), and $Tel(\Omega_{01}B\mathscr{S}) \simeq \Omega_{01}B\mathscr{S}$ as the telescope structure map has a homotopy inverse. □

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