Ribbon Braids and related operads

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Abstract

This thesis consists of two parts, both being concerned with operads related to the ribbon braid groups.

In the first part, we define a notion of semidirect product for operads and use it to study the framed n-discs operad (the semidirect product $f\mathcal{D}_n = \mathcal{D}_n \rtimes SO(n)$ of the little n-discs operad with the special orthogonal group). This enables us to deduce properties of $f\mathcal{D}_n$ from the corresponding properties for $\mathcal{D}_n$.

We prove an equivariant recognition principle saying that algebras over the framed n-discs operad are n-fold loop spaces on $SO(n)$-spaces. We also study the operations induced on homology, showing that an $H(f\mathcal{D}_n)$-algebra is a higher dimensional Batalin-Vilkovisky algebra with some additional operators when n is even. Contrastingly, for n odd, we show that the Gerstenhaber structure coming from the little n-discs does not give rise to a Batalin-Vilkovisky structure.

We give a general construction of operads from families of groups. We then show that the operad obtained from the ribbon braid groups is equivalent to the framed 2-discs operad. It follows that the classifying spaces of ribbon braided monoidal categories are double loop spaces on S^1-spaces.

The second part of this thesis is concerned with infinite loop space structures on the stable mapping class group. Two such structures were discovered by Tillmann. We show that they are equivalent, constructing a map between the spectra of deloops. We first construct an "almost map", i.e a map between simplicial spaces for which one of the simplicial identities is satisfied only up to homotopy. We show that there are higher homotopies and deduce the existence of a rectification. We then show that the rectification gives an equivalence of spectra.

Acknowledgements

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And of course I wish to thank Ulrike, who has been what I want to call a perfect supervisor. I am very happy to be one of her many daughters.

Introduction

This thesis is centred on two main topics: the first is to study and describe the framed discs operad, and the second to compare the two infinite loop space structures known on the stable mapping class group. The framed discs and the mapping class group, as operads, have a common relative: the ribbon braid group operad. Hence the title.

The little discs operad, created by Boardman and Vogt in the early seventies, represents the first important application of the theory of operads, as it is relevant to the study of iterated loop spaces. The framed discs operad, which carries an additional rotational structure, had not been much studied. The original motivation was to understand the following fact: algebras over the little discs operad are iterated loop spaces, which are themselves algebras over the framed discs operad. So an extra structure seemed to be coming for free. This led us to formulate an equivariant recognition principle, to describe the operations induced on homology, and also to show the equivalence between the two-dimensional framed discs and a ribbon braid groups operad. Our main tool is a notion of semidirect products for operads.

The second part of this work answers a question left open in [40]. The mapping class group $\Gamma_{g,1}$ of a surface $F_{g,1}$ of genus g with one boundary component is of special interest as, when $g > 0$, the classifying space of $\Gamma_{g,1}$ is homotopy equivalent to the moduli space of Riemann surfaces of topological type $F_{g,1}$. U. Tillmann showed the existence of an infinite loop space structure on $B\Gamma_\infty^+$, the classifying space of the stable mapping class group after plus construction in two different ways. The first proof uses a disjoint union structure while the second relies on pairs of pants multiplication. The first structure actually lives on the first deloop, which makes the comparison less obvious. We construct a map between the spectra of deloops and show that it is an equivalence.

In chapter 1, after reviewing the main definitions and examples used in the text, we give in detail a construction of operads built out of families of groups. The outline was given by U. Tillmann in [40]. We give here explicit conditions for families of groups to give rise to an operad this way. When fed with symmetric groups, the machinery produces Barratt and Eccles E_∞ operad Γ [1]. We are interested in the operads obtained from braid and ribbon braid groups (for chapter 1 and 2), and the mapping class groups (for chapter 3). The construction provides categorical operads, of which we take the classifying spaces to get topological operads.

In section 1.3, we define and study a notion of semidirect products for operads. This notion arose naturally from studying the framed discs operad. We originally used the name "twisted operads" (in [42]). However, it was independently discovered by M. Markl [21], and he uses the more appropriate name of semidirect product operads, so we follow his terminology. We define semidirect products in any symmetric monoidal category $\mathcal{S}$. If M is a cocommutative bimonoid in $\mathcal{S}$ (Hopf type object), the category of M-modules is again a symmetric monoidal category. For any operad $\mathcal{P}$ in that category, one can define a new operad $\mathcal{P} \rtimes M$ in $\mathcal{S}$, the semidirect product of $\mathcal{P}$ and M. For example, the framed discs operad is the semidirect product $\mathcal{D}_n \rtimes SO(n)$, where $\mathcal{D}_n$ denotes the little discs operad.

We will use throughout chapters 1 and 2 the following property of semidirect products: an object X is a $\mathcal{P} \rtimes M$-algebra if and only if it is a $\mathcal{P}$-algebra in the category of M-modules (proposition 1.3.5).

In section 1.4, we show that the algebras over the braided and ribbon braided categorical operads constructed in section 1.2, are precisely braided and ribbon braided strict monoidal categories (theorems 1.4.4 and 1.4.7). We use Joyal and Street coherence results for braided and ribbon braided monoidal categories, as well as the formalism of semidirect products to deduce the ribbon braid case from the braid one.

In section 1.5, we show that the topological ribbon operad is equivalent to the framed little 2-discs operad. To show this, we follow Fiedorowicz's ideas, extending his similar result for the braid and little discs operad. We define a notion of R_∞ operads, which is a ribbon braid version of E_∞ operads. As in the case of E_∞ operads, one shows that any two R_∞ operads are equivalent. The equivalence stated above then follows from the fact that the universal covers of the framed discs (theorem 1.5.8) and ribbon operads are R_∞ operad.

We also give a characterisation of fE_2 *operads*, i.e. operads equivalent to the framed little 2-discs (theorem 1.5.16).

In the second chapter, we use the tools and results of the first chapter to give a description of the algebras over the framed discs operad and the algebraic structure of their homology.

May's recognition principle says that algebras over the little n-discs operad are n-fold loop spaces after group completion. In section 2.1, we show that algebras over the framed n-discs operad are n-fold loop spaces on $SO(n)$-spaces (theorem 2.1.1). This is done by setting the original recognition principle in the category of $SO(n)$-spaces, and using our theory of semidirect products.

It follows then from sections 1.4 and 1.5 that the classifying space of a ribbon braided monoidal category, after group completion, is a double loop space on an S^1-space (theorem 2.1.2).

In section 2.2, we give a description of the homology of the framed n-discs operad (theorem 2.2.7). We show that in the even-dimensional case, one gets a Batalin-Vilkovisky structure with some additional differential operators. In the odd case, the Gerstenhaber structure coming from the little discs does not give rise to a Batalin-Vilkovisky structure. Once more, the formalism of semidirect products allows us to deduce the homology from the known results for the little discs. The homology of $SO(n)$ gives the new operators. The relations they satisfy with the Gerstenhaber structure are obtained directly from the geometry, by analysing the action $SO(n)$ induces on the homology of the little discs. We give thus a conceptual proof of a result which was known in the case $n = 2$ [9].

In the first annex to chapter 2, we attempt and fail to construct $f\mathcal{D}_2$-structures on $\mathcal{D}_2$-spaces. The idea was to try to use the S^1-action appearing in the recognition principle, or the action existing on the free $\mathcal{D}_2$-algebra on any space X.

In the second annex, we give a theory of *twisted monads*, which are a generalisation of semidirect products of topological operads. The motivation is to construct a monad "$\Omega^2\Sigma^2 \rtimes S^1$" which would be the group completion of $fD_2 = D_2 \rtimes S^1$. Double loop spaces are algebras over $\Omega^2\Sigma^2 \rtimes S^1$ and proposition 2.4.3 tells us that one can deloop the S^1-action existing on those algebras.

With this new monad, we formulate a *framed recognition principle*, which is a variant of the equivariant recognition principle (theorem 2.1.1) and yields the same result. It is obtained by working with fD_2 and $\Omega^2\Sigma^2 \rtimes S^1$ instead of D_2 and $\Omega^2\Sigma^2$.

Some parts of chapters 1 and 2 appear in a preprint which is joint work with Paolo Salvatore [29]. I include only results which I personally worked on. Chapter 1 and the equivariant recognition principle originate in my work, whereas section 2.2

originates in Paolo's work. My results were improved through my collaboration with Paolo and I want to thank him for that.

In Chapter 3, we show that the two infinite loop space structures discovered by Tillmann on the classifying space of the stable mapping class group are equivalent. The first proof of the existence of an infinite loop space structure uses a cobordism 2-category $\mathcal{S}$: the objects are one dimensional manifolds, the *1-morphisms* are cobordisms between them and *2-morphisms* are diffeomorphisms of the cobordisms. This category has a natural symmetric monoidal structure given by disjoint union. The result follows from the fact that $\Omega B\mathcal{S}$ is equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$. The second proof uses an operad $\mathcal{M}$ associated to this category, considering only the surfaces with n "incoming" and one "outgoing" boundary components. This operad is an infinite loop space operad.

In section 3.1, we describe the construction of the infinite loop space operad $\mathcal{M}$, following [40] and spelling out the details needed further on in the text. At the end of the section, we give explicitly the spectrum of deloops we will work with and show that it is equivalent to the one produced in [40] (proposition 3.1.4).

In section 3.2, we give a description of the category $\mathcal{S}$, adapted to our needs. Our category is a variant of [39], which we modify to make it more like $\mathcal{M}$ to help the comparison.

In section 3.3, we compare the two infinite loop space structures. To do this, we first construct in 3.3.1 an "almost map" between the spectra. The spaces defining the spectra are realisations of simplicial spaces. We construct a map which is almost simplicial in the sense that the relation $\delta_p f_p = f_{p-1}\delta_p$ is satisfied only up to homotopy, all the other required relations being satisfied (proposition 3.3.1).

In 3.3.2, we rectify the map obtained in the previous section using ideas of Dwyer and Kan. A commutative diagram can be thought of as a functor from a discrete category to Top. A "not quite commutative" diagram can then be thought of as a functor from a *thicker* category to Top, i.e. with larger morphism spaces. We construct the categories relevant to our situation and show in theorem 3.3.4 that the data given in the previous sections provide a map from this category to Top. This theorem shows the existence of higher homotopies. It follows (corollary 3.3.5) that our map can be rectified, i.e. we obtain an actual map between spaces equivalent to the ones we started with.

These ideas and techniques of rectifications were kindly explained to the author by Dwyer at the conference in Skye in June 2001.

In 3.3.3, we show that the rectified map is a map of spectra. Finally, in 3.3.4 we show that it is an equivalence.

This last chapter is a first attempt to write up a new result. We believe that all the important points are correct but the exposition and the technical details should not be considered as in final form.

Contents

List of Figures

Chapter 1

Operads, discs, braids and ribbons

1.1 Basic definitions and examples

This first section is meant as a reference section. Some of the main definitions and examples used throughout the text are given here. In 1.1.1, we recall what symmetric monoidal categories are and introduce notations for the main categories we use. We begin 1.1.2 by an informal discussion of operads. It is followed by a formal definition in a general context. However, the informal definition should be enough for the understanding of the text. We also give details of some examples of operads relevant to the text. In 1.1.3, monads and their algebras are defined. The monad associated to an operad is constructed. We also recall the two-sided bar construction, which will be used on several occasions.

1.1.1 Categories

Throughout the text, we will be working in various categories. Most of them will be monoidal categories, often symmetric. Also, most of them have an internal Hom-functor, i.e. the set of morphisms between any two objects of the category is again an object of the category. For a category $\mathcal{S}$, we will denote by $\mathcal{S}(A, B)$ the set of morphisms from A to B. We recall here what a symmetric monoidal category is and give a list of the main category we will be working with.

A *monoidal category* $(\mathcal{S}, \boxtimes, a, 1)$ is a category $\mathcal{S}$ equipped with a tensor product

$$\boxtimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}, \qquad (A, B) \mapsto A \boxtimes B,$$

which is associative, i.e. there is a natural isomorphism

$$a : (A \boxtimes B) \boxtimes C \xrightarrow{\cong} A \boxtimes (B \boxtimes C)$$

satisfying a pentagonal condition [19], and unital, i.e. there are natural isomorphisms

$$A \boxtimes 1 \xrightarrow{\cong} A \xleftarrow{\cong} 1 \boxtimes A,$$

satisfying a triangular condition.

The triangular and pentagonal conditions are such that all diagrams involving the associativity and the unit that *should* commute actually commute.

A *strict monoidal category* is a monoidal category for which the associativity and unit morphisms are the identity.

A *symmetric monoidal category* is a monoidal category equipped with an involutive natural isomorphism

$$A \boxtimes B \xrightarrow{\cong} B \boxtimes A.$$

We will denote (symmetric) monoidal categories simply by $(\mathcal{S}, \boxtimes)$ or even just $\mathcal{S}$ when the product structure we are working with is clear.

Here is a list of most of the categories we will be using:

- $\mathsf{Top}=(\mathsf{Top},\ \times)$ is the symmetric monoidal category of compactly generated Hausdorff topological spaces, with $\times$ the product of spaces. It will be called the category of topological spaces;

- Top_* is the category of based topological spaces;

- $G-\mathsf{Top}$, for G a group, is the symmetric monoidal category of G-spaces, where the G-action on a product of G-spaces is given by the diagonal action;

- $\mathsf{Simp}=(\mathsf{Simp},\times)$ is the symmetric monoidal category of simplicial spaces with $(X_* \times Y_*)_p = X_p \times Y_p$;

- $\mathsf{CAT}=(\mathsf{CAT},\times)$ is the symmetric monoidal category of small categories;

- $\mathsf{gVect}=(\mathsf{gVect},\otimes)$ and $\mathsf{dgVect}=(\mathsf{dgVect},\otimes)$ are the symmetric monoidal categories of graded and differential graded vector spaces with tensor product;

- $H-\mathsf{Mod}$, for H a graded Hopf algebra, is the category of (differential) graded vector spaces equipped with an H-module structure. The comultiplication of H induces a monoidal structure on $H-\mathsf{Mod}$.

1.1.2 Operads

For this section, let $(\mathcal{S}, \boxtimes)$ be a symmetric monoidal category. We think of $\mathcal{S}$ as being Top, Simp, CATor dgVect.

An operad in $\mathcal{S}$ is a sequence $\mathcal{P}(k)$, $k \in \mathbb{N}$, of objects of $\mathcal{S}$ such that $\mathcal{P}(k)$ admits an action of the symmetric group Σ_k for each k. One can think of $\mathcal{P}(k)$ as a set of k-ary operations and the symmetric group action as permuting the entries. The operad is then equipped with composition maps of the form

$$\gamma : \mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k).$$

The k operations of arity $n_1, \ldots, n_k$ are "plugged in" the operation of arity k to give an operation of arity $n_1 + \ldots + n_k$ (see figure 1.1).

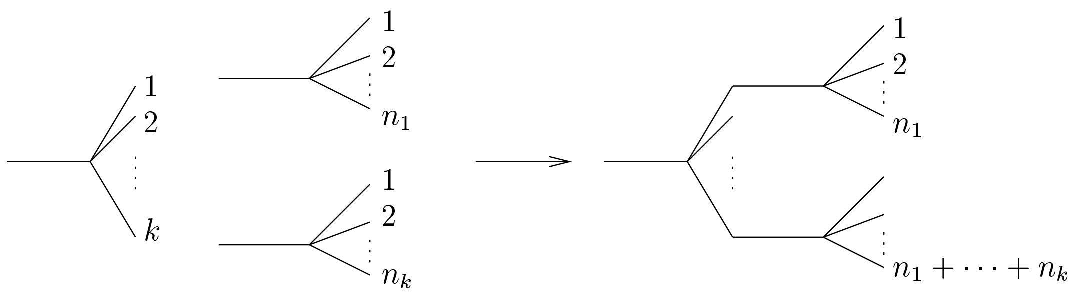


Figure 1.1: Composition in an operad

The basic example, of which operads are a generalisation, is the endomorphism operad $\mathcal{E}nd_X$ of an object X of $\mathcal{S}$. It is defined by

$$\mathcal{E}nd_X(k) = \{f : X^{\boxtimes k} \longrightarrow X\},$$

where the symmetric group acts by permuting the entries, and the maps γ are given by composition of morphisms. In the definition of an operad, we require the maps γ to satisfy natural associative, unit and Σ-equivariance conditions which are properties of this example. By Σ-*equivariant*, we mean equivariant with respect to the action of appropriate symmetric groups.

The meaning of the life of an operad is in the existence of its algebras. An algebra over an operad $\mathcal{P}$ is an object X for which the operations encoded by the operad "make sense", i.e. X is equipped with maps

$$\theta_k : \mathcal{P}(k) \boxtimes X^{\boxtimes k} \longrightarrow X,$$

which are again associative, unital and Σ-equivariant in an appropriate sense.

For example, X is an $\mathcal{E}nd_X$-algebra. In fact, a $\mathcal{P}$-algebra structure on a space X is the same as a morphism of operads $\mathcal{P} \to \mathcal{E}nd_X$.

The simplest example of an operad in Top, the category of topological spaces, is given by the operad $\mathcal{C}om$, where $\mathcal{C}om(k) = \{*\}$ for all k, with trivial Σ_k-action and composition maps. An algebra over $\mathcal{C}om$ is a commutative monoid. The multiplication is given by $\mathcal{C}om(2) \times X^2 \to X$ and the unit by $\mathcal{C}om(0) \to X$. The commutativity is a consequence of the Σ-equivariance condition. The associativity and unit properties of the operad map imply those properties for the multiplication defined above. The associativity also implies that the operations coming from $\mathcal{C}om(k)$ for $k > 2$ are actually induced by $\mathcal{C}om(2)$.

Similarly, we can consider the operad $\mathcal{A}ss \equiv \Sigma$ of (associative) monoids, where $\mathcal{A}ss(k) = \Sigma(k) = \Sigma_k$, with Σ_k-action by right multiplication. We give the operad structure maps in example 1.1.2. This well-known operad is usually denoted $\mathcal{A}ss$ as its algebras are associative, not necessarily commutative, algebras. Indeed, there are two multiplications (i.e. two binary operations) μ_1, μ_2 and the Σ-equivariance condition requires that $\mu_1(x, y) = \mu_2(y, x)$. If X is an associative algebra with multiplication μ, define $\mu_1 = \mu$ and $\mu_2(x, y) = \mu(y, x)$. We will however refer to this operad as Σ because we want to emphasise the groups involved. We will in fact define similar operads with braids and ribbon braid groups.

We give below a general definition of operads, valid in any symmetric monoidal category. A reader not familiar with (heavy) diagrams, might want to skip the diagrams involved in the definition below, which only express in details the natural associative, unital and Σ-equivariance properties mentioned above. On the other hand, a reader who wants more details about this section (and more diagrams) should refer to [25].

Let $(\mathcal{S}, \boxtimes)$ be a symmetric monoidal category with unit object 1.

Definition 1.1.1. *An* operad *in $\mathcal{S}$ consists of a sequence of objects $\mathcal{P}(k)$, $k \geq 0$, a unit map $\eta : 1 \to \mathcal{P}(1)$, a right action of the symmetric group Σ_k on $\mathcal{P}(k)$ for each k, and product maps*

$$\gamma : \mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k),$$

which are associative, unital and Σ-equivariant, i.e. the following diagrams commute:

(i) associativity:

$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes \big(\boxtimes_{s=1}^{k} \mathcal{P}(n_s)\big) \boxtimes \big(\boxtimes_{r=1}^{n} \mathcal{P}(j_r)\big) \arrow[r, "\gamma\boxtimes\mathrm{id}"] \arrow[d, "\mathrm{shuffle}"'] & \mathcal{P}(n) \boxtimes \big(\boxtimes_{r=1}^{n} \mathcal{P}(j_r)\big) \arrow[dd, "\gamma"] \\
\mathcal{P}(k) \boxtimes \big(\boxtimes_{s=1}^{k} \big(\mathcal{P}(n_s) \boxtimes \boxtimes_{t=1}^{i_s} \mathcal{P}(j_{i_{s-1}+t})\big)\big) \arrow[d, "\mathrm{id}\boxtimes(\boxtimes_s \gamma)"'] & \\
\mathcal{P}(k) \boxtimes \big(\boxtimes_{s=1}^{k} \mathcal{P}(h_s)\big) \arrow[r, "\gamma"] & \mathcal{P}(j),
\end{tikzcd}$$

where $n = n_1 + \ldots + n_k$, $j = j_1 + \ldots + j_n$, $h_s = j_{i_{s-1}+1} + \ldots + j_{i_s}$, *with* $i_s = j_1 + \ldots + j_s$.

(ii) unit:

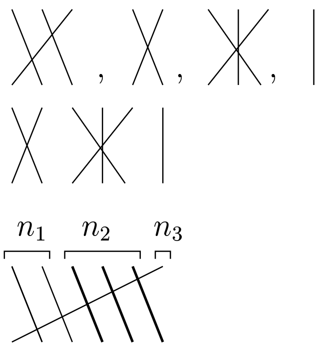
$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes 1^k \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\boxtimes\eta^k"'] & \mathcal{P}(k) \\
\mathcal{P}(k) \boxtimes \mathcal{P}(1)^k \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \boxtimes \mathcal{P}(k) \arrow[r, "\simeq"] \arrow[d, "\eta\boxtimes\mathrm{id}"'] & \mathcal{P}(k) \\
\mathcal{P}(1) \boxtimes \mathcal{P}(k) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

(iii) Σ-equivariance:

$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k) \arrow[r, "\sigma\boxtimes\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathcal{P}(k) \boxtimes \mathcal{P}(n_{\sigma(1)}) \boxtimes \ldots \boxtimes \mathcal{P}(n_{\sigma(k)}) \arrow[d, "\gamma"] \\
\mathcal{P}(n_1 + \cdots + n_k) \arrow[r, "\sigma(n_1,\ldots,n_k)"] & \mathcal{P}(n_1 + \cdots + n_k)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k) \arrow[r, "\mathrm{id}\boxtimes\tau_1\boxtimes\ldots\boxtimes\tau_k"] \arrow[d, "\gamma"'] & \mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k) \arrow[d, "\gamma"] \\
\mathcal{P}(n_1 + \cdots + n_k) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_k"] & \mathcal{P}(n_1 + \cdots + n_k),
\end{tikzcd}$$

where $\sigma \in \Sigma_k$, $\tau_i \in \Sigma_{n_i}$, $\sigma(n_1, \ldots, n_k) \in \Sigma_{n_1+\cdots+n_k}$ *permutes* k *blocks as* σ *permutes* k *letters, and* $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_{n_1+\cdots+n_k}$ *is the block sum (see figure 1.2).*



$\sigma, \tau_1, \tau_2, \tau_3$

$\tau_1 \oplus \tau_2 \oplus \tau_3$

$\sigma(n_1, n_2, n_3)$

Figure 1.2: Block sum and block permutation

A morphism of operads *is a sequence of* Σ_k*-equivariant maps,* $k \geq 0$*, which respect the unit and commute with operad structure maps* γ*.*

NOTATION: We will sometimes write $\gamma_{\mathcal{P}}$ for the operad map γ of $\mathcal{P}$ when we want to make clear which operad we are using.

In May's original definition of an operad, which was in the category of topological spaces, $\mathcal{P}(0)$ was required to be trivial ($\mathcal{P}(0) = \{*\}$). These operads are called *unital* as their algebras come equipped with a base point or unit ($\mathcal{P}(0) \to X$).

Most of the topological operads we will encounter have this property, but not all of them (the mapping class group operad defined in chapter 3 is not unital).

We give now a couple of examples of topological operads, selected for their relevance to the text.

Example 1.1.2. *The symmetric groups operad* Σ
The symmetric groups $\{\Sigma_k\}_{k\in\mathbb{N}}$ give rise to an operad Σ the following way :

$$\Sigma(k) = \Sigma_k$$

and the operad map $\gamma : \Sigma_k \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \longrightarrow \Sigma_{n_1+\cdots+n_k}$ is defined by

$$\gamma(\sigma, \tau_1, \ldots, \tau_k) = \sigma(n_1, \ldots, n_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

where $\sigma(n_1, \ldots, n_k)$ and $(\tau_1 \oplus \cdots \oplus \tau_k)$ are defined as above (see figure 1.2).

The little n-discs operad, due to Boardman and Vogt, was one of the first important example in the theory of operads. Its importance comes from its relevance to the study of iterated loop spaces.

Example 1.1.3. *The little n-discs operad* $\mathcal{D}_n$
Let D^n be the n-dimensional open unit disc (in $\mathbb{R}^n$). Define the spaces $\mathcal{D}_n(k)$, for $k \in \mathbb{N}$, to be the space of embeddings

$$f : \coprod_{1\leq i\leq k} D^n \longrightarrow D^n$$

of k disjoint discs in a disc, where f is a composition of translations and dilations only.

The operad structure map is by composition :

$$\gamma(f, g_1, \ldots, g_k) = \coprod_{n_1+\cdots+n_k} D^n \overset{g_1\sqcup\cdots\sqcup g_k}{\longrightarrow} \coprod_k D^n \overset{f}{\longrightarrow} D^n,$$

where $\sqcup$ denotes the disjoint union of the maps. Thinking of an element of $D(k)$ as a disc with k holes, the composition can be thought of as "sticking in the k discs g_i in the k holes of f".

Operads encode operations. An algebra over an operad is an object having those operations as part of its structure.

Definition 1.1.4. *Let $\mathcal{P}$ be an operad in the category $\mathcal{S}$. An* algebra over $\mathcal{P}$ *or* $\mathcal{P}$-algebra *is an object X of $\mathcal{S}$ together with maps*

$$\theta : \mathcal{P}(k) \boxtimes X^{\boxtimes k} \longrightarrow X$$

for each $k \geq 0$ which are associative, unital and Σ-equivariant in an appropriate sense.

If $\mathcal{S}$ has an internal Hom functor, a $\mathcal{P}$-algebra is precisely an object X together with a morphism of operads $\mathcal{P} \to \mathcal{E}nd_X$, where $\mathcal{E}nd_X(k) = \mathcal{S}(X^{\boxtimes k}, X)$ and $\gamma_{\mathcal{E}nd_X}$ is the composition of morphisms in $\mathcal{S}$.

So an algebra over an operad $\mathcal{P}$ is an element of the category $\mathcal{S}$ for which $\mathcal{P}(k)$ encodes a set of k-ary operations, for each k, in a coherent way, i.e. coherent with the composition of operations (associativity) and permutation of the variables (Σ-equivariance).

We have already mentioned that X is clearly an algebra over $\mathcal{E}nd_X$. Also, we explained how the Σ-algebras are exactly the associative algebras. We explain here what are the algebras over the little n-discs.

Example 1.1.5. Any n-fold loop space is an algebra over the little discs operad $\mathcal{D}_n$: thinking of $\Omega^n Y$ as $\{f : (D^n, \partial D^n) \longrightarrow (Y, *)\}$, the maps $\theta_k : \mathcal{D}_n(k) \times (\Omega^n Y)^k \to \Omega^n Y$ are obtained by evaluating the ith n-fold loop on the ith embedded disc and sending the rest of D^n to the base point.

The recognition principle actually tells us that those are the only $\mathcal{D}_n$-algebras.

Theorem 1.1.6. *Recognition principle ([27], Theorem 13.1) If X is a $\mathcal{D}_n$-algebra, the group completion of X is weakly homotopy equivalent to an n-fold loop space $\Omega^n Y$, as $\mathcal{D}_n$-algebras.*

1.1.3 Monads

The concept of monad is related to the concept of operad in many ways. Operads can be defined as monads in the category of Σ-collections [10]. We are interested here in the free algebra functor associated to an operad, which has a monad structure induced by the operad structure.

We are also interested in certain monads not coming from operads.

Definition 1.1.7. *A* monad (M, μ, η) *in a category* $\mathcal{S}$ *is a functor* $M : \mathcal{S} \longrightarrow \mathcal{S}$ *together with natural transformations* $\mu : MM \longrightarrow M$ *and* $\eta : Id \longrightarrow M$ *such that the following diagrams commute:*

$$\begin{tikzcd} M \arrow[r, "\eta M"] \arrow[dr, "id"'] & MM \arrow[d, "\mu"] & M \arrow[l, "M\eta"'] \arrow[dl, "id"] \\ & M & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} MMM \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & MM \arrow[d, "\mu"] \\ MM \arrow[r, "\mu"] & M \end{tikzcd}$$

As for operads, we have a notion of an algebra over a monad:

Definition 1.1.8. *An* algebra over a monad M, *or* M-algebra, (X, ξ) *is an object* X *of* $\mathcal{S}$ *together with a map* $\xi : MX \longrightarrow X$ *such that the following diagrams commute:*

$$\begin{tikzcd} X \arrow[r, "\eta_M"] \arrow[dr, "id"'] & MX \arrow[d, "\xi"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} MMX \arrow[r, "\mu"] \arrow[d, "M\xi"'] & MX \arrow[d, "\xi"] \\ MX \arrow[r, "\xi"] & X. \end{tikzcd}$$

For an operad $\mathcal{P}$ in $\mathcal{S}$, the free algebra over $\mathcal{P}$ in $\mathcal{S}$ is given by

$$F_{\mathcal{P}}X := \coprod \mathcal{P}(k) \boxtimes_{\Sigma_k} X^k.$$

The operad structure of $\mathcal{P}$ induces a monad structure on $F_{\mathcal{P}}$.

For topological operads with a pointed 0-space, i.e. with $* \in \mathcal{P}(0)$, it is more natural to consider a reduced version of this free monad. Indeed, algebras over a such operads are pointed spaces via the map $\mathcal{P}(0) \to X$. We consider the free monad in the category of pointed spaces (rather than spaces):

Example 1.1.9. Let $\mathcal{P}$ be a unital operad in Top. Define the monad P associated to $\mathcal{P}$ as follows: for a space X with base point $*_X$,

$$PX = \frac{\coprod_{n \geq 0} \mathcal{P}(n) \times_{\Sigma_n} X^n}{\approx},$$

where $\approx$ is the base-point relation $(\sigma_i c, \underline{x}) \approx (c, s_i \underline{x})$ defined for all $c \in \mathcal{P}(j)$, $\underline{x} = (x_1, \ldots, x_{j-1}) \in X^{j-1}$ and $0 \leq i \leq j$, where $\sigma_i c = \gamma(c, 1, \ldots, 1, *, \ldots, 1)$ and $s_i \underline{x} = (x_1, \ldots, x_i, *_X, x_{i+1}, \ldots, x_{j-1})$.

The multiplication μ and unit map η of the monad are induced by the maps γ and η of the operad.

Proposition 1.1.10. *[25] Let* $\mathcal{P}$ *be a unital operad. Consider the category of* $\mathcal{P}$*-algebras as a subcategory of* Top_*. *Then the identity functor on* Top_* *restricts to an isomorphism of categories between the category of algebras over the operad* $\mathcal{P}$ *and the category of algebras over the monad* P.

Not all monads come from an operad. Here is an example of a non-operadic topological monad which will be used further on.

Example 1.1.11. We consider the following functors in Top_*: let Ω be the based loop functor and Σ the (reduced) suspension. We think of the iteration of those functors as

$$\Omega^n X = \{(D^n, \partial D^n) \to (X, *)\}$$

$$\Sigma^n X = \frac{D^n \times X}{\partial D^n \times X \sqcup D^n \times *}.$$

For each $n \geq 1$, the functor $\Omega^n\Sigma^n$ is a monad with unit map $\eta : X \to \Omega^n\Sigma^n X$ given by $\eta(x) = \{t \mapsto [t, x]\}$ and multiplication $\mu : \Omega^n\Sigma^n\Omega^n\Sigma^n X \to \Omega^n\Sigma^n X$ induced by the evaluation of the "internal iterated loop" on the "external iterated suspension":
$\mu(\{t \mapsto [\sigma_1(t), \{s \mapsto [\sigma_2(t, s), x(t, s)]\}]\}) = \{t \mapsto [\sigma_2(t, \sigma_1(t)), x(t, \sigma_1(t))]\}$.

1.1.3.1 Two-sided bar construction

This construction will be used several times in the text. We give here a summary of May's definition in [27].

Let $\mathcal{S}, \mathcal{S}'$ be two categories and let (M, μ, η) be a monad in $\mathcal{S}$.

An *M-functor* (F, λ) is a functor $F : \mathcal{S} \to \mathcal{S}'$ with a natural transformation $\lambda : FM \to F$ such that the diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta_M"] \arrow[dr, "\mathrm{id}"'] & FM \arrow[d, "\lambda"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FMM \arrow[r, "F\mu"] \arrow[d, "\lambda"'] & FM \arrow[d, "\lambda"] \\ FM \arrow[r, "\lambda"] & F \end{tikzcd}$$

commute.

Note that any monad M is an M-functor. More generally, if $M \to M'$ is a morphism of monads, M' is an M-functor.

We will also consider the following example: the iterated suspension Σ^n is an $\Omega^n\Sigma^n$-functor. The natural transformation $\lambda : \Sigma^n\Omega^n\Sigma^n \to \Sigma^n$ is defined similarly to the multiplication of $\Omega^n\Sigma^n$.

We will only be interested in the case $\mathcal{S} = \mathcal{S}' = \mathsf{Top}$. Consider a monad M in Top, an M-functor F and an M-algebra (X, ξ). Define the simplicial space $B_*(F, M, X)$ by

$$B_q(F, M, X) = FM^qX,$$

with face and degeneracy operators given by
$\partial_0 = \lambda : FM^qX \to FM^{q-1}X$
$\partial_i = FM^{i-1}\mu, \quad \mu : M^{q-i+1}X \to M^{q-i}X, \ 0 < i < q$
$\partial_q = FM^{q-1}\xi, \quad \xi : MX \to X,$
$s_i = FM^i\eta, \quad \eta : M^{q-i}X \to M^{q-i+1}X, \ 0 \geq i \geq q.$

A morphism $(\pi, \phi, f) : B_*(F, M, X) \to B_*(F', M', X')$ is a triple consisting of a morphism of monads $\phi : M \to M'$, a morphism of M-functors $\pi : F \to \phi^*(F')$ and a morphism $f : X \to \phi^*(X')$ of M-algebras, where ϕ^* gives the natural pull-back structure.

Definition 1.1.12. *Let X_* be a simplicial space, i.e. a simplicial set in* Top. *The* geometric realisation *of X, denoted $|X|$ is the topological space*

$$|X| = \sum_{q\geq 0} X_q \times \Delta_q / \approx,$$

where Δ_q is the standard topological q-simplex

$$\Delta_q = \{t_0, \ldots, t_q) | 0 \leq t_i \leq 1, \sum t_i = 1\} \subset \mathbb{R}^{q+1}$$

and $\approx$ is defined for $x \in X_q$ by
$(\partial_i x, (t_0, \ldots, t_{q-1})) \approx (x, (t_0, \ldots, t_{i-1}, 0, \ldots, t_{q-1}))$, *and*
$(s_i x, (t_0, \ldots, t_{q+1})) \approx (x, (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{q+1}))$.

The geometric realisation is functorial and respects the product of spaces:

Proposition 1.1.13. *Let X, Y be two simplicial spaces. Then the map $|\pi_1| \times |\pi_2| : |X \times Y| \to |X| \times |Y|$ is a natural homeomorphism.*

Definition 1.1.14. *We call*

$$B(F, M, X) := |B_*(F, M, X)|$$

the two-sided bar construction *(or* double bar construction*) of F, M and X.*

We will use the following properties of the bar construction:

Proposition 1.1.15. *Let M be a monad, X an M-algebra, F an M-functor and G a functor. We have*

(i) $B_*(GF, M, X) = GB_*(F, M, X)$;

(ii) $B(M, M, X) \xrightarrow{|\xi^{*+1}|} X$ is a strong deformation retract.

1.2 Operads from families of groups

We construct operads from certain types of families of groups. The construction given here is adaptable. In section 1.5, we will replace the symmetric groups by the ribbon braid groups, and in chapter 3, we will replace groups by groupoids. This section provides a detailed study of a construction given in [40].

The operads of this section are obtained through a categorical construction: from groups, we construct categories, which form an operad in CAT. The classifying space of those categories provides then a topological operad.

Let $N : \mathsf{CAT} \to \mathsf{Simp}$ be the nerve functor and $| \ |$ the geometric realisation.

Proposition 1.2.1. *Let CAT-Op, Simp-Op and Top-Op be the categories of categorical, simplicial and topological operads respectively. Then N and $| \ |$ induce functors*

$$\mathsf{CAT}-Op \xrightarrow{N} \mathsf{Simp}-Op \xrightarrow{| \ |} \mathsf{Top}-Op.$$

Moreover, we have the following relations on the algebras:

Proposition 1.2.2. *Let $\mathcal{P}$ be a categorical operad and $\mathcal{A}$ a category. Then $\mathcal{A}$ is a $\mathcal{P}$-algebra if and only if $N\mathcal{A}$ is a $N\mathcal{P}$-algebra.*
Let $\mathcal{D}$ be a simplicial operad and S a $\mathcal{D}$-algebra. Then $|S|$ is a $|\mathcal{D}|$-algebra.

Let G be a group and H a subgroup of G. Let $\mathcal{C}_H^G$ denote the category with objects G/H, the left cosets of H, and morphisms given by left multiplication by elements of G. The set of morphisms $\mathcal{C}_H^G(aH, bH)$ is in one-one correspondence with H: an element $h \in H$ can be identified with left multiplication by bha^{-1}. We will think of a morphism from $\alpha = aH$ to $\beta = bH$ in G/H as an element $g \in G$ such that $\pi(g)\alpha = \beta$, where $\pi : G \longrightarrow G/H$ is the projection on the quotient. We will denote such a morphism by $\beta \xleftarrow{g} \alpha$.

If H is a normal subgroup of G, G/H acts freely on the right by multiplication on objects and on morphisms by $(\alpha_0 \xleftarrow{g} \alpha_1)\beta = \alpha_0\beta \xleftarrow{g} \alpha_1\beta$.

A p-simplex in $N\mathcal{C}_H^G$ is of the form $\alpha_0 \xleftarrow{g_1} \dots \xleftarrow{g_p} \alpha_p$. Let $B\mathcal{C}_H^G = |N\mathcal{C}_H^G|$ be the classifying space of this category.

An example we have in mind is the braid group β_k sitting over the symmetric group Σ_k:

Example 1.2.3. Let $C_n(\mathbb{R}^2)$ be the configuration space of n points in the plane, i.e. the space of unordered subsets of $\mathbb{R}^2$ of cardinality n. The *braid group* β_n is the fundamental group of $C_n(\mathbb{R}^2)$. It can be alternatively described as the set of deformable strings from n points to n other points.

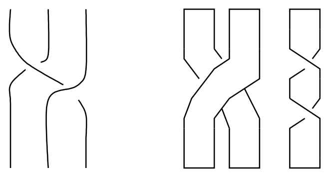


Figure 1.3: Braid and ribbon

There is a natural projection $\beta_n \twoheadrightarrow \Sigma_n$, sending a braid to the permutation it induces on the points. We call *pure braid group* the kernel $P\beta_n$ of this projection:

$$P\beta_n \hookrightarrow \beta_n \twoheadrightarrow \Sigma_n.$$

Taking $G = \beta_k$ and $H = P\beta_k$, the category $\mathcal{C}^{\beta_k}_{P\beta_k}$ has objects $G/H = \Sigma_k$. A morphism between two permutations σ and τ is a braid whose underlying permutation is $\tau\sigma^{-1}$.

1.2.0.2 The category operad

Let (G_k, H_k), $k \in \mathbb{N}$, be pairs of groups as above, with $H_k \lhd G_k$ and $G_k/H_k = \Sigma_k$. Denote by π_k or simply $\pi : G_k \longrightarrow \Sigma_k$ the projections on the quotient spaces.

Suppose that for all $k, n_1, \ldots, n_k \in \mathbb{N}$, we have a map

$$\omega : G_k \times G_{n_1} \times \cdots \times G_{n_k} \longrightarrow G_{n_1+\cdots+n_k}$$

which preserves the product on $\{G_k \times G_{n_{i_1}} \times \cdots \times G_{n_{i_k}} | n_{i_j} \in \mathbb{N}\}$ when it is defined, i.e. when the following product makes sense:

$$(f, g_1, \ldots, g_k).(f', g'_1, \ldots, g'_k) = (f.f', g_{(\pi f')(1)}.g'_1, \ldots, g_{(\pi f')(k)}.g'_k).$$

Suppose that ω satisfies the unit and associativity conditions of an operad structure map (definition 1.1.1, conditions (i) and (ii)) and that the following diagram is commutative :

$$\begin{tikzcd}
G_k \times G_{n_1} \times \cdots \times G_{n_k} \arrow[r, "\omega"] \arrow[d, "\pi_k \times \pi_{n_1} \times \cdots \times \pi_{n_k}"'] & G_{n_1+\cdots+n_k} \arrow[d, "\pi_{n_1+\cdots+n_k}"] \\
\Sigma_k \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \arrow[r, "\gamma_\Sigma"] & \Sigma_{n_1+\cdots+n_k}
\end{tikzcd}$$

where γ_Σ is the operad map of the symmetric groups operad (example 1.1.2).

Define $\Gamma : \mathcal{C}_{H_k}^{G_k} \times \mathcal{C}_{H_{n_1}}^{G_{n_1}} \times \cdots \times \mathcal{C}_{H_{n_k}}^{G_{n_k}} \longrightarrow \mathcal{C}_{H_{n_1+\cdots+n_k}}^{G_{n_1+\cdots+n_k}}$ on objects by

$$\Gamma(\sigma, \tau_1, \ldots, \tau_k) = \gamma_\Sigma(\sigma, \tau_1, \ldots, \tau_k);$$

and on morphisms by

$$\Gamma(\sigma_1 \xleftarrow{f} \sigma_0, \tau_1^1 \xleftarrow{g_1} \tau_0^1, \ldots, \tau_1^k \xleftarrow{g_k} \tau_0^k) = \omega(f, g_{\sigma_0^{-1}(1)}, \ldots, g_{\sigma_0^{-1}(k)}).$$

Proposition 1.2.4. Γ *is a functor inducing a categorical operad structure on the sequence* $\{\mathcal{C}_{H_n}^{G_n}\}_{n\in N}$.

Corollary 1.2.5. $\{B\mathcal{C}_{H_n}^{G_n}\}_{n\in N}$ *is a topological operad.*

Remark 1.2.6. For any space X, Σ_j acts on X^j by permuting the components,

$$(x_1, \ldots, x_j) \xrightarrow{\sigma.} (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)}).$$

It is a left action, so $(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)}) \xrightarrow{\tau.} (x_{\sigma^{-1}(\tau^{-1}(1))}, \ldots, x_{\sigma^{-1}(\tau^{-1}(j))})$.

Proof of the proposition. The functoriality of Γ is a consequence of the commutativity of the diagram above and the multiplication preserving property of ω.

The group Σ_k acts (freely) on the right of $\mathcal{C}_{H_k}^{G_k}$. The conditions for Γ to be an operad structure map are satisfied on objects as it restricts to the symmetric groups operad. On morphisms, the unit and associativity conditions come as a consequence of those properties for ω, and the Σ-equivariance so obtained using remark 1.2.6. □

Example 1.2.7. The E_∞-operad $\Gamma = \{B\mathcal{C}_{\{e\}}^{\Sigma_k}\}$ (constructed by Barrat and Eccles in [1]) is a special case of the above construction, taking the pairs of groups $(\Sigma_k, \{e\})$, where $\{e\}$ is the trivial group and with the map $\omega = \gamma_\Sigma$.

Example 1.2.8. The braid categorical operad $\mathcal{C}^\beta = \{\mathcal{C}_{P\beta_k}^{\beta_k}\}$ and its classifying space $B\mathcal{C}^\beta = \{B\mathcal{C}_{P\beta_k}^{\beta_k}\}$ are obtained from the pairs $(\beta_k, P\beta_k)$, with the map ω defined by

$$\omega(a, b_1, \ldots, b_k) = a(n_1, \ldots, n_k)(b_1 \oplus \cdots \oplus b_k),$$

where $a(n_1, \ldots, n_k)$ and $(b_1 \oplus \cdots \oplus b_k)$ are defined as in the symmetric case (see figure 1.2).

We will denote by $R\beta_k$ the *ribbon braid group* on k elements, the fundamental group of the configuration space of k unordered particles in $\mathcal{R}\beta^2$ with label in S^1. One can think of an element of $R\beta_k$ as a braid on k ribbons, where full twists of the ribbons are allowed (see figure 1.3).

The *pure ribbon braid group* $PR\beta_k$ is the kernel of the surjection $R\beta_k \twoheadrightarrow \Sigma_k$.

The groups $R\beta_k$ and $PR\beta_k$ are isomorphic to $\beta_k \rtimes \mathbb{Z}^k$ and $P\beta_k \times \mathbb{Z}^k$ respectively, where $\mathbb{Z}^k$ encodes the number of twists on each ribbon.

Example 1.2.9. The ribbon categorical operad $\mathcal{C}^{R\beta} = \{\mathcal{C}^{R\beta_k}_{PR\beta_k}\}$ and its classifying space $B\mathcal{C}^{R\beta} = \{B\mathcal{C}^{R\beta_k}_{PR\beta_k}\}$ are obtained from the pairs $(R\beta_k, PR\beta_k)$, with the map ω defined by

$$\omega(r, s_1, \ldots, s_k) = r(n_1, \ldots, n_k)(s_1 \oplus \cdots \oplus s_k),$$

similarly to the symmetric and braid case.

We will describe algebras over the braid and ribbon braid operads in section 1.4.

Remark 1.2.10. For all n, we have functors

$$\mathcal{C}^{\beta_n}_{P\beta_n} \xrightarrow{I} \mathcal{C}^{R\beta_n}_{PR\beta_n} \xrightarrow{\Pi} \mathcal{C}^{\Sigma_n}_{\{e\}},$$

giving rise to morphisms of operads in the categorical and topological cases. Both functors are the identity on objects. On morphisms I is the inclusion of the braid group in the ribbon braid obtained by replacing the strings by flat ribbons, and Π is the canonical projection.

1.3 Semidirect product of operads

Studying the framed discs operad $f\mathcal{D}_n$, which is a variant of the little discs we already encountered, lead us to consider a notion of semidirect product for operads. This notion was introduced independently by Markl in [21]. We introduce the notion by giving a detailed description of this first example.

Example 1.3.1. The framed discs operad, $f\mathcal{D}_n$, is defined similarly to the little n-discs operad $\mathcal{D}_n$ (example 1.1.3), with the only difference that $f\mathcal{D}_n(k)$ consists of the embeddings from the disjoint union of k n-discs to a discs, obtained by composing translations, dilations and rotations. The composition of embeddings provides again an operad structure.

One can think of an element of $f\mathcal{D}_n(k)$ as a configuration of k discs in a disc, each of the k discs having a distinguished marked point (see figure 1.4). The composition maps are then defined by "plugging in" the discs matching the marked points.

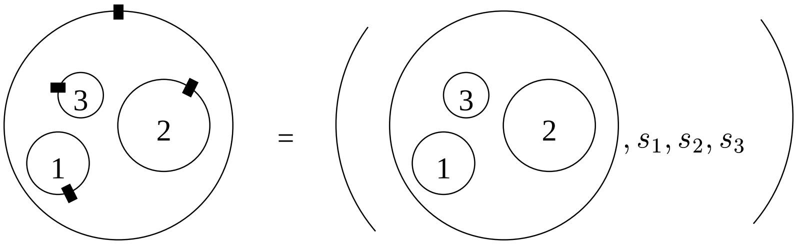


Figure 1.4: Element of $f\mathcal{D}_2(3) = \mathcal{D}_2(3) \times (S^1)^3$

As spaces, $f\mathcal{D}_n(k) = \mathcal{D}_n(k) \times (SO(n))^k$, the ith element of $SO(n)$ encoding the rotation of the ith disc. In fact, as operads, we have a structure of semidirect product. We write $f\mathcal{D}_n = \mathcal{D}_n \rtimes SO(n)$. Indeed, the composition maps are twisted by an action of $SO(n)$ on $\mathcal{D}_n$:

$$\gamma : (\mathcal{D}_n(k) \times (SO(n))^k) \times (\mathcal{D}_n(n_1) \times (SO(n))^{n_1}) \times \cdots \times (\mathcal{D}_n(n_k) \times (SO(n))^{n_k}) \\ \longrightarrow \mathcal{D}_n(n_1 + \cdots + n_k) \times (SO(n))^{n_1+\cdots+n_k}$$

is given by

$$\gamma((a, \underline{g}), (b_1, \underline{h}^1), \ldots, (b_k, \underline{h}^k)) = (\gamma_{\mathcal{A}}(a, g_1 b_1, \ldots, g_k b_k), g_1\underline{h}^1, \ldots, g_k\underline{h}^k),$$

where $\underline{h}^i = (h_1^i, \ldots, h_{n_i}^i)$ and $g_1.\underline{h}^i = (g_1 h_1^i, \ldots, g_1 h_{n_i}^i)$. So each of the k copies of $SO(n)$ acts by rotation on the corresponding $\mathcal{D}_n(n_i)$.

A notion of semidirect product for operads, inspired by this example, can be defined in a general context. In Top, one can define an operad $\mathcal{P} \rtimes G$ for an operad

$\mathcal{P}$ and a group G whenever $\mathcal{P}$ admits a G-action such that its structure maps are G-equivariant. Studying the operad $\mathcal{P} \rtimes G$ is then made easier by the following fact: a space X is a $\mathcal{P} \rtimes G$-algebra if and only if X is a $\mathcal{P}$-algebra in the category of G-spaces. As a result, working with $\mathcal{P} \rtimes G$ comes down to working with $\mathcal{P}$ but in the category of G-spaces. This is will allow us to deduce results for $f\mathcal{D}_n$ from known results for $\mathcal{D}_n$.

In this section, we will define semidirect products for operads in the general setting as we will need the notion in $\mathsf{CAT}$and gVectas well as in Top.

1.3.1 Definition and properties

Let $\mathcal{S}$ be a symmetric monoidal category and let $(M, \mu, \eta, c, \epsilon)$ be a *bimonoid* in $\mathcal{S}$. So M is an object of $\mathcal{S}$ equipped with an associative, unital multiplication ($\mu : M \boxtimes M \to M$, $\eta : 1 \to M$) and a coassociative, counital comultiplication ($c : M \to M \boxtimes M$, $\epsilon : M \to 1$) which is a morphism of algebras, i.e. such that

$$\begin{tikzcd} M \boxtimes M \arrow[r, "c \boxtimes c"] \arrow[d, "\mu"'] & M \boxtimes M \boxtimes M \boxtimes M \arrow[r, "\mathrm{id} \boxtimes \tau \boxtimes \mathrm{id}"] & M \boxtimes M \boxtimes M \boxtimes M \arrow[d, "\mu \boxtimes \mu"] \\ M \arrow[rr, "c"] & & M \boxtimes M \end{tikzcd}$$

Example 1.3.2. (1) In Top, take G a topological group with $c : G \to G \times G$ defined by $c(g) = (g, g)$.

(2) In gVect, consider the homology of a topological group, $H_*(G)$, which is a Hopf algebra and in particular a bimonoid. The comultiplication is defined by $c(x) = 1 \otimes x + x \otimes 1$, where $1 \in H_0(G)$ is the generator in the component of the unit of G.

(3) In CAT, consider the category $\mathcal{Z}$ with one object and $\mathbb{Z}$ as set of morphisms. The group structure of $\mathbb{Z}$ induces a bimonoid structure on $\mathcal{Z}$ (as in (1)).

For a bimonoid M, the category of M-modules, denoted M-Mod, is a monoidal category. We will consider the case where M is cocommutative, so that the category M-Mod is symmetric. We can thus consider operads and their algebras in M-Mod. We will call those operads *M-operads*.

An M-operad $\mathcal{P}$ can be considered as an operad in $\mathcal{S}$ by forgetting the M-module structures. The M-equivariance of the operad structure maps of $\mathcal{P}$ is given by the commutation of the following diagram:

$$\begin{tikzcd}
M \boxtimes \mathcal{P}(k) \boxtimes X^{\boxtimes k} \arrow[rr, "\mathrm{id}\boxtimes\theta_k"] \arrow[d, "\mathrm{shuffle}\circ c^k"'] & & M \boxtimes X \arrow[dd, "\Psi"] \\
M \boxtimes \mathcal{P}(k) \boxtimes M \boxtimes X \boxtimes \ldots \boxtimes M \boxtimes X \arrow[d, "\Phi\boxtimes\Psi^{\boxtimes k}"'] & & \\
\mathcal{P}(k) \boxtimes X^{\boxtimes k} \arrow[rr, "\theta_k"] & & X.
\end{tikzcd}$$

Definition 1.3.3. *Let $\mathcal{P}$ be an M-operad. Define $\mathcal{P} \rtimes M$, the* semidirect product of $\mathcal{P}$ and M, *to be the following operad in $\mathcal{S}$: for $k \in \mathbb{N}$,*

$$(\mathcal{P} \rtimes M)(k) = \mathcal{P}(k) \boxtimes M^{\boxtimes k}$$

with Σ_k acting diagonally on the right, permuting the components of $M^{\boxtimes k}$ and acting on $\mathcal{P}(k)$, and the map
$\gamma : (\mathcal{P} \rtimes M)(k) \boxtimes (\mathcal{P} \rtimes M)(n_1) \boxtimes \ldots \boxtimes (\mathcal{P} \rtimes M)(n_k) \longrightarrow (\mathcal{P} \rtimes M)(n_1 + \cdots + n_k)$
given by

$$\begin{tikzcd}
\big(\mathcal{P}(k) \boxtimes M^{\boxtimes k}\big) \boxtimes \big(\mathcal{P}(n_1) \boxtimes M^{\boxtimes n_1}\big) \boxtimes \ldots \boxtimes \big(\mathcal{P}(n_k) \boxtimes M^{\boxtimes n_k}\big) \arrow[d, "shuffle\circ((id\boxtimes c^{\boxtimes k})\boxtimes id\boxtimes\ldots\boxtimes id)"] \\
\big(\mathcal{P}(k) \boxtimes \big(M \boxtimes \mathcal{P}(n_1)\big) \boxtimes \ldots \boxtimes \big(M \boxtimes \mathcal{P}(n_k)\big)\big) \boxtimes \big(M \boxtimes M^{\boxtimes n_1} \boxtimes \ldots \boxtimes M \boxtimes M^{\boxtimes n_k}\big) \arrow[d, "(id\boxtimes\Phi^{\boxtimes k})\boxtimes(\mu^{\boxtimes n}\circ shuffle\circ(c^{n_1}\boxtimes id\ldots\boxtimes c^{n_k}\boxtimes id))"] \\
\big(\mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \ldots \boxtimes \mathcal{P}(n_k)\big) \boxtimes \big(M^{\boxtimes n_1} \boxtimes \ldots \boxtimes M^{\boxtimes n_k}\big) \arrow[d, "\gamma_{\mathcal{P}}\boxtimes id"] \\
\mathcal{P}(n_1 + \cdots + n_k) \boxtimes M^{\boxtimes n_1+\cdots+n_k},
\end{tikzcd}$$

where $\Phi : M \boxtimes \mathcal{P}(k) \to \mathcal{P}(k)$ gives the M-module structure of $\mathcal{P}$ and $\gamma_{\mathcal{P}}$ its operad structure map. To see precisely what the shuffle does, see the description of γ in example 1.3.1, where it is given on elements.

The unit in $\mathcal{P} \rtimes M(1)$ is formed of the units of $\mathcal{P}$ and M.

The associativity of $\gamma_{\mathcal{P}\rtimes M}$ follows from the associativity of μ, $\gamma_{\mathcal{P}}$ and the M-equivariance of $\gamma_{\mathcal{P}}$. This can be seen in an enormous diagram which would not fit in the margin.

Remark 1.3.4. Any monoid M gives rise to an operad $\mathcal{M}$, where $\mathcal{M}(k) = M^k$ and the maps γ are defined as the "right half" of $\gamma_{\mathcal{P}\rtimes M}$. If G is a topological group, explicitly we have $\gamma_G : G^k \times G^{n_1} \times \ldots \times G^{n_k}$ is defined by $\gamma_G(\underline{g}, \underline{h}^1, \ldots, \underline{h}^k) = (g_1 h_1^1, \ldots, g_1 h_{n_1}^1, \ldots, g_k h_{n_k}^k)$.

So, in a sense, $\gamma_{\mathcal{P} \rtimes M} = \gamma_{\mathcal{P}} \rtimes \gamma_M$.

Proposition 1.3.5. *Let $\mathcal{P}$ and M be as above. An object X of $\mathcal{S}$ is a $\mathcal{P} \rtimes M$-algebra if and only if X is a $\mathcal{P}$-algebra in M-Mod, i.e. if and only if X is an M-module and a $\mathcal{P}$-algebra with M-equivariant structure maps.*

Proof. Suppose X is a $\mathcal{P}$-algebra in $M - \mathsf{Mod}$ with structure map $\theta_{\mathcal{P}}$, and let $\Psi : M \boxtimes X \to X$ denote the M-module structure of X. Define

$$\theta_{\mathcal{P} \rtimes M} : \mathcal{P} \boxtimes M^{\boxtimes k} \boxtimes X^{\boxtimes k} \xrightarrow{\text{sh}} \mathcal{P} \boxtimes (M \boxtimes X)^{\boxtimes k} \xrightarrow{\text{id} \boxtimes \Psi^k} \mathcal{P} \boxtimes X^{\boxtimes k} \xrightarrow{\theta_{\mathcal{P}}} X.$$

The associativity of $\theta_{\mathcal{P} \rtimes M}$ is a consequence of the equivariance of the $\mathcal{P}$-algebra structure maps.

Conversely, if X is a $\mathcal{P} \rtimes M$-algebra, define the M-module structure on X by

$$\Psi : M \boxtimes X \xrightarrow{\eta_{\mathcal{P}}} \mathcal{P}(1) \boxtimes M \boxtimes X \xrightarrow{\theta_{\mathcal{P} \rtimes M}} X$$

and the $\mathcal{P}$-algebra structure on X by

$$\theta_{\mathcal{P}} : \mathcal{P}(k) \boxtimes X^{\boxtimes k} \xrightarrow{\eta_M^{\boxtimes k}} \mathcal{P}(k) \boxtimes M^{\boxtimes k} \boxtimes X^{\boxtimes k} \xrightarrow{\theta_{\mathcal{P} \rtimes M}} X,$$

where $\eta_{\mathcal{P}}$ and η_M are the unit maps of $\mathcal{P}$ as an operad and M as a monoid respectively. The M-equivariance of $\theta_{\mathcal{P}}$ follows, with some efforts, from the associativity of $\theta_{\mathcal{P} \rtimes M}$. Note also that this definition of Ψ and $\theta_{\mathcal{P}}$ gives again $\theta_{\mathcal{P} \rtimes M} = \theta_{\mathcal{P}} \circ (\text{id} \boxtimes \Psi^k) \circ \text{shuffle}$. $\square$

Proposition 1.3.6. *A morphism of $\mathcal{P} \rtimes M$-algebras is a morphism of $\mathcal{P}$-algebras which is an M-equivariant map.*

Proof. Let X, Y be $\mathcal{P} \rtimes M$-algebras and $f : X \to Y$ be a $\mathcal{P}$-algebra map and an M-module map. Then f is a $\mathcal{P} \rtimes M$-algebra map as the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{P}(k) \boxtimes M^{\boxtimes k} \boxtimes X^{\boxtimes k} & \xrightarrow{\text{id} \boxtimes (\Psi_X)^{\boxtimes k}} & \mathcal{P}(k) \boxtimes X^{\boxtimes k} & \xrightarrow{\theta_{\mathcal{P}}} & X \\
\downarrow{\scriptstyle \text{id} \boxtimes f^{\boxtimes k}} & & \downarrow{\scriptstyle \text{id} \boxtimes f^{\boxtimes k}} & & \downarrow{\scriptstyle f} \\
\mathcal{P}(k) \boxtimes M^{\boxtimes k} \boxtimes Y^{\boxtimes k} & \xrightarrow{\text{id} \boxtimes (\Psi_Y)^{\boxtimes k}} & \mathcal{P}(k) \boxtimes Y^{\boxtimes k} & \xrightarrow{\theta_{\mathcal{P}}} & Y
\end{array}$$

Conversely, suppose that f is a $\mathcal{P} \rtimes M$-algebra map. Then, in particular, f is an $\mathcal{P}$-algebra map and an M-equivariant map as those structures on X and Y can be expressed in terms of the $\mathcal{P} \rtimes M$-algebra structure. $\square$

Let $\mathcal{P}$ be a topological operad with a base point $* \in \mathcal{P}(0)$. If moreover $\mathcal{P}$ is an M-operad with the M-action preserving $*$, we can describe the free monad associated to a $\mathcal{P} \rtimes M$ in terms of the monad associated to $\mathcal{P}$.

Let $X \wedge Y$ denote the smash product of X and Y, i.e.

$$X \wedge Y = \frac{X \times Y}{(*_X \times Y) \cup (X \times *_Y)},$$

and let X_+ denote X with a formally added base point.

Proposition 1.3.7. *For $\mathcal{P}$ and M as described above and for any space X,*

$$(P \rtimes M)X \cong P(M_+ \wedge X).$$

In particular, $\quad fD_nX \cong D_n(SO(n)_+ \wedge X).$

Proof.

$$(P \rtimes M)X = \frac{\coprod_k \mathcal{P} \rtimes M(k) \times_{\Sigma_k} X^k}{\approx_{P \rtimes M}}$$

$$= \frac{\coprod_k (\mathcal{P}(k) \times M^k) \times_{\Sigma_k} X^k}{\approx_{P \rtimes M}} = \frac{\coprod_k \mathcal{P}(k) \times_{\Sigma_k} (M_+ \wedge X)^k}{\approx_P}.$$

The smash product occurring in the last equality comes from the base point relation ($\approx_{P \rtimes M}$ gives a relation whenever the base point of X appears, regardless of the element of M associated to it). $\square$

1.3.2 Examples

We first give a "trivial" example. Recall from remark 1.3.4 that any monoid M give rise to an operad $\mathcal{M}$ with $\mathcal{M}(k) = M^k$. If $\mathcal{P}$ is the trivial operad, $\mathcal{M}$ is isomorphic to $\mathcal{P} \rtimes M$. We express this precisely in the topological case:

Proposition 1.3.8. *Let M be a topological monoid and $\mathcal{M}$ its associated operad. Then $\mathcal{M} \cong \mathcal{C}om \rtimes M$. In particular, an $\mathcal{M}$-algebra is an M-space with an M-equivariant commutative multiplication.*

Our main example of semidirect product is the framed discs operad: $f\mathcal{D}_n = \mathcal{D}_n \rtimes SO(n)$ defined in example 1.3.1. We will in fact consider a more general version of this semidirect product.

Proposition 1.3.9. *The natural action of the orthogonal group $O(n)$ on $\mathcal{D}_n(k)$, for all k, is such that the operad structure maps are $O(n)$-equivariant.*

Proof. The group $O(n)$ acts on $\mathcal{D}_n(k)$ by the restriction of its action on $\mathbb{R}^n$. The equivariance of the maps $\gamma_{\mathcal{D}_n}$ follows from the fact that the action restricts to the same action on sub-discs of the disc (after rescaling). So one can act equivalently before or after plugging in the discs. $\square$

Example 1.3.10. Let G be a topological group and $\phi : G \to O(n)$ an n-dimensional orthogonal representation of G. One can construct the semidirect product

$$\mathcal{D}_n \rtimes_\phi G,$$

often just denoted by $\mathcal{D}_n \rtimes G$, where G acts on $\mathcal{D}_n$ through its orthogonal representation. A $\mathcal{D}_n \rtimes G$-algebra is a G-space X which is a $\mathcal{D}_n$-algebra with structure maps satisfying

$$g\theta_k(c, x_1, \dots, x_k) = \theta_k(\phi(g)c, gx_1, \dots, gx_k),$$

where $c \in \mathcal{D}_n(k)$, $x_i \in X$ and $g \in G$.

We will use the following examples of algebras over $\mathcal{D}_n \rtimes_\phi G$:

Example 1.3.11. Let Y be a pointed G-space. Let D_nY be the free $\mathcal{D}_n$-algebra on Y in Top_*, and let $\Omega^n Y = \{(D^n, \partial D^n) \to (Y, *)\}$. We already saw in example 1.1.5 that the space $\Omega^n Y$ carries a natural $\mathcal{D}_n$-algebra structure.

Let $\phi : G \longrightarrow O(n)$ be a continuous group homomorphism. The spaces D_nY and $\Omega^n Y$ are $\mathcal{D}_n \rtimes_\phi G$-algebras, with the action of $g \in G$

on $[c; y_1, \dots, y_k] \in D_nY$, where $c \in \mathcal{D}_n(k)$, $y_i \in Y$, given by

$$g[c; y_1, \dots, y_k] = [\phi(g)c; gy_1, \dots, gy_k],$$

and on $[y(t)] \in \Omega^n Y$, where $t \in D^n$ and $[y(t)]$ denotes the n-fold loop $t \mapsto y(t)$, given by

$$g[y(t)] = [gy(\phi(g)^{-1}(t))].$$

If $\mathcal{P}$ is a topological operad, its homology $H(\mathcal{P}) := \{H_*(\mathcal{P}(k))\}_{k\in\mathbb{N}}$, with any coefficient, is again an operad. We consider here field coefficients because we want an isomorphism.

Proposition 1.3.12. *Let G be a topological group and $\mathcal{P}$ a G-operad. Then for any field k, we have the following isomorphism:*

$$H(\mathcal{P} \rtimes G, k) \cong H(\mathcal{P}) \rtimes H(G).$$

This will allow us, in chapter 2, to identify the homology of the framed discs operads.

Another example is given by the ribbon operad we constructed in section 1.2:

Example 1.3.13. *In CAT, we have* $\quad \mathcal{C}^{R\beta} = \mathcal{C}^{\beta} \rtimes \mathcal{Z}$,
where $\mathcal{Z}$ is the category with one object and $\mathbb{Z}$ as set of morphisms.

Clearly, $\mathcal{C}^{R\beta_k}_{PR\beta_k} = \mathcal{C}^{\beta_k}_{P\beta_k} \times \mathcal{Z}^k$. The action of $\mathcal{Z}$ on $\mathcal{C}^{\beta_k}_{P\beta_k}$ is trivial on objects. On morphisms, $\mathbb{Z} \times \mathcal{C}^{\beta}(\sigma, \tau) \to \mathcal{C}^{\beta}(\sigma, \tau)$ is defined by $z.b = t^z b$, where t the full twist of the k strings in $P\beta_k$. The braid t is actually the generator of the centre of β_k.

We will use this fact to describe algebras over $\mathcal{C}^{R\beta}$ in section 1.4.

Proposition 1.3.14. *Let $\mathcal{M}$ be a cocommutative bimonoid in $\mathsf{CAT}$and $\mathcal{C}$ be an $\mathcal{M}$-operad. Then*

$$B(\mathcal{C} \rtimes \mathcal{M}) \cong B\mathcal{C} \rtimes B\mathcal{M}.$$

In particular, $B\mathcal{C}^{R\beta} \cong B\mathcal{C}^{\beta} \rtimes B\mathbb{Z}$.

1.4 Braided and ribbon braided categories

Braided monoidal categories are "not quite symmetric" monoidal categories. More precisely, they are equipped with a natural isomorphism $c_{A,B} : A \boxtimes B \to B \boxtimes A$ which is not required to be involutive. Instead, c is required to satisfy braid type relations.

Braided monoidal categories arise in the theory of quantum groups and their associated link invariants [30]. Braided categories also appear in higher dimensional category theory. For example, a 3-category with only one object and one 1-morphism is a braided monoidal category [17].

Ribbon braided categories are braided monoidal categories with an additional twist, i.e. a natural isomorphism, $\tau_A : A \to A$, compatible with the braiding. Those categories give the right general setting for the introduction of dual objects in a braided monoidal category [35].

It has been known for a long time that the group completion of the classifying spaces of symmetric monoidal categories are infinite loop spaces ([34, 26]). In our notations, symmetric (strict) monoidal categories are precisely the algebras over the operad $\mathcal{C}^{\Sigma} = \{\mathcal{C}^{\Sigma_k}_{\{e\}}\}$. The result follows from the fact that $\Gamma = B\mathcal{C}^{\Sigma}$ is an E_∞ operad. In particular, it detects infinite loop spaces.

In this section, we show that algebras over the operads $\mathcal{C}^{\beta}$ and $\mathcal{C}^{R\beta}$ are precisely braided and ribbon braided (strict) monoidal categories respectively. Following Fiedorowicz's ideas, we will use this later on to relate the classifying spaces of these categories to double loop spaces.

We use Joyal and Street coherence results for braided and ribbon braided categories [15] as well as the formalism of semidirect products to deduce the ribbon braided case from the braided one. Note that Joyal and Street call ribbon braided categories "balanced categories". Not knowing why these authors chose this terminology, and being confronted with the fact that symmetric and braided monoidal categories are what they are, we could not resist calling ribbon braided categories by their obvious name, natural from our point of view.

1.4.1 Braided categories

Recall from section 1.1.1 that a monoidal category $(\mathcal{A}, \boxtimes)$ is a category $\mathcal{A}$ equipped with an associative, unital product $\boxtimes$. We will denote by a the associativity isomorphisms and by 1 the unit.

Definition 1.4.1. *[15] Let $(\mathcal{A}, \boxtimes)$ be a monoidal category. A* braiding *for $\mathcal{A}$ is a natural family of isomorphisms*

$$c = c_{A,B} : A \boxtimes B \longrightarrow B \boxtimes A$$

in $\mathcal{A}$ such that the two following diagrams commute (see figure 1.4.1 for an illustration of those relations):

$$\begin{tikzcd}
 & (B \boxtimes A) \boxtimes C \arrow[r, "a"] & B \boxtimes (A \boxtimes C) \arrow[dr, "id \boxtimes c"] & \\
(A \boxtimes B) \boxtimes C \arrow[ur, "c \boxtimes id"] \arrow[dr, "a"'] & & & B \boxtimes (C \boxtimes A) \\
 & A \boxtimes (B \boxtimes C) \arrow[r, "c"'] & (B \boxtimes C) \boxtimes A \arrow[ur, "a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \boxtimes (C \boxtimes B) \arrow[r, "a^{-1}"] & (A \boxtimes C) \boxtimes B \arrow[dr, "c \boxtimes id"] & \\
A \boxtimes (B \boxtimes C) \arrow[ur, "id \boxtimes c"] \arrow[dr, "a^{-1}"'] & & & (C \boxtimes A) \boxtimes B \\
 & (A \boxtimes B) \boxtimes C \arrow[r, "c"'] & C \boxtimes (A \boxtimes B) \arrow[ur, "a^{-1}"'] &
\end{tikzcd}$$

We call $(\mathcal{A}, \boxtimes, c)$ a braided monoidal category *(or shortly a* braided category*). It is called a* braided strict monoidal category *if the monoidal structure is strict, i.e. if the associativity isomorphisms are all the identity.*

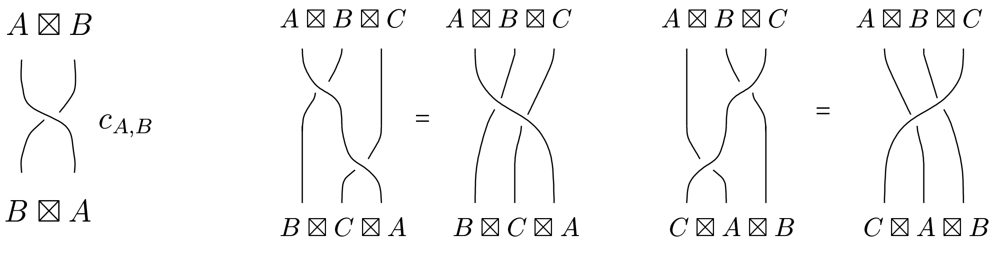


Figure 1.5: Braiding and braid relations

Let $\mathfrak{B}$ be the category with $\mathbb{N}$ as set of elements and β_n, the braid group on n strings, as set of morphisms from n to n.

Then $\mathfrak{B}$ is a braided monoidal category. The product is given by addition on the objects, and by block sum on the morphisms, i.e. putting the braids "side by side". The braiding $b_{n,m}$ is given by the element of β_{n+m} shown in figure 1.6.

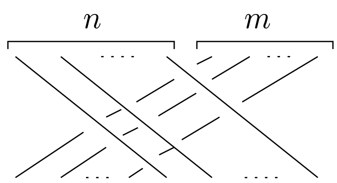


Figure 1.6: Braiding $b_{n,m}$ of $\mathfrak{B}$

Now, for any category $\mathcal{A}$, one can construct a braided monoidal category $(\mathfrak{B}\int\mathcal{A}, \otimes, b)$, which is a wreath product of $\mathcal{A}$ and $\mathfrak{B}$. The objects of $\mathfrak{B}\int\mathcal{A}$ are finite sequences of objects of $\mathcal{A}$. An arrow $(\alpha, f_1, \ldots, f_n) : A_1 \ldots A_n \longrightarrow B_1 \ldots B_n$

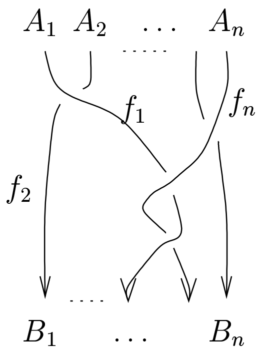


Figure 1.7: Arrow in $\mathfrak{B}\int\mathcal{A}$

consists of $\alpha \in \beta_n$ and $f_i \in Hom_{\mathcal{A}}(A_i, B_{\pi(\alpha)(i)})$, where $\pi : \beta_n \twoheadrightarrow \Sigma_n$ is the natural projection. An arrow is thus a braid having its strings labelled by arrows of $\mathcal{A}$ (see figure 1.4.1). The tensor product is defined on objects and morphisms by block sum.

The braiding of $\mathfrak{B}\int\mathcal{A}$ is defined by

$$c'_{A_1\ldots A_n, B_1\ldots B_m} = (b_{n,m}, id_{A_1}, \ldots, id_{A_n}, id_{B_1}, \ldots, id_{B_m}).$$

So it is the braiding given in figure 1.6 with identity morphisms on the strings.

Let $(\mathcal{A}, \boxtimes, c)$ be a braided strict monoidal category. By a result of Joyal and Street (in [15]), there exists a unique strict monoidal functor $T : \mathfrak{B}\int\mathcal{A} \longrightarrow \mathcal{A}$ such that the following triangle commutes and $T(c') = c$, for c' the braiding of $\mathfrak{B}\int\mathcal{A}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[dr, "id"'] & \mathfrak{B}\int\mathcal{A} \arrow[d, dashed, "T"] \\ & \mathcal{A}. \end{tikzcd}$$

Let α be an element of β_n. For each $A_1 \ldots A_n \in \mathfrak{B} \int \mathcal{A}$, α defines an arrow

$$f_\alpha = (\alpha, id, \ldots, id) \ : \ A_1 \ldots A_n \longrightarrow A_{\pi(\alpha)^{-1}(1)} \ldots A_{\pi(\alpha)^{-1}(n)}$$

where the strings are labelled by identities in $\mathcal{A}$.

Define the following arrow in $\mathcal{A}$:

$$d_\alpha = T(f_\alpha) : A_1 \boxtimes \ldots \boxtimes A_n \longrightarrow A_{\pi(\alpha)^{-1}(1)} \boxtimes \ldots \boxtimes A_{\pi(\alpha)^{-1}(n)}$$

We will use the following properties of d_α:

Since $f_{\alpha_1} \circ f_{\alpha_2} = f_{\alpha_1\alpha_2}$ and $T(f_{\alpha_1} \circ f_{\alpha_2}) = T(f_{\alpha_1}) \circ T(f_{\alpha_2})$, we have the relation

$$d_{\alpha_1} \circ d_{\alpha_2} = d_{\alpha_1\alpha_2}. \tag{1.1}$$

As in $\mathfrak{B}\int\mathcal{A}$, $f_\alpha \circ (id, g_1, \ldots, g_n) = (id, g_{\pi(\alpha)^{-1}(1)}, \ldots, g_{\pi(\alpha)^{-1}(n)}) \circ f_\alpha$ for g_i morphisms in $\mathcal{A}$, we have that

$$d_\alpha \circ (g_1 \boxtimes \ldots \boxtimes g_n) = (g_{\pi(\alpha)^{-1}(1)} \boxtimes \ldots \boxtimes g_{\pi(\alpha)^{-1}(n)}) \circ d_\alpha. \tag{1.2}$$

Another property of the maps f_α is that
$f_\alpha((A_1 \boxtimes \ldots \boxtimes A_{j_1}) \ldots (A_i \boxtimes \ldots \boxtimes A_j)) = f_{\alpha(j_1,\ldots,j_k)}(A_1 \ldots A_j)$,
where $i = j_1 + \cdots + j_{k-1} + 1$. It follows that

$$d_\alpha((A_1 \boxtimes \ldots \boxtimes A_{j_1}) \boxtimes \ldots \boxtimes (A_i \boxtimes \ldots \boxtimes A_j)) = d_{\alpha(j_1,\ldots,j_k)}(A_1 \boxtimes \ldots \boxtimes A_j). \tag{1.3}$$

And for $\alpha_i \in \beta_{j_i}$, with $j_1 + \cdots + j_k = j$, we have that

$$f_{\alpha_1\boxtimes\ldots\boxtimes\alpha_k}(A_1 \ldots A_j) = f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_k}(A_1 \ldots A_{j_1} \otimes \cdots \otimes A_{j_1+\cdots+j_{k-1}+1} \ldots A_j),$$

where $\boxtimes$ and $\otimes$ are the products in $\mathfrak{B}$ and $\mathfrak{B}\int\mathcal{A}$ respectively. Since T is a monoidal functor, it follows that

$$d_{\alpha_1} \boxtimes \ldots \boxtimes d_{\alpha_k} = d_{\alpha_1\boxtimes\ldots\boxtimes\alpha_k}. \tag{1.4}$$

Let $\mathcal{C}^\beta$ be the operad defined in section 1.2.

Lemma 1.4.2. *Let $(\mathcal{A}, \boxtimes, c)$ be a braided strict monoidal category. Then for all $j \geq 0$ there exists a Σ_j-equivariant functor $\boldsymbol{d}_j : \mathcal{C}^{\beta_j}_{P\beta_j} \times \mathcal{A}^j \longrightarrow \mathcal{A}$ defined on objects and morphisms respectively by*

$$\boldsymbol{d}_j(\sigma, A_1, \ldots, A_j) = A_{\sigma^{-1}(1)} \boxtimes \ldots \boxtimes A_{\sigma^{-1}(j)}$$

and

$$\boldsymbol{d}_j(\tau \xleftarrow{\alpha} \sigma, f_1, \ldots, f_j) = d_\alpha \circ (f_{\sigma^{-1}(1)} \boxtimes \ldots \boxtimes f_{\sigma^{-1}(j)})$$

where $\sigma, \tau \in \Sigma_j$, $\alpha \in \mathcal{C}^{\beta_j}_{P\beta_j}(\sigma, \tau)$, $A_i \in \mathcal{A}$ and f_i are arrows of $\mathcal{A}$.

Proof. The functoriality of $\mathbf{d}_j$ is a result of the two first properties of d_α given above.

The Σ_j-equivariance of $\mathbf{d}_j$ on objects is a direct consequence of the remark 1.2.6.

We check here the equivariance on morphisms. Let $\rho \in \Sigma_j$. We have

$\mathbf{d}_j(\tau\rho \xleftarrow{\alpha\rho} \sigma\rho, f_1, \ldots, f_j) = d_\alpha \circ (f_{(\sigma\rho)^{-1}(1)} \boxtimes \ldots \boxtimes f_{(\sigma\rho)^{-1}(j)})$.

And

$\mathbf{d}_j(\tau \xleftarrow{\alpha} \sigma, f_{\rho^{-1}(1)}, \ldots, f_{\rho^{-1}(j)}) = d_\alpha \circ (f_{\rho^{-1}(\sigma^{-1}(1))} \boxtimes \ldots \boxtimes f_{\rho^{-1}(\sigma^{-1}(j))})$. $\square$

Lemma 1.4.3. *For $\mathcal{A}$ and $\mathbf{d}_j$ as above, the following diagram is commutative for all $j \geq 0, k \geq 0$ and $j_i \geq 0$ with $\Sigma j_i = j$:*

$$\begin{tikzcd}
\mathcal{C}^{\beta_k}_{P\beta_k} \times \mathcal{C}^{\beta_{j_1}}_{P\beta_{j_1}} \times \cdots \times \mathcal{C}^{\beta_{j_k}}_{P\beta_{j_k}} \times \mathcal{A}^j \arrow[rr, "\Gamma_\beta \times id"] \arrow[dd, "id \times \mu"'] & & \mathcal{C}^{\beta_j}_{P\beta_j} \times \mathcal{A}^j \arrow[d, "\mathbf{d}_j"] \\
 & & \mathcal{A} \\
\mathcal{C}^{\beta_k}_{P\beta_k} \times \mathcal{C}^{\beta_{j_1}}_{P\beta_{j_1}} \times \mathcal{A}^{j_1} \times \cdots \times \mathcal{C}^{\beta_{j_k}}_{P\beta_{j_k}} \times \mathcal{A}^{j_k} \arrow[rr, "id \times \mathbf{d}_{j_1} \times \cdots \times \mathbf{d}_{j_k}"] & & \mathcal{C}^{\beta_k}_{P\beta_k} \times \mathcal{A}^k \arrow[u, "\mathbf{d}_k"']
\end{tikzcd}$$

where μ is the shuffle and Γ_β is the operad functor of $\mathcal{C}^\beta$.

Proof. The commutativity on objects follows from the commutativity of certain permutations. We check more precisely the commutativity on morphisms.

$\mathbf{d}_j(\Gamma_\beta \times 1(\tau \xleftarrow{\alpha} \sigma, \nu_1 \xleftarrow{\delta_1} \mu_1, \ldots, \nu_k \xleftarrow{\delta_k} \mu_k, f_1, \ldots, f_j))$

$= \mathbf{d}_j(\zeta \xleftarrow{\beta} \lambda, f_1, \ldots, f_j)$,

$= d_\beta \circ (f_{\lambda^{-1}(1)} \boxtimes \ldots \boxtimes f_{\lambda^{-1}(k)})$,

where $\beta = \alpha(j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(k)})(\delta_{\sigma^{-1}(1)} \boxtimes \ldots \boxtimes \delta_{\sigma^{-1}(k)})$

and $\lambda = \sigma(j_1, \ldots, j_k)(\mu_1 \oplus \cdots \oplus \mu_k)$.

$\mathbf{d}_k((1 \times \mathbf{d}_{j_1} \times \cdots \times \mathbf{d}_{j_k})$

$(\tau \xleftarrow{\alpha} \sigma, (\nu_1 \xleftarrow{\delta_1} \mu_1, f_1, \ldots, f_{j_1}), \ldots, (\nu_k \xleftarrow{\delta_k} \mu_k, f_{j_1+\cdots+j_{k-1}+1}, \ldots, f_j)))$

$= \mathbf{d}_k(\tau \xleftarrow{\alpha} \sigma, d_{\delta_1} \circ (f_{\mu_1^{-1}(1)} \boxtimes \ldots \boxtimes f_{\mu_1^{-1}(j_1)}),$

$\ldots, d_{\delta_k} \circ (f_{j_1+\cdots+j_{k-1}+\mu_k^{-1}(1)} \boxtimes \ldots \boxtimes f_{j_1+\cdots+j_{k-1}+\mu_k^{-1}(j_k)}))$

$= d_\alpha \circ ((d_{\delta_{\sigma^{-1}(1)}} \circ (f_{\mu^{-1}_{\sigma^{-1}(1)}(1)} \boxtimes \ldots \boxtimes f_{\mu^{-1}_{\sigma^{-1}(1)}(j_{\sigma^{-1}(1)})})) \boxtimes \ldots \boxtimes (d_{\delta_{\sigma^{-1}(k)}}$

$\circ(f_{j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)-1}+\mu^{-1}_{\sigma^{-1}(k)}(1)} \boxtimes \ldots \boxtimes f_{j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)-1}+\mu^{-1}_{\sigma^{-1}(k)}(j_{\sigma^{-1}(k)})}))).$

Using 1.3 and defining $\tilde{\alpha} = \alpha(j_1, \ldots, j_k)$,

$= d_{\tilde{\alpha}} \circ ((d_{\delta_{\sigma^{-1}(1)}} \boxtimes \ldots \boxtimes d_{\delta_{\sigma^{-1}(k)}}) \circ ((f_{\mu^{-1}_{\sigma^{-1}(1)}(1)} \boxtimes \ldots \boxtimes f_{\mu^{-1}_{\sigma^{-1}(1)}(j_1)}) \boxtimes \ldots \boxtimes$

$(f_{j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)-1}+\mu_{\sigma^{-1}(k)}{-1}(1)} \boxtimes \ldots \boxtimes f_{j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)-1}+\mu^{-1}_{\sigma^{-1}(k)}(j_{\sigma^{-1}(k)})}))),$

which, by 1.4,

$= d_{\tilde{\alpha}} \circ d_{\delta_{\sigma^{-1}(1)} \boxtimes \ldots \boxtimes \delta_{\sigma^{-1}(k)}}$

$$\circ(f_{\mu^{-1}_{\sigma^{-1}(1)}(1)} \boxtimes \ldots \boxtimes f_{j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)-1}+\mu^{-1}_{\sigma^{-1}(k)}(j_{\sigma^{-1}(k)})})),$$

and by 1.1

$$= d_{\tilde{\alpha}.(\delta_{\sigma^{-1}(1)}\boxtimes\ldots\boxtimes\delta_{\sigma^{-1}(k)})} \circ (f_{\lambda^{-1}(1)} \boxtimes \ldots \boxtimes f_{\lambda^{-1}(j)}) \qquad \square$$

We then have the following result :

Theorem 1.4.4. *Let $\mathcal{A}$ be category and $\mathcal{C}^{\beta} = \{\mathcal{C}^{\beta_j}_{P\beta_j}\}_{j\in\mathbb{N}}$ be the braid categorical operad. Then $\mathcal{A}$ is a braided strict monoidal category if and only if $\mathcal{A}$ is a $\mathcal{C}^{\beta}$-algebra.*

Corollary 1.4.5. *If $(\mathcal{A}, \boxtimes, c)$ is a braided strict monoidal category, then $B\mathcal{A}$ is a $B\mathcal{C}^{\beta}$-algebra.*

Proof of the theorem. If $\mathcal{A}$ is a braided strict monoidal category, then lemma 1.4.2 and 1.4.3 imply that $\mathcal{A}$ is a $\mathcal{C}$-algebra, the maps $\mathbf{d}_j$ giving the action of the operad (the unit condition is clear, lemma 1.4.2 gives the Σ-equivariance and lemma 1.4.3 the associativity condition).

For the converse, let $\mathcal{A}$ be a category and θ be the action of $\mathcal{C}$ on $\mathcal{A}$:

$$\theta_j : \mathcal{C}^{\beta_j}_{P\beta_j} \times \mathcal{A}^j \longrightarrow \mathcal{A}.$$

Define the product on $\mathcal{A}$ on objects by $A \boxtimes B = \theta_1(id, A, B)$, where id is the identity element of Σ_2, and on morphisms by $f \boxtimes g = \theta_1(id \xleftarrow{id_\beta} id, f, g)$. The unit is given by $1 = \theta_0(1)$.

The functoriality of θ_j and its associativity property imply that $(\mathcal{A}, \boxtimes, 1)$ is a strict monoidal category.

The braiding is then defined by

$$c_{A,B} = \theta_2(\sigma \xleftarrow{b} id, id_A, id_B) : A \boxtimes B \longrightarrow B \boxtimes A,$$

where σ is the non-trivial element of Σ_2 and b is the generator of β_2.

Using the functoriality, the Σ-equivariance and the associativity property of θ, one can show that the two diagrams of definition 1.4.1 commute. $\square$

1.4.2 Ribbon braided categories

We want a similar result for ribbon braided categories, i.e. we want to prove that the ribbon braid operad $\mathcal{C}^{R\beta} = \{\mathcal{C}^{R\beta_n}_{PR\beta_n}\}_{n\in\mathbb{N}}$ detects ribbon braided monoidal category.

We will use the proposition 1.3.5 and the example 1.3.13 to deduce this result from the result just obtained in the braid case.

We first recall from [15] the definition of a ribbon braided category.

Definition 1.4.6. *Let* $(\mathcal{A}, \boxtimes, c)$ *be a braided monoidal category. A* twist *for* $\mathcal{A}$ *is a natural family of isomorphisms*

$$\tau = \tau_A : A \longrightarrow A$$

such that $\tau_1 = id_1$ *and the following diagram commutes (see figure 1.4.2 for an illustration of this relation):*

$$\begin{tikzcd} A \boxtimes B \arrow[r, "c_{A,B}"] \arrow[d, "\tau_{A\boxtimes B}"'] & B \boxtimes A \arrow[d, "\tau_B \boxtimes \tau_A"] \\ A \boxtimes B & B \boxtimes A \arrow[l, "c_{B,A}"] \end{tikzcd}$$

$(\mathcal{A}, \boxtimes, c, \tau)$ *is called a* ribbon braided *monoidal category (or shortly* ribbon braided category*). If the monoidal structure is strict, it is called a* ribbon braided strict monoidal category*.*

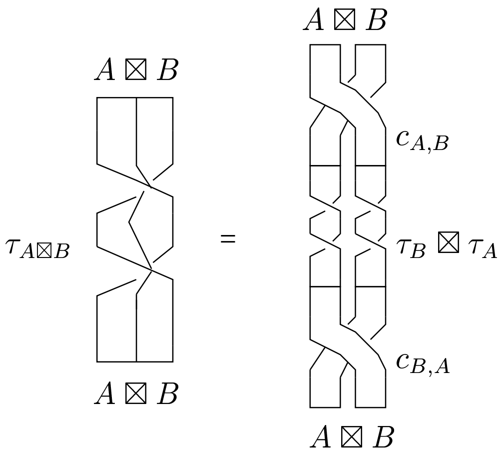


Figure 1.8: Compatibility between the twist and the braiding

Theorem 1.4.7. *Let* $\mathcal{A}$ *a be category and* $\mathcal{C}^{R\beta} = \{\mathcal{C}^{R\beta_j}_{PR\beta_j}\}_{j\in\mathbb{N}}$ *be the ribbon braid cat-operad. Then* $\mathcal{A}$ *is a ribbon braided strict monoidal category if and only if* $\mathcal{A}$ *is a* $\mathcal{C}^{R\beta}$*-algebra.*

Corollary 1.4.8. *If* $(\mathcal{A}, \boxtimes, c)$ *is a ribbon braided strict monoidal category, then* $B\mathcal{A}$ *is a* $B\mathcal{C}^{R\beta}$*-algebra.*

Proof of the theorem. Recall that $\mathcal{Z}$ denotes the category with only one object $*$ and with $\mathbb{Z}$ as set of morphisms.

Using proposition 1.3.5 and example 1.3.13, we have that

$$\mathcal{A} \text{ is an } \mathcal{C}^{R\beta}\text{-algebra}$$
$$\Longleftrightarrow$$
$$\mathcal{A} \text{ is a } \mathcal{C}^{\beta}\text{-algebra which has a } \mathcal{Z}\text{-action commuting with } \theta_{\mathcal{C}^\beta}.$$

Thus, using now theorem 1.4.4, the theorem 1.4.7 can be reformulated as

$$\mathcal{A} \text{ is a ribbon braided strict monoidal category}$$
$$\Longleftrightarrow$$
$$\mathcal{A} \text{ is a braided strict monoidal category which has a } \mathcal{Z}\text{-action commuting with } \theta_{\mathcal{C}^\beta}.$$

Suppose first that $(\mathcal{A}, \boxtimes, c, \tau)$ is a ribbon braided strict monoidal category. Define $\Psi : \mathcal{Z} \times \mathcal{A} \to \mathcal{A}$ to be the identity on objects, $\Psi(*, A) = A$, and on a morphism $f : A \to B$ by

$$\Psi(z, f) = f \circ (\tau_A)^z.$$

This defines an action of $\mathcal{Z}$ on $\mathcal{A}$ as $(\tau_A)^{z'} \circ (\tau_A)^z = (\tau_A)^{z'+z}$.

We now have to check that this action commutes with $\theta_{\mathcal{C}^\beta}$. Note that

$$\tau_A = \Psi(1, id_A)$$

and

$$\Psi(z, f) = f \circ \Psi(z, id_A). \qquad (*)$$

We consider $\theta_{\mathcal{C}^\beta} : \mathcal{C}^\beta(k) \times \mathcal{A}^k \longrightarrow \mathcal{A}$. In the case $k = 1$, the commutativity of Ψ and $\theta_{\mathcal{C}^\beta}$ is trivial because $\theta_{\mathcal{C}^\beta}$ is trivial. We study the case $k = 2$. Observe that we only have to worry about morphisms as the action is trivial on objects. As τ is a twist, we have that

$$\tau_{A\boxtimes B} = c_{B,A} \circ (\tau_B \boxtimes \tau_A) \circ c_{A,B}, \qquad (**)$$

which translates in our case to

$$\Psi(1, \theta_{\mathcal{C}^\beta}(id, id_A, id_B)) = \theta_{\mathcal{C}^\beta}(b, id_B, id_A) \circ \theta_{\mathcal{C}^\beta}(id, \tau_B, \tau_A) \circ \theta_{\mathcal{C}^\beta}(b, id_A, id_B).$$

Using the Σ-equivariance and the functoriality of $\theta_{\mathcal{C}^\beta}$, we deduce from the last equality that

$$\Psi(1, \theta_{\mathcal{C}^\beta}(id, id_A, id_B)) = \theta_{\mathcal{C}^\beta}(\Phi(1, id), \Psi(1, id_A), \Psi(1, id_B)),$$

where Φ gives the action of $\mathbb{Z}$ on $\mathcal{C}^\beta$. This provides the commutativity required for $\theta_{\mathcal{C}^\beta}(id, id_A, id_B)$ and $\Psi(1, -)$. The equation $(**)$ inverted gives the commutation for $\Psi(-1, -)$. As Ψ is an action, we can deduce the commutativity above for all $z \in \mathbb{Z}$.

Now, if b is any braid in β_2, and $f : A \longrightarrow B,\ g : C \longrightarrow D$ any morphisms in $\mathcal{A}$, we have by $(*)$

$\Psi(z, \theta_{\mathcal{C}^\beta}(b, f, g)) = \theta_{\mathcal{C}^\beta}(b, f, g) \circ \Psi(z, \theta_{\mathcal{C}^\beta}(id, id_A, id_C))$
$\quad = \theta_{\mathcal{C}^\beta}(b, f, g) \circ \theta_{\mathcal{C}^\beta}(\Phi(z, id), \Psi(z, id_A), \Psi(z, id_C))$
$\quad = \theta_{\mathcal{C}^\beta}(\Phi(z, b), \Psi(z, f), \Psi(z, g))$.

Hence, the case $k = 2$ is proved. For $k > 2$, we first remark that any braid in β_k can be obtained by compositions (internal and external) of the operad structure functor $\Gamma_{\mathcal{C}^\beta}$ with itself applied exclusively on element of β_1 and β_2 (this is a form of quadraticity for the operad $\mathcal{C}^\beta$). This can be seen by writing a braid in terms of the canonical generators of β_k. The result follows then by remarking that

$\theta_{\mathcal{C}^\beta}(\Gamma_{\mathcal{C}^\beta}(b', b_1, b_2), f_1, \ldots, f_j) = \theta_{\mathcal{C}^\beta}(b', \theta_{\mathcal{C}^\beta}(b_1, f_1, \ldots, f_i), \theta_{\mathcal{C}^\beta}(b_2, f_{i+1}, \ldots, f_j))$

and

$\Psi(z, \theta_{\mathcal{C}^\beta}(b \circ b', f, g)) = \Psi(z, \theta_{\mathcal{C}^\beta}(b, f, g)) \circ \Psi(z, \theta_{\mathcal{C}^\beta}(b', id_A, id_C))$.

Suppose now that $(\mathcal{A}, \boxtimes, c)$ is a braided strict monoidal category which has a $\mathcal{Z}$-action given by a functor Ψ and commuting with $\theta_{\mathcal{C}^\beta}$. We want to show that in this case $\mathcal{A}$ is a ribbon braided monoidal category.

Define a twist on $A \in \mathcal{A}$ by

$$\tau_A = \Psi(1, id_A) : A \longrightarrow A.$$

As Ψ is an action, this is a natural family of isomorphisms. We have to check that

$$\tau_{A \boxtimes B} = c_{B,A} \circ (\tau_B \boxtimes \tau_A) \circ c_{A,B},$$

for all $A, B \in \mathcal{A}$. This is easily checked by translating it in terms of $\theta_{\mathcal{C}^\beta}$ and Ψ as above, using this time the fact that we know that Ψ commutes with $\theta_{\mathcal{C}^\beta}$. $\square$

1.5 Framed discs and ribbons

This section gives an extension of results of Z. Fiedorowicz for the braid groups [7].

Fiedorowicz proved that the braid groups give rise to an operad equivalent to the little discs. To do this, he defined "B_∞ operads", which are braid equivalents of the E_∞ operads. By definition, E_∞ operads are operads with contractible spaces and a free symmetric group action. They are equivalent to the infinite little discs operad $\mathcal{D}_\infty$. Any operad map between two E_∞ operads is an equivalence of operads [26]. Considering the product $\mathcal{P} \times \mathcal{Q}$ of E_∞ operads, and the maps $\mathcal{P} \leftarrow \mathcal{P} \times \mathcal{Q} \to \mathcal{Q}$, one deduces that any two E_∞ operads are equivalent. Algebras over E_∞ operads are infinite loop spaces after group completion.

Fiedorowicz's B_∞ operads are braid operads, i.e. like usual operads but with braid group actions instead of symmetric group actions, which have contractible spaces and free braid group actions. By an argument similar to the above, he shows that any two B_∞ operads are equivalent. He then shows that $\tilde{D}_2$, the universal cover of the little discs operad and a certain braid operad are B_∞ operads. A corollary of the equivalence thus obtained is that the classifying space of a braided monoidal categories is a double loop spaces after group completion.

Fiedorowicz also uses B_∞ operads to characterise E_2 operads, i.e. operads equivalent to the little discs $\mathcal{D}_2$.

In this section, we extend these results to the ribbon braid groups, exhibiting an equivalence between the framed discs operad and the ribbon operad $B\mathcal{C}^{R\beta}$. We also provide a description of "fE_2 operads", i.e. operads equivalent to the framed little 2-discs. We will deduce a description of the classifying spaces of ribbon braided monoidal categories in section 2.1.

1.5.1 Ribbon operads

Definition 1.5.1. *A* ribbon operad $\mathcal{P}$ *is a collection of spaces* $\{\mathcal{P}(j)\}_{j\in\mathbb{N}}$ *with* $\mathcal{P}(0) = \{*\}$*, an identity element* 1 *in* $\mathcal{P}(1)$*, a right action of the ribbon braid group* $R\beta_j$ *on* $\mathcal{P}(j)$ *for all* j*, and with maps*

$$\gamma : \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

such that the unit and associativity conditions of usual operads are satisfied (definition 1.1.1, (i) and (ii)), as well as the following equivariance conditions :

$$\gamma(c^\sigma, d_1, \ldots, d_k) = \gamma(c, d_{\pi(r)^{-1}(1)}, \ldots, d_{\pi(r)^{-1}(k)})^{r(j_1,\ldots,j_k)}$$

and

$$\gamma(c, d_1^{s_1}, \ldots, d_k^{s_k}) = \gamma(c, d_1, \ldots, d_k)^{(s_1 \oplus \cdots \oplus s_k)},$$

for all $c \in \mathcal{P}(k), d_i \in \mathcal{P}(j_i), r \in \mathcal{R}\beta_k, s_i \in \mathcal{R}\beta_{j_i}$, where $\pi : R\beta_k \twoheadrightarrow \Sigma_k$ is the natural projection.

A ribbon operad $\mathcal{P}$ is called unital *if $\mathcal{P}(0) = \{*\}$.*

A morphism of ribbon operads is a family of maps $\phi_j : \mathcal{P}(j) \longrightarrow \mathcal{Q}(j)$ which are $R\beta_j$-equivariant and commute with $\gamma_{\mathcal{P}}$ and $\gamma_{\mathcal{Q}}$.

To avoid confusion, we will sometimes call the usual operads "symmetric operads". On the other hand, we will sometimes use "operad" for ribbon operad when it is not confusing.

Remark 1.5.2. As in the symmetric case, it is possible to define ribbon braid operads in a more general context. We will actually encounter a categorical ribbon operad.

Note that any symmetric operad is a ribbon operad, with the ribbon braid groups acting on the spaces through their projection to the symmetric groups. Remark that the converse is not true, as there is no natural group inclusion of the symmetric groups in the ribbon braid groups.

Hence, the symmetric operad of endomorphisms of X, $\mathcal{E}nd_X$, where $\mathcal{E}nd_X(j) = \{f : X^j \longrightarrow X\}$, can be considered as a ribbon operad.

We can thus use it to define algebras over a ribbon operad like in the symmetric case:

Definition 1.5.3. *Let $\mathcal{P}$ be a ribbon operad. A $\mathcal{P}$-algebra structure on a space X is a morphism of ribbon operads $\phi : \mathcal{P} \longrightarrow \mathcal{E}nd_X$.*

As in the symmetric case, the last definition is equivalent to requesting that there exist maps $\theta_j : \mathcal{P}(j) \times X^j \longrightarrow X$ satisfying unit and associativity conditions (exactly the same as before) and such that $\theta_j(c^r, x_1, \ldots, x_j) = \theta_j(c, x_{\pi(r)^{-1}(1)}, \ldots, x_{\pi(r)^{-1}(k)})$, for all $r \in R\beta_j$.

Example 1.5.4. The ribbon braid groups give rise to the following ribbon operad :

$$\mathcal{R}\beta = \{R\beta_j\}_{j \in \mathbb{N}},$$

with the action of $R\beta_j$ given by right multiplication and with the operad maps defined by $\gamma(r, s_1, \ldots, s_k) = r(j_1, \ldots, j_k)(s_1 \oplus \cdots \oplus s_k)$.

Definition 1.5.5. *A ribbon operad $\mathcal{P}$ is called an R_∞* operad *if the ribbon braid groups act freely and properly on $\mathcal{P}$ and if each space $\mathcal{P}(k)$ is contractible.*

Let $\mathcal{ERB}_n$ be the translation category of $R\beta_n$, i.e. $R\beta_n$ is the set of objects and there is only one morphism $\tau\sigma^{-1}$ from σ to τ. In the notation of section 1.2, we have the groups $G_n = R\beta_n$ and $H_n = \{e\}$, and $\mathcal{ERB}_n = \mathcal{C}_{\{e\}}^{R\beta_n}$. Note that we now have $G_n/H_n = R\beta_n$, so the ribbon braid groups will play the role of the symmetric groups in the construction.

Example 1.5.6. $E\mathcal{RB} = \{B\mathcal{C}_{\{e\}}^{R\beta_n}\}_{n\in\mathbb{N}}$, the sequence of the classifying spaces of the categories $\mathcal{ERB}_n$, forms an R_∞ operad.

It is well-known that the spaces $E\mathcal{RB}_n$ are contractible. Also, the $R\beta_n$-action, defined by right multiplication, as the symmetric case, is free. The operad structure is defined on the categorical level, i.e. by a functor Γ on $\{\mathcal{ERB}_n\}$. We need to specify Γ only on objects, on which it is defined to be the operad map on the ribbon braid groups given in example 1.5.4.

The categorical ribbon operad $\mathcal{ERB}$ acts on any ribbon braided strict monoidal category $(\mathcal{A}, \boxtimes, c, \tau)$ through the action of $\mathcal{C}^{R\beta}$ (theorem 1.4.7) via the natural projection $\mathcal{ERB} \longrightarrow \mathcal{C}^{R\beta}$.

Note that this projection induces a covering map on the classifying spaces.

Example 1.5.7. Let $\mathbb{R}$ be the set of the real numbers. As $\mathbb{R}$ is a group, we know that it gives rise to a symmetric operad (see remark 1.3.4). We define here a ribbon operad with $\mathbb{R}$. Take again the collection of sets $\mathbb{R} = \{\mathbb{R}^j\}_{j\in\mathbb{N}}$ and the maps

$$\gamma : \mathbb{R}^k \times \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \longrightarrow \mathbb{R}^{n_1+\cdots+n_k}$$

$$(\underline{r}, \underline{x}^1, \ldots, \underline{x}^k) \mapsto (x_1^1 + r_1, \ldots, x_{n_1}^1 + r_1, \ldots, x_1^k + r_k, \ldots, x_{n_k}^k + r_k),$$

and define the action of the ribbon braid groups by

$$(r_1, \ldots, r_k)^s = (r_{\pi(s)(1)} + z_1, \ldots, r_{\pi(s)(k)} + z_k),$$

where z_i is the number of twists on the ith string.

1.5.2 Framed discs

Let $f\tilde{\mathcal{D}}_2(k)$ denote the universal cover of $f\mathcal{D}_2(k)$. The proof of the following proposition is an adaptation of Fiedorowicz's proof for the universal cover of the little cubes.

Theorem 1.5.8. *The sequence of spaces $f\tilde{\mathcal{D}}_2 = \{f\tilde{\mathcal{D}_2}(k)\}_{k\in\mathbb{N}}$ forms an R_∞-operad.*

Proof. Recall that $\mathcal{D}_2(k)$ is a $K(P\beta_k, 1)$. So $f\mathcal{D}_2(k) = \mathcal{D}_2(k) \times (S^1)^k$ is a $K(PR\beta_k, 1)$ and its universal cover $f\tilde{\mathcal{D}}_2 = \tilde{\mathcal{D}}_2 \times \mathbb{R}^k$ is contractible.

We want to lift the operad structure of $f\mathcal{D}_2$ to a ribbon operad structure on $f\tilde{\mathcal{D}}_2$. As there is no possible consistent choice of basepoints for the spaces $f\mathcal{D}_2(k)$, we consider instead the contractible subspaces defined by the horizontal inclusion of the "non-symmetric operad" $\mathcal{D}_1^0$ of unlabelled little intervals, in $f\mathcal{D}_2$ shown in figure 1.9, seeing $\mathcal{D}_1^0$ as the component of $\mathcal{D}_1$ with the intervals ordered in the canonical way, from left to right.

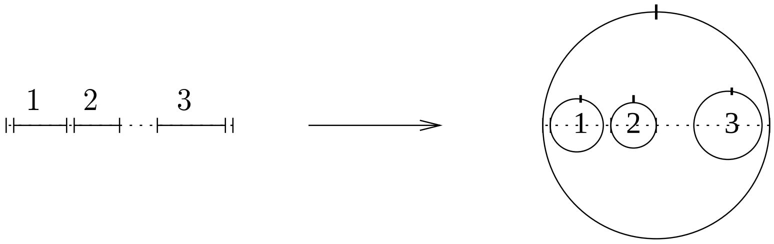


Figure 1.9: Horizontal inclusion of $\mathcal{D}_1^0(3)$ in $f\mathcal{D}_2(3)$

Denote by $p : f\tilde{\mathcal{D}}_2(k) \to f\mathcal{D}_2(k)$ the universal covering. We thus have that $p^{-1}(\mathcal{D}_1^0(k)) \subset f\tilde{\mathcal{D}}_2(k)$ is a disjoint union of contractible components, each homeomorphic to $\mathcal{D}_1^0(k)$ via p. For each k, choose one of these components and denote it by $\tilde{\mathcal{D}_1^0}(k)$.

Now, define $\tilde{\gamma}$ on $f\tilde{\mathcal{D}}_2$ to be the unique lifting of $\gamma \circ p$

$$\begin{tikzcd}
f\tilde{\mathcal{D}}_2(k) \times f\tilde{\mathcal{D}}_2(n_1) \times \cdots \times f\tilde{\mathcal{D}}_2(n_k) \arrow[r, dashed, "\tilde{\gamma}"] \arrow[d, "p"'] & f\tilde{\mathcal{D}}_2(n_1 + \cdots + n_k) \arrow[d, "p"] \\
f\mathcal{D}_2(k) \times f\mathcal{D}_2(n_1) \times \cdots \times f\mathcal{D}_2(n_k) \arrow[r, "\gamma"] & f\mathcal{D}_2(n_1 + \cdots + n_k)
\end{tikzcd}$$

which takes $\tilde{\mathcal{D}_1^0}(k) \times \tilde{\mathcal{D}_1^0}(n_1) \times \cdots \times \tilde{\mathcal{D}_1^0}(n_k)$ to $\tilde{\mathcal{D}_1^0}(n_1 + \cdots + n_k)$.

Define the unit element $1 \in f\tilde{\mathcal{D}}_2(1)$ to be the unique element in $1 \in \tilde{\mathcal{D}_1^0}(1)$ such that $p(1) = 1 \in f\mathcal{D}_2(1)$.

We are now left to define an action of $R\beta_k$ on $f\tilde{\mathcal{D}}_2(k)$. We define the action of each generator $r_i, t_j \in R\beta_k$ for $i = 1, \ldots, k-1$ and $j = 1, \ldots, k$ (see figure 1.10).

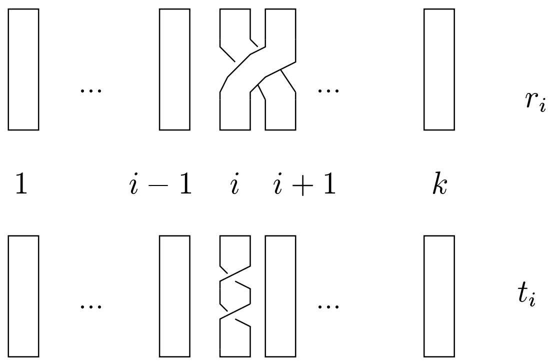


Figure 1.10: Generators of $R\beta_k$

For r_i, consider the path α_i in $f\mathcal{D}_2(k)$, from a point x to its translate $x\sigma_i$, where $\sigma_i = \pi(r_i) \in \Sigma_k$, as shown in figure 1.11.

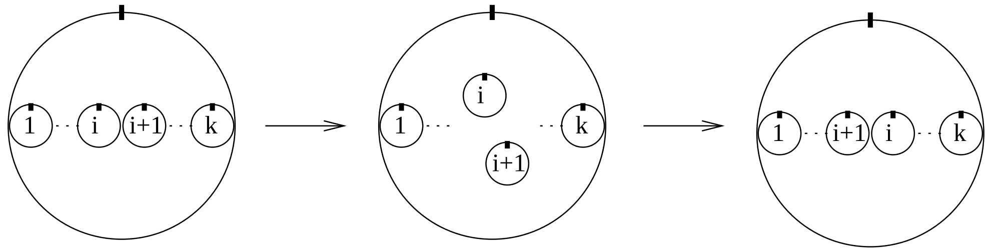


Figure 1.11: Path α_i in $f\mathcal{D}_2(k)$

Let $\tilde{x}$ be the point in $\tilde{\mathcal{D}_1^0}(k)$ such that $p(\tilde{x}) = x$ and let $\tilde{\alpha_i}$ be the lift of α_i starting at $\tilde{x}$. Define the action of r_i on $f\tilde{\mathcal{D}_2}(k)$ to be the unique lift of the map $p \circ \sigma_i : f\tilde{\mathcal{D}_2}(k) \to f\mathcal{D}_2(k)$ which takes x to $\tilde{\alpha_i}(1)$.

Simiarly, define the action of t_i by considering the lift at $\tilde{x}$ of the path β_i from x to x shown in figure 1.12.

This data provides $f\tilde{\mathcal{D}_2}$ with a ribbon operad structure as all commutative diagrams lift to commutative diagrams □

1.5.3 Ribbon monads

We want to show that R_∞ operads are all equivalent in the sense that their categories of algebras are equivalent. We do this by comparing their associated monads.

Definition 1.5.9. *Let $\mathcal{P}$ be a unital ribbon operad. For any pointed space X, define*

$$CX = \frac{\coprod_{n\geq 0} \mathcal{P}(n) \times_{R\beta_n} X^n}{\approx},$$

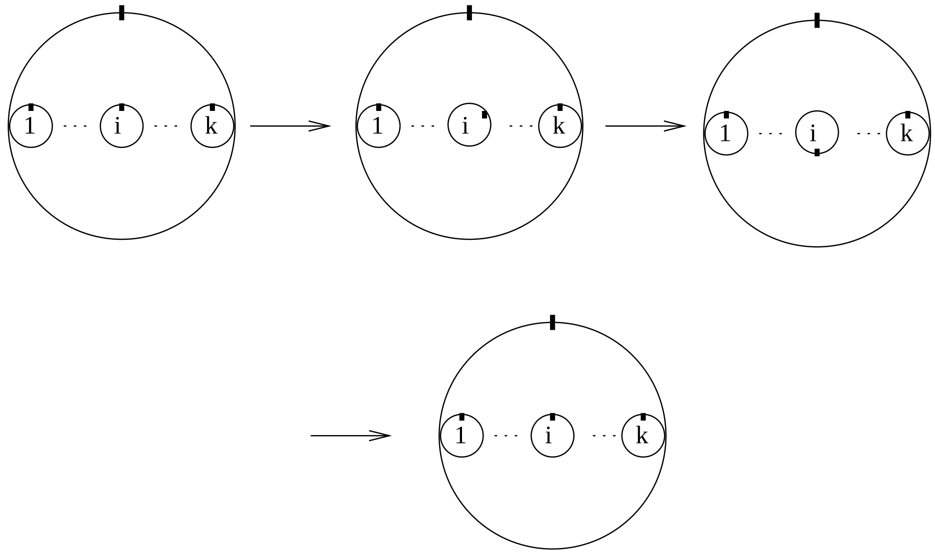


Figure 1.12: Path β_i in $f\mathcal{D}_2(k)$

where $R\beta_n$ acts on X^n via its projection to Σ_n and the equivalence relation $\approx$ is defined as in the symmetric case by $(\sigma_i c, y) \approx (c, s_i y)$ for $c \in \mathcal{P}(j)$, $y \in X^{j-1}$ and $0 \leq i \leq j$, where $\sigma_i c = \gamma(c, 1, \ldots, 1, *, \ldots, 1)$ and $s_i y = (y_1, \ldots, y_i, *, y_{i+1}, \ldots, y_{j-1})$.

The ribbon operad structure of $\mathcal{P}$ induces a monad structure on P.

A morphism of ribbon operads induces a map of the associated monads.

Example 1.5.10. The monad associated to $f\tilde{\mathcal{D}}_2$ as a ribbon operad is the same as the monad associated two $f\mathcal{D}_2$ as a symmetric operad:

$$f\tilde{D}_2 X = \left(\coprod_{n\geq 0}(\tilde{\mathcal{D}}_2(n) \times \mathbb{R}^n) \times_{R\beta_n} X^n\right)\Big/ \approx$$
$$= (\coprod_{n\geq 0}(\tilde{\mathcal{D}}_2(n) \times (S^1)^n) \times_{\beta_n} X^n)\Big/ \approx$$
$$= (\coprod_{n\geq 0}(\mathcal{D}_2(n) \times (S^1)^n) \times_{\Sigma_n} X^n)\Big/ \approx \qquad = fD_2 X$$

A similar equality holds for the ribbon operad $ER\beta = \{BC^{R\beta_n}_{\{e\}}\}$ and the operad $B\mathcal{C}^{R\beta}$:

$$ER\beta X = \left(\coprod_{n\geq 0} ER\beta(n) \times_{R\beta_n} X^n\right)\Big/ \approx$$
$$= \left(\coprod_{n\geq 0} B\mathcal{C}^{R\beta_n}_{PR\beta_n} \times_{\Sigma_n} X^n\right)\Big/ \approx \qquad = B\mathcal{C}^{R\beta} X.$$

R_∞ operads have essentially the same properties as E_∞ operads, having contractible spaces carrying a free group action. The next result is an extension of a result of J.P. May in the E_∞ case (proposition 3.4 p22 in [27]). The proof is essentially identical to May's proof .

Proposition 1.5.11. *Let $\psi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of R_∞-operads. Then the induced map $\psi : PX \longrightarrow QX$ is a weak homotopy equivalence for any connected pointed space X, with $(X, *)$ an NDR-pair.*

Now, remark that the product of any two R_∞ operads $\mathcal{P}$ and $\mathcal{Q}$ is again an R_∞ operad. And the projection maps $\mathcal{P} \leftarrow \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{Q}$ are morphisms of R_∞ operads.

Proposition 1.5.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be R_∞ operads. For any $\mathcal{P}$-algebra X, there exists a $\mathcal{Q}$-algebra X' with X' weakly homotopy equivalent to X.*

Proof. Define X' to be the double bar construction $B(Q, P \times Q, X)$ (see section 1.1.3.1). We have the following equivalences:

$$X \xleftarrow{\simeq} B(P \times Q, P \times Q, X) \xrightarrow{\simeq} B(Q, P \times Q, X) = X'.$$

□

By example 1.5.10, we have the following corollary:

Corollary 1.5.13. *If X is an $f\mathcal{D}_2$-algebra (respectively a $BC^{R\beta}$-algebra), there exists a space X' weakly equivalent to X such that X' is a $BC^{R\beta}$-algebra (respectively an $f\mathcal{D}_2$-algebra).*

1.5.4 fE_2-spaces

We want to characterise operads equivalent to the framed 2-discs. We consider the following notion of equivalence:

Definition 1.5.14. *An operad map $\mathcal{A} \rightarrow \mathfrak{B}$ is an* equivalence *if each map $\mathcal{A}(k) \rightarrow \mathfrak{B}(k)$ is a Σ_k-equivariant homotopy equivalence.*

An operad $\mathcal{A}$ is a E_n-operad (respectively fE_n-operad) if there is a chain of equivalences connecting $\mathcal{A}$ to $\mathcal{D}_n$ (respectively $f\mathcal{D}_n$).

Fiedorowicz gave the following characterisation of the little discs operad:

Theorem 1.5.15. *[8] An operad $\mathcal{A}$ is an E_2 operad if and only if its operad structure lifts to a B_∞ operad structure on its universal cover $\tilde{\mathcal{A}}$.*

We will prove here a similar statement for fE_2 operads:

Theorem 1.5.16. *An operad $\mathcal{A}$ is an fE_2 operad if and only if its operad structure lifts to an R_∞ operad structure on its universal cover $\tilde{\mathcal{A}}$.*

We have encountered two examples of operads having an R_∞ structure on their universal cover: $f\mathcal{D}_2$, which is clearly an fE_2 operad, and the ribbon braid operad $BC^{R\beta}$.

Lemma 1.5.17. *If $\mathcal{P}$ is an R_∞ operad, the sequence of quotient spaces $\{\mathcal{P}(n)/PR\beta_n\}$ forms a symmetric operad equivalent to the framed discs.*

Proof. As the operad maps γ are $R\beta$-equivariant, they induce operad maps on the quotient spaces having the same associative and unital properties. The $R\beta_n$-action on $\mathcal{P}(n)$ induces an $R\beta_n/PR\beta_n$ i.e. Σ_n-action on $\mathcal{P}(n)/PR\beta_n$ and the maps γ induced on the quotient spaces are Σ-equivariant.

The equivalences of R_∞ operads are $R\beta$-equivariant and so induce equivalences on the quotients:

$$\begin{tikzcd}
\mathcal{P} \arrow[d] & \mathcal{P} \times f\tilde{\mathcal{D}}_2 \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & f\tilde{\mathcal{D}}_2 \arrow[d] \\
\mathcal{P}/PR\beta & \mathcal{P} \times_{PR\beta} f\tilde{\mathcal{D}}_2 \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & f\mathcal{D}_2
\end{tikzcd}$$

□

Proof of the theorem From the lemma, we know that if the operad structure of $\mathcal{P}$ lifts to an R_∞ structure on its universal cover, $\mathcal{P}$ is an fE_2 operad. We are left to show that if $\mathcal{P}$ is an fE_2 operad, one can lift its operad structure to its universal cover $\tilde{\mathcal{P}}$. This will then be an R_∞ structure as $\mathcal{P}$ is Σ-equivariantly equivalent to $f\mathcal{D}_2$.

To lift the operad structure, we need a consistent choice of base-points in $\{\mathcal{P}(k)\}$. As $\mathcal{P}$ is equivalent to $f\mathcal{D}_2$, we know that there exists an equivalence of operads

$$Wf\mathcal{D}_2 \xrightarrow{\simeq} \mathcal{P},$$

where $Wf\mathcal{D}_2$ is a cofibrant resolution of $f\mathcal{D}_2$ constructed by Boardman and Vogt [2, 41]. As the W-construction is functorial, using the map given in figure 1.9 we have the following maps of operads

$$W\mathcal{D}_1^0 \longrightarrow Wf\mathcal{D}_2 \longrightarrow \mathcal{P},$$

where $W\mathcal{D}_1^0(k)$ is a contractible space for each k. Now we can proceed as in theorem 1.5.8 with $W\mathcal{D}_1^0$ playing for $\mathcal{P}$ the role of $\mathcal{D}_1^0$ for $f\mathcal{D}_2$. □

Chapter 2

Equivariant recognition principle and Batalin-Vilkovisky algebras

2.1 Equivariant recognition principle

Let $\phi : G \longrightarrow O(n)$ be an orthogonal representation of a group G and let X be a grouplike $\mathcal{D}_n \rtimes G$-algebra, i.e. the components of X form a group by the product induced by any element of $\mathcal{D}_n(2)$. By proposition 1.3.5, as X is a $\mathcal{D}_n \rtimes G$-algebra, it is in particular a $\mathcal{D}_n$-algebra.

Note that we dropped ϕ from the notation of the semidirect product.

May introduced a deloop functor B_n from $\mathcal{D}_n$-algebras to pointed spaces defined by $B_n X := B(\Sigma^n, D_n, X)$, where B is the double bar construction (see section 1.1.3.1), and Σ the (reduced) suspension. May's recognition principle (theorem 1.1.6) says that X is equivalent to $\Omega^n B_n X$. The equivalence is a weak homotopy equivalence and a map of $\mathcal{D}_n$-algebras. May also showed that, conversely, B_n applied to an n-fold loop space $\Omega^n Y$ provides an equivalent delooping ($B_n \Omega^n Y \simeq Y$).

In what follows, we consider the behaviour of Ω^n and B_n with respect to G-actions and provide a recognition principle for algebras over $\mathcal{D}_n \rtimes G$.

Let $\mathcal{D}_n \rtimes G-\mathsf{Top}_{gl}$, $\mathcal{D}_n \rtimes G-\mathsf{Top}_0$ and $G-\mathsf{Top}^*_n$ be the categories of grouplike, connected $\mathcal{D}_n \rtimes G$-algebras and n-connected pointed G-spaces respectively. Those three categories are closed model categories with weak homotopy equivalences as weak equivalences [31]. For a model category $\mathcal{C}$, we will denote by $\mathrm{Ho}(\mathcal{C})$ its associated homotopy category, obtained by inverting the weak equivalences.

For any G-space Y, we have seen in example 1.3.11 that $\Omega^n Y$ has a $\mathcal{D}_n \rtimes G$-algebra structure induced by the diagonal action of G. On the other hand, we will define a

G-action on $B_n X$ for any $\mathcal{D}_n \rtimes G$-algebra X. Hence, Ω^n and B_n will be functors between the categories of pointed G-spaces and of $\mathcal{D}_n \rtimes G$-algebras.

Theorem 2.1.1. *For each continuous homomorphism $\phi : G \longrightarrow O(n)$, we have functors*

$$\Omega^n_\phi = \Omega^n : G-\mathsf{Top}^*_{n-1} \longrightarrow \mathcal{D}_n \rtimes G-\mathsf{Top}_{gl}$$

$$B^\phi_n = B_n : \mathcal{D}_n \rtimes G-\mathsf{Top}_{gl} \longrightarrow G-\mathsf{Top}^*_{n-1}$$

which induce an equivalence of homotopy categories

$$Ho(G-\mathsf{Top}^*_{n-1}) \simeq Ho(\mathcal{D}_n \rtimes G-\mathsf{Top}_{gl}).$$

This equivalence restricts to

$$Ho(G-\mathsf{Top}^*_{n}) \simeq Ho(\mathcal{D}_n \rtimes G-\mathsf{Top}_{0}).$$

Proof. May's recognition principle is obtained through the following maps:

$$X \longleftarrow B(D_n, D_n, X) \xrightarrow{\alpha} B(\Omega^n\Sigma^n, D_n, X) \longrightarrow \Omega^n B(\Sigma^n, D_n, X) = \Omega^n B_n X,$$

where all maps are $\mathcal{D}_n$-maps between $\mathcal{D}_n$-spaces. When X is a $\mathcal{D}_n \rtimes G$-algebra, we want to define G-actions on the spaces involved which induce $\mathcal{D}_n \rtimes G$-algebra structures and such that all maps are G-maps.

The functors D_n, Σ^n and Ω^n restrict to functors in the category of G-spaces, where, for any G-space Y, we define the action on D_nY, Σ^nY and Ω^nY diagonally as in example 1.3.11. Hence for any G-space Y the G-action on $\Omega^n\Sigma^nY$ is given by

$$g[\sigma(t), y(t)] = [\phi(g)\sigma(\phi(g)^{-1}t), gy(\phi(g)^{-1}t)],$$

where $g \in G,\ t, \sigma(t) \in D^n$ and $y(t) \in Y$. One gets a $\mathcal{D}_n \rtimes G$-algebra structure on $\Omega^n\Sigma^nY$ such that May's map $\alpha : D_nY \longrightarrow \Omega^n\Sigma^nY$ is a G-map, and thus a $\mathcal{D}_n \rtimes G$-map, where the $\mathcal{D}_n \rtimes G$-structure on D_nY is given in example 1.3.11.

We extend these actions on the simplicial spaces $B(D_n, D_n, X)$, $B(\Omega^n\Sigma^n, D_n, X)$ and $\Omega^n B(\Sigma^n, D_n, X)$ as follows.

Recall from 1.1.3.1 that the double bar construction $B(F, C, X)$ is defined simplicially, for a monad C, a left C-functor F and a C-algebra X by $B(F, C, X) = |B_*(F, C, X)|$, where $B_p(F, C, X) = FC^pX$, with boundary and degeneracy maps using the left functor, monad and algebra structure maps. The group G acts then

on $B_p(F, C, X)$ through its action on the functors F and C, which comes to "rotate everything". For example, the action of $g \in G$ on a 1-simplex of $B(\Omega^n\Sigma^n, D_n, X)$ is given by $g[\sigma(t), c(t), x_1(t), \dots, x_k(t)]$
$= [\phi(g)\sigma(\phi(g)^{-1}t), \phi(g)c(\phi(g)^{-1}t), gx_1(\phi(g)^{-1}t), \dots, gx_k(\phi(g)^{-1}t)]$.

With these actions, all maps above are G-maps between $\mathcal{D}_n \rtimes G$-spaces and B_nX is equipped with an explicit G-action.

On the other hand, we have a weak homotopy equivalence [27, 5]

$$B_n\Omega^nY = B(\Sigma^n, D_n, \Omega^nY) \xrightarrow{|\delta_0^*|} \Sigma^n\Omega^nY \xrightarrow{e} Y$$

for any $(n-1)$-connected space Y. If Y is a G-space, then this composite is a G-map with the actions on $B_n\Omega^nY$ and $\Sigma^n\Omega^nY$ defined as above. □

Consider the monoid $\mathcal{R}\beta \times_{\mathbb{Z}} E\mathbb{Z} \subset \tilde{\mathcal{F}}_2(1) \times_{PR\beta_1} |\tilde{R}|(1)$. There are monoid maps

$$S^1 \cong (\mathcal{R}\beta \times_{\mathbb{Z}} *) \xleftarrow{\simeq} \mathcal{R}\beta \times_{\mathbb{Z}} E\mathbb{Z} \xrightarrow{\simeq} (* \times_{\mathbb{Z}} E\mathbb{Z}) \cong B\mathbb{Z}.$$

So any S^1-space or $B\mathbb{Z}$-space is canonically an $\mathcal{R}\beta \times_{\mathbb{Z}} E\mathbb{Z}$-space. The above maps are restrictions of the operad maps $f\mathcal{D}_2 \xleftarrow{\simeq} \tilde{\mathcal{F}}_2 \times_{PR\beta} |\tilde{R}| \xrightarrow{\simeq} |R|$ in arity 1. Using our recognition principle, theorem 1.4.7 and theorem 1.5.13 we obtain the following:

Theorem 2.1.2. *The nerve of a ribbon braided monoidal category $\mathcal{C}$, after group completion, is weakly homotopy equivalent to a double loop space Ω^2Y on an S^1-space Y. The S^1-action on Y is induced by the twist on $\mathcal{C}$ and the equivalence given by $\mathcal{R}\beta \times_{\mathbb{Z}} E\mathbb{Z}$-equivariant maps.*

Proof. Let $\mathcal{C}$ be a ribbon braided monoidal category and let $\mathcal{C}'$ be the strictification of $\mathcal{C}$ as a monoidal category. The category $\mathcal{C}'$ then inherits a ribbon braided structure from the one existing on $\mathcal{C}$. Its nerve $|\mathcal{C}'|$ is an $|R|$-algebra. The space $|\mathcal{C}|$ is not necessarily an $|R|$-algebra, but it admits a $B\mathbb{Z}$-action induced by the twist on $\mathcal{C}$, and the equivalence $|\mathcal{C}| \xrightarrow{\simeq} |\mathcal{C}'|$ is $B\mathbb{Z}$-equivariant.

Now the space $X = B(fD_2, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|)$ is weakly homotopy equivalent to $|\mathcal{C}'|$ and is an $f\mathcal{D}_2$-algebra. The equivalence is obtained through the following diagram of weak equivalences in $\mathcal{R}\beta \times_{\mathbb{Z}} E\mathbb{Z}\mathsf{Top}$.

$$\begin{tikzcd}
B(fD_2, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|) & B(f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|) \arrow[l] \arrow[d] \\
 & B(|R|, |R|, |\mathcal{C}'|) \arrow[r] & |\mathcal{C}'|,
\end{tikzcd}$$

The group completion of X is then equivalent to a double loop space Ω^2Y, where $Y = B(\Sigma^2, D_2, X)$ and the S^1-action on X now induces one on Y, as explained in theorem 2.1.1. □

2.2 Algebraic semidirect products and Batalin--Vilkovisky algebras

We work for this section in dgVect, the category of chain complexes over a field k (possibly with trivial differential). For an element x of a chain complex, we denote by $|x|$ its degree. We call operads in this category differential graded operads, or *dg-operads*. We will consider only dg-operads P with $P(0) = 0$, which comes to working without units for the algebras.

2.2.1 Semidirect products of algebraic operads

Let H be a graded associative cocommutative Hopf algebra over k. The category of differential graded H-modules, denoted H-Mod, is a symmetric monoidal category with product the ordinary tensor product. We call operads in this category *operads of H-modules.*

For P an operad of H-modules, we have seen in section 1.3 that one can construct a dg-operad $P \rtimes H$ with $(P \rtimes H)(n) = P(n) \otimes H^{\otimes n}$.

Recall from proposition 1.3.12 that if $\mathcal{A}$ is an operad in $G-$Top there is a natural isomorphism of homology operads $H(\mathcal{A} \rtimes G) \cong H(\mathcal{A}) \rtimes H(G)$.

Note that, in our convention, we consider $H((\mathcal{A} \rtimes G)(k))$ only for $k > 0$, setting $H(\mathcal{A} \rtimes G)(0)$ to be 0.

2.2.1.1 Quadratic semidirect products

Suppose now that P is a quadratic dg-operad, namely has binary generators and 3-ary relations [11]. We will restrict ourselves to the case where $P(1) = k$, concentrated in dimension 0. Explicitly $P = F(V)/(R)$, where $F(V)$ is the free operad generated by a $k[\Sigma_2]$-module of binary operations V and (R) is the ideal generated by a $k[\Sigma_3]$-submodule $R \subset F(V)(3)$.

Proposition 2.2.1. *Let H be a cocommutative Hopf algebra and $P = F(V)/(R)$ a quadratic operad. Then P is an operad of H-modules if and only if*

(i) V is a $(H, k[\Sigma_2])$-bimodule;

(ii) $R \subseteq F(V)(3)$ is a $(H, k[\Sigma_3])$-sub-bimodule.

In this case, we will call P a quadratic operad of H-modules.

Proof. An element of the free operad on V is described as a sum of trees with vertices labelled by V [22]. Define the action of H on such a tree by acting on the labels of the vertices, using the comultiplication of H. This is well defined as H is cocommutative. It induces an H-module structure on $F(V)$, which induces one on $P(n)$ for all n by condition (ii). The operad structure maps are then H-equivariant by construction. $\square$

Let $c : H \to H \otimes H$ be the comultiplication. For $g \in H$ we write informally $(c \otimes id)(c(g)) = \sum_i g'_i \otimes g''_i \otimes g'''_i$.

Proposition 2.2.2. *Let $P = F(V)/(R)$ be a quadratic operad of H-modules as above. A chain complex X is an algebra over $P \rtimes H$ if and only if*

(i) X is an H-module

(ii) X is a P-algebra

(iii) for each $g \in H$, $v \in V$ and $x, y \in X$,

$$g(v(x,y)) = \sum_i (-1)^{|g''_i||v|+|g'''_i|(|v|+|x|)} g'_i(v)(g''_i(x), g'''_i(y)).$$

Proof. The H-equivariance of the algebra map $\theta_2 : P(2) \otimes X \otimes X \longrightarrow X$ is given by the commutativity of the following diagram:

$$\begin{tikzcd}
H \otimes P(2) \otimes X \otimes X \arrow[r, "\mathrm{id}\otimes\theta_2"] \arrow[d, "\mathrm{shuffle}\circ(c\otimes\mathrm{id})\circ c"] & H \otimes X \arrow[dd, "\phi"] \\
H \otimes P(2) \otimes H \otimes X \otimes H \otimes X \arrow[d, "\psi\otimes\phi\otimes\phi"] & \\
P(2) \otimes X \otimes X \arrow[r, "\theta_2"] & X,
\end{tikzcd}$$

where ϕ and ψ give the action of H on X and $P(2)$ respectively. This diagram translates, for the generators of $P(2)$, into condition (iii) of the proposition. The H-equivariance of the structure maps θ_k for $k > 2$ is a consequence of the fact that V generates $P(k)$, that the operadic composition is H-equivariance and that the map θ is associative. $\square$

2.2.2 Batalin-Vilkovisky algebras

From now on we work over a field k of characteristic 0. As first application we give a conceptual proof of a theorem of Getzler [9]. Recall that a Batalin-Vilkovisky algebra X is a graded commutative algebra with a linear endomorphism $\Delta : X \to X$ of degree 1 such that $\Delta^2 = 0$ and for each $x, y, z \in X$ the following BV-axiom holds.

$$\begin{aligned}\Delta(xyz) = \Delta(xy)z + (-1)^{|x|}x\Delta(yz) + (-1)^{(|x|+1)|y|}y\Delta(xz) - \Delta(x)yz \\ -(-1)^{|x|}x\Delta(y)z - (-1)^{|x|+|y|}xy\Delta(z) = 0.\end{aligned} \tag{2.1}$$

Theorem 2.2.3. *[9] Let $H(fD_2) = H(D_2) \rtimes H(SO(2))$ be the homology of the framed little 2-discs operad. An $H(fD_2)$-algebra is exactly a Batalin-Vilkovisky algebra.*

Proof. Let X be an algebra over $H(fD_2)$. By proposition 2.2.2 (condition (i)), X is an $H(SO(2))$-module. As an algebra, $H(SO(2)) = k[\Delta]/\Delta^2$, where $\Delta \in H_1(SO(2))$ is the fundamental class. This provides X with an operator Δ of degree 1 satisfying $\Delta^2 = 0$. Condition (ii) of proposition 2.2.2 tells us that X is an algebra over $H(D_2)$. The operad $H(D_2)$, called the Gerstenhaber operad, was identified by F. Cohen [4]. This operad is quadratic, generated by the operations $* \in H_0(D_2(2))$ and $b \in H_1(D_2(2))$, corresponding to the class of a point and the fundamental class under the $SO(2)$-equivariant homotopy equivalence $D_2(2) \simeq S^1$. The class $*$ induces a graded commutative product on X, while b induces a Lie bracket of degree 1, i.e. a Lie algebra structure on on ΣX, the suspension of X, defined by $(\Sigma X)_i = X_{i-1}$, with bracket $[x, y] = (-1)^{|x|}b(x, y)$. Cohen proved that the product and the bracket satisfy the following Poisson relation:

$$[x, y * z] = [x, y] * z + (-1)^{|y|(|x|+1)}y * [x, z]\,. \tag{2.2}$$

In order to unravel condition (iii) of proposition 2.2.2, we must understand the effect in homology of the $SO(2)$-action on $D_2(2) \simeq S^1$. Clearly $\Delta(*) = b$ because the rotation of the generator in degree 0 gives precisely the fundamental 1-cycle. Moreover $\Delta(b) = 0$ by dimension. As Δ is primitive, condition (iii) (or the diagram in the proof) applied respectively to $(\Delta, *, x, y)$ and (Δ, b, x, y) provides the following relations:

$$\Delta(x * y) = \Delta(*)(x, y) + \Delta(x) * y + (-1)^{|x|}x * \Delta(y)\,; \tag{2.3}$$

$$\Delta(b(x, y)) = \Delta(b)(x, y) - b(\Delta(x), y) + (-1)^{|x|+1}b(x, \Delta(y))\,. \tag{2.4}$$

As $\Delta(*) = b$, equation 2.3 expresses the bracket in terms of the product and Δ :

$$[x, y] = (-1)^{|x|}\Delta(x * y) - (-1)^{|x|}\Delta(x) * y - x * \Delta(y)\,. \tag{2.5}$$

If we substitute this expression into the Poisson relation 2.2 we get exactly the BV-axiom 2.1. We can re-wright equation 2.4 as

$$\Delta[x, y] = [\Delta(x), y] + (-1)^{|x|+1}[x, \Delta(y)] \tag{2.6}$$

which says that Δ is a derivation with respect to the bracket. To conclude we must show that 2.4 and the Lie algebra axioms follow from the BV-axiom. This is shown in Proposition 1.2 of [9]. □

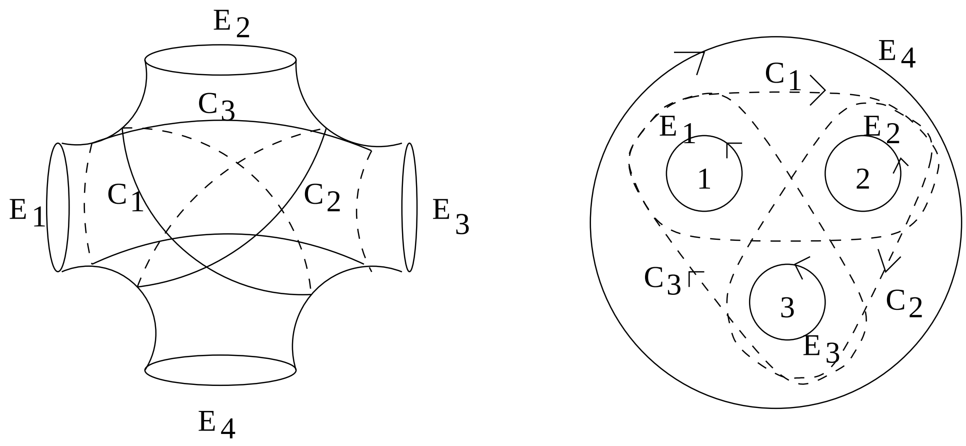


Figure 2.1: Lantern relation

Remark 2.2.4. The lantern relation, introduced by Johnson for its relevance to the mapping class group of surfaces [14] is defined by the following equation: $T_{E_4} = T_{E_1}T_{E_2}T_{E_3}T_{C_1}T_{C_2}T_{C_3}$, where T_C denotes the Dehn twist along the curve C. See figure 2.1 for the relevant curves on a sphere with four holes, or equivalently on a disc with three holes. The mapping class group of a sphere with 4 ordered holes, is the group of path components of orientation preserving diffeomorphisms which fix pointwise the boundary. The group is isomorphic to the pure ribbon braid group $PR\beta_3$. The lantern relation is thus a relation in $PR\beta_3$ and gives rise to a relation in $H_1(f\mathcal{D}_2(3))$ which is the abelianisation of $PR\beta_3$. It was noted by Tillmann that, with this interpretation, one gets precisely the BV-axiom 2.1. Indeed, up to signs, the curve E_1 represents the operation $(x, y, z) \mapsto \Delta x * y * z$. Moreover E_2 corresponds to $x * \Delta y * z$, E_3 to $x*y*\Delta z$, E_4 to $\Delta(x*y*z)$, C_1 to $\Delta(x*y)*z$, C_2 to $x*\Delta(y*z)$ and C_3 to $y*\Delta(x*z)$.

This geometric interpretation shows that any $H(f\mathcal{D}_2)$-algebra is a BV-algebra.

We will use alternatively the notations e_n and $H(D_n)$ for the homology of the little n-discs operad, by which we mean

$$\begin{cases} e_n(k) := H_*(\mathcal{D}_n(k)) & k \geq 1 \\ e_n(0) = 0. \end{cases}$$

Algebras over the operad e_n, $n \geq 2$, are called *n-algebras*. By assumption they have no units. F. Cohen's study of $H(\mathcal{D}_n)$ in [5] implies that an n-algebra X is a differential graded commutative algebra with a Lie bracket of degree $n-1$, i.e.

$(L1)$ $[x, y] + (-1)^{(|x|+n-1)(|y|+n-1)}[y, x] = 0,$

$(L2)$ $\partial[x, y] = [\partial x, y] + (-1)^{|x|+n-1}[x, \partial y],$

$(L3)$ $[x, [y, z]] = [[x, y], z] + (-1)^{(|x|+n-1)(|y|+n-1)}[y, [x, z]],$

satisfying the Poisson relation

$(P1)$ $[x, y * z] = [x, y] * z + (-1)^{|y|(|x|+n-1)} y * [x, z]\,.$

Gerstenhaber algebras correspond to the case $n = 2$.

Note that $\mathcal{D}_n(2)$ is $SO(n)$-equivariantly homotopic to S^{n-1}. In an n-algebra, the product comes from the generating class $* \in H_0(\mathcal{D}_n(2)) \cong k$ and the bracket from the fundamental class $b \in H_{n-1}(\mathcal{D}_n(2)) \cong k$, if we define $[x, y] = (-1)^{(n-1)|x|}b(x, y)$. The operad e_n is quadratic [10].

In order to determine the homology operad $H(f\mathcal{D}_n)$, we need to know the Hopf algebra structure of $H(SO(n))$ and the effect in homology of the action of $SO(n)$ on $\mathcal{D}_n(2)$. For dimensional reasons, one always has $\delta(b) = 0$ for each $\delta \in \tilde{H}(SO(n))$. On the other hand $\delta(*) = \pi_*(\delta)$ is the action in homology of the evaluation map $\pi : SO(n) \to S^{n-1}$, by $\mathcal{D}_n(2) \simeq S^{n-1}$.

Let us give two further examples:

Example 2.2.5. *(i) An $H(f\mathcal{D}_3)$-algebra is a 3-algebra together with an endomorphism δ of degree 3 such that $\delta^2 = 0$ and δ is a derivation both with respect to the product and the bracket.*

(ii) An $H(f\mathcal{D}_4)$-algebra is a commutative dg-algebra together with two linear endomorphisms α, β of degree 3, such that $\alpha^2 = 0$, $\beta^2 = 0$, $\alpha\beta = -\beta\alpha$, α and the product satisfy the BV-axiom 2.1, and β is a derivation with respect to the product.

Proof. (i) $H(SO(3)) = \wedge(\delta)$ is the free exterior algebra generated by the fundamental class $\delta \in H_3(SO(3))$, and $\pi_*(\delta) = 0$ by dimension. By proposition 2.2.2, X is an $H(f\mathcal{D}_3)$-algebra if and only if X is a 3-algebra admitting an $H(SO(3))$-module structure, i.e. an operator δ of degree 3 with $\delta^2 = 0$, such that the following relations hold:

$$\delta(x * y) = \delta x * y + (-1)^{|x|} x * \delta y \tag{2.7}$$

$$\delta[x, y] = [\delta x, y] + (-1)^{|x|}[x, \delta y]\,, \tag{2.8}$$

where those equations are obtained by plugging $(\delta, *, x, y)$ and (δ, b, x, y) in turn in the diagram of proposition 2.2.2. In this case the operator b is equal to the bracket and lies in degree 2 and $\delta(b) = 0$.

(ii) The evaluation fibration $SO(3) \to SO(4) \to S^3$ splits as a product. So $H(SO(4)) = \wedge(\alpha, \beta)$, with both generators in degree 3. The class α comes from the basis, so $\pi_*(\alpha) = b$, whereas β comes from the fibre, so $\pi_*(\beta) = 0$.

As in the previous case, we know that an $H(f\mathcal{D}_4)$-algebra X is a 4-algebra with two operators α and β both in degree 3, satisfying $\alpha^2 = 0 = \beta^2$ and $\alpha\beta = -\beta\alpha$ and relations obtained by plugging $(\alpha, *, x, y)$, (α, b, x, y), $(\beta, *, x, y)$ and (β, b, x, y) in turn in the diagram. Using the identification $b(x, y) = (-1)^{|x|}[x, y]$, this gives the following equations:

$$\alpha(x * y) = (-1)^{|x|}[x, y] + \alpha x * y + (-1)^{|x|} x * \alpha y \tag{2.9}$$

$$\alpha[x, y] = [\alpha x, y] + (-1)^{|x|+1}[x, \alpha y] \tag{2.10}$$

$$\beta(x * y) = \beta x * y + (-1)^{|x|} x * \beta y \tag{2.11}$$

$$\beta[x, y] = [\beta x, y] + (-1)^{|x|+1}[x, \beta y]\,. \tag{2.12}$$

Note that equations 2.9 and 2.10 correspond precisely to the equations we had for Δ and the bracket in theorem 2.2.3. So, by the same calculations, we know that α and the product form a Batalin-Vilkovisky algebra of higher degree, i.e. α and $*$ satisfy equation 2.1 but the operator α is now in degree 3. There is an additional operator β of degree 3. Equation 2.11 says that β is a derivation with respect to the product. Using equation 2.9, one can rewrite equation 2.12 in terms of α, β and the product. This shows that equation 2.12 is a consequence of equation 2.11. ∎

2.2.3 General case

We need a lemma in order to state the general case.

Lemma 2.2.6. *For $n \geq 1$, over a field of characteristic 0, the Hopf algebra $H(SO(2n)) = \bigwedge(\beta_1, \ldots, \beta_{n-1}, \alpha_{2n-1})$ is the free exterior algebra on primitive generators $\beta_i \in H_{4i-1}(SO(2n))$ and $\alpha_{2n-1} \in H_{2n-1}(SO(2n))$. Moreover, $\pi_*(\beta_i) = 0$ for all i and $\pi_*(\alpha_{2n-1}) = b \in H_{2n-1}(S^{2n-1})$ is the fundamental class.*

The Hopf algebra $H(SO(2n+1)) = \bigwedge(\beta_1, \ldots, \beta_n)$ is the free exterior algebra on primitive generators $\beta_i \in H_{4i-1}(SO(2n+1))$, and $\pi_*(\beta_i) = 0$ for all i.

Proof. The homology Serre spectral sequence of the principal fibration $SO(n) \to SO(n+1) \to S^n$ collapses at the E_2 term if n is odd; if n is even then there is a non-trivial differential $d(b) = \alpha_{n-1}$ [24]. □

If a Hopf algebra H acts trivially, via the counit, on an operad P, we call the semidirect product just the *direct product* and denote it by $P \times H$. Note that a $P \times H$-algebra is an H-module X with a P-algebra structure satisfying an H-equivariance condition which is trivial only if H acts trivially on X. In particular, any P-algebra is a $P \times H$-algebra with the trivial H-module structure.

Let us denote by BV_n, for n even, the Batalin-Vilkovisky operad with the operator Δ in degree $n-1$. Hence a BV_n-algebra is a differential graded commutative algebra with an operator Δ of degree $n-1$ such that $\Delta^2 = 0$ and the BV-equation (2.1) holds.

Note that there is no non-trivial Σ_2-equivariant map from $H_0(\mathcal{D}_{2n+1}(2))$ to $H_{2n}(\mathcal{D}_{2n+1}(2))$. So $(2n+1)$-algebras do not give rise to BV-structures like in the even case.

Theorem 2.2.7. *For $n \geq 1$ there are isomorphisms of operads*

$$H(f\mathcal{D}_{2n+1}) \cong H(\mathcal{D}_{2n+1}) \times H(SO(2n+1))$$

and

$$H(f\mathcal{D}_{2n}) \cong BV_{2n} \times H(SO(2n-1)).$$

Hence an $H(f\mathcal{D}_{2n+1})$-algebra is a $(2n+1)$-algebra together with endomorphisms β_i of degree $4i-1$ for $i = 1, \ldots, n$ such that $\beta_i^2 = 0$, $\beta_i\beta_j = -\beta_j\beta_i$ for each i, j, and each β_i is a $(2n+1)$-algebra derivation, i.e. a derivation both with respect to the product and the bracket.

On the other hand, an $H(f\mathcal{D}_{2n})$-algebra is a BV_{2n}-algebra together with endomorphisms β_i of degree $4i-1$ for $i=1,\dots,n-1$ squaring to 0 and anti-commuting as in the odd case, which moreover anti-commute with the BV operator Δ and are derivations with respect to the product.

Proof. We have seen that the theorem is true for $H(f\mathcal{D}_n)$, with $n = 2,3,4$. The general case differs from example 2.2.5 (i) and (ii), for odd and even case respectively, only by the presence of more operators, all of the same type in the odd case, and of the type β in the even case. The degree of the operators varies, but they are all of odd degree and so the equations they satisfy do not change.

More precisely, in the odd case, by lemma 2.2.6, we know that the $H(SO(2n+1))$-module structure gives operators β_i, $i=1,\dots,n$, squaring to 0 and anti-commuting. Having the same properties as δ in example 2.2.5 (i), they all satisfy equation 2.7 and 2.8, replacing δ by β_i.

The even case is similarly deduced from example 2.2.5 (ii). □

We already saw that iterated loop spaces are algebras over the framed discs operad. We deduce the following example:

Example 2.2.8. *The homology of an n-fold loop space $H(\Omega^n(X))$ is an algebra over $H(f\mathcal{D}_n)$.*

Another interesting class of algebras over $H(fD_n)$ is given by the space $\Lambda^n(X)$ of unbased maps from S^n to a space X. Chas and Sullivan showed that the homology of a free loop space ΛM on an oriented manifold M is a Batalin-Vilkovisky algebra [3]. Sullivan and Voronov generalised it to higher dimension and have a geometrical proof involving the so-called cacti operad.

Theorem 2.2.9. *[36] Let M be a d-dimensional oriented manifold. Then the d-fold desuspended homology $\Sigma^{-d}H(\Lambda^n M)$ of the unbased mapping space from the n-sphere into M is an algebra over $H(fD_{n+1})$.*

2.3 Annex 1: How $\mathcal{D}_2$-algebras are almost $f\mathcal{D}_2$-algebras

Or attempts to construct S^1-actions.

Consider the two following facts:
1) the group completion of any $\mathcal{D}_n$-algebra is an n-fold loop space, and n-fold loop spaces are $f\mathcal{D}_n$-algebra.
2) the free $\mathcal{D}_n$-algebra on any space X is an $f\mathcal{D}_n$-algebra.
Our first attempt to understand the situation, before looking at an equivariant recognition principle, was to try to use these facts to construct an $f\mathcal{D}_n$-structure on any $\mathcal{D}_n$-algebra X, or at least on a space equivalent to X, without group completing. For simplicity, we consider here only the case $n = 2$, thus trying to construct an S^1-action compatible with the existing $\mathcal{D}_2$-structure.

In 2.3.2, using the first fact, i.e May's recognition principle, we try to carry the S^1-action existing on the double loop space $\Omega^2 B(\Sigma^2, D_2, X)$ back to $B(D_2, D_2, X)$, which is a space equivalent to X. Extending the map $D_2X \to \Omega^2\Sigma^2$ to a map $fD_2X \to \Omega^2\Sigma^2$, we first show how the pull back of the action naturally lives on $B(fD_2, D_2, X)$ rather than on $B(D_2, D_2, X)$. After this pessimistic observation, we explain how the natural action one would construct on $B(D_2, D_2, X)$ leads to requiring the existence of an action on X precisely of the form we are trying to construct! In other words, we arrive at the modest conclusion that if X is an $f\mathcal{D}_2$-algebra, so is $B(D_2, D_2, X)$. We actually construct the $f\mathcal{D}_2$-structure used in the equivariant recognition principle (theorem 2.1.1).

In 2.3.3, we try to construct an S^1-action on X using the second fact mentioned above. The idea is to use the map $D_2X \to X$. We show that finding any section/splitting of this map is actually useless in the sense that the induced action on X will not be of the right form.

We begin the section by giving an explicit description of $f\mathcal{D}_2$ in terms of complex numbers. It is followed by a detailed description of the situation.

As our two attempts failed, the question of finding, for any $\mathcal{D}_n$-algebra, an equivalent $f\mathcal{D}_n$-algebra, stays open.

2.3.1 Detailed description

The framed little 2-discs operad $f\mathcal{D}_2$ can be described explicitly as follows: Considering D^2 as the unit disc in the complex plane, an element of $f\mathcal{D}_2(k)$ is determined by a pair of vectors $(\underline{s}, \underline{a}) \in \mathbb{C}^k \times \mathbb{C}^k$, where $(\underline{s}, \underline{a})$ represents the embedding which sends the ith disk D^2 to $s_i.D^2 + a_i$. So for each i, s_i encodes the dilation and rotation applied to the ith disc and a_i encodes the translation. Note that, of course, not any couple $(\underline{s}, \underline{a})$ defines an element of $f\mathcal{D}_2$. In these notations, the operad structure map is given by $\gamma((\underline{s}, \underline{a}), (\underline{t}^1, \underline{b}^1), \ldots, (\underline{t}^k, \underline{b}^k))$
$= ((s_1 t_1^1, s_1 b_1^1 + a_1), \ldots, (s_1 t_{n_1}^1, s_1 b_{n_1}^1 + a_1), \ldots, (s_k t_{n_k}^k, s_k b_{n_k}^k + a_k))$
Note that $\mathcal{D}_2$ is the sub-operad consisting of the $(\underline{s}, \underline{a}) \in f\mathcal{D}_2$ with $(\underline{s}, \underline{a}) \in \mathbb{R} \times \mathbb{C}$.

As $f\mathcal{D}_2 = \mathcal{D}_2 \rtimes S^1$, to construct an $f\mathcal{D}_2$-structure on a $\mathcal{D}_2$-algebra, we need (and it is sufficient) to give an action of the circle S^1 which is compatible with the existent $\mathcal{D}_2$-structure in the following sense : if X is a $\mathcal{D}_2$-algebra, with structure maps $\theta_{\mathcal{D}_2}$, for any $c \in \mathcal{D}_2(k)$, $x_i \in X$ and $s \in S^1$, the action must satisfy

$$s(\theta_{\mathcal{D}_2}(c, x_1, \ldots, x_k)) = \theta_{\mathcal{D}_2}(sc, sx_1, \ldots, sx_k),$$

where the action of S^1 on an element of $\mathcal{D}_2$ is by rotation of the discs around its centre. In the above notations, for $c = (\underline{r}, \underline{a}) \in \mathbb{R}^k \times \mathbb{C}^k$, $s.(\underline{r}, \underline{a}) = (\underline{r}, s.\underline{a})$.

We have such an action on $D_2 X$, the free $\mathcal{D}_2$-algebra on the pointed space X (see example 1.3.11). For $c \in \mathcal{D}_2(k)$, $s \in S^1$ and $x_i \in X$, the action is defined by

$$s(c, x_1, \ldots, x_k) = (sc, x_1, \ldots, x_k).$$

Also, any double loop space admits an $f\mathcal{D}_2$-structure. In this case, the S^1-action on $f : D^2 \longrightarrow X$ is given by

$$s[f(t)] \;\longmapsto\; [f(t.s^{-1})],$$

still considering D^2 and S^1 as a subsets of the complex numbers.

Using May's recognition principle, we deduce that the group completion of any $\mathcal{D}_2$-space X is weakly homotopy equivalent to an $f\mathcal{D}_2$-space.

2.3.2 May's circle action

May defined a map of monads $\alpha : D_2 \longrightarrow \Omega^2\Sigma^2$. This map can be extended to fD_2, i.e. there exists a monad map $\beta : fD_2 \longrightarrow \Omega^2\Sigma^2$ such that the following diagram commutes :

$$\begin{tikzcd} D_2 \arrow[d, "i"'] \arrow[dr, "\alpha"] & \\ fD_2 \arrow[r, "\beta"] & \Omega^2\Sigma^2, \end{tikzcd}$$

where i is induced by the natural inclusion of $\mathcal{D}_2$ into $f\mathcal{D}_2$. The maps α and β are given by

$$\alpha = \theta_{\mathcal{D}_2} \circ \eta_{\Omega^2\Sigma^2} \quad \text{and} \quad \beta = \theta_{f\mathcal{D}_2} \circ \eta_{\Omega^2\Sigma^2},$$

where $\theta_{\mathcal{D}_2}$ and $\theta_{f\mathcal{D}_2}$ are the $\mathcal{D}_2$- and $f\mathcal{D}_2$-algebra structure maps on $\Omega^2\Sigma^2 X$ (as a double loop space). Explicitly, $\beta : fD_2X \to \Omega^2\Sigma^2 X$ is given by

$$\beta((\underline{s}, \underline{a}), x_1, \ldots, x_k) = \left[t \in D^2 \mapsto \begin{cases} [s_i^{-1}(t - a_i), x_i] & \text{if } t \text{ is in the } i^{th} \text{ disk} \\ \star & \text{otherwise} \end{cases} \right]$$

for $(\underline{s}, \underline{a}) \in f\mathcal{D}_2(k)$ and $x_i \in X$.

The map α is given by the same formula with $\underline{s} \in \mathbb{R}^k$ (multiplication by s_i^{-1} being then a dilation without rotation).

Note that α and β are injective.

Let $C \longrightarrow D$ be a morphism of monads and let X be a C-algebra. A map of double bar constructions $B(D, C, X) \mapsto B(D', C', X')$ is given by a morphism of monads $\psi : C \longrightarrow C'$, a natural transformation $D \longrightarrow D'$ and a morphism of C-algebras $X \longrightarrow \psi^* X'$ such that the following diagram commute

$$\begin{tikzcd} DC \arrow[r] \arrow[d] & D'C' \arrow[d] \\ DD \arrow[r] & D'D'. \end{tikzcd}$$

In our case, this diagram will always commute because we work only with the inclusions $D_2 \hookrightarrow fD_2 \hookrightarrow \Omega^2\Sigma^2$ and their composition. Also, we will always be in the case $X = X'$ and use the pull back structure on X (from $X' = X$). In the case $X = \Omega^2 Y$, the D_2- and fD_2-algebra structures we defined are in fact the pull back structure of the $\Omega^2\Sigma^2$-structure on $\Omega^2 Y$.

Let X be a $\mathcal{D}_2$-algebra. Recall from section 2.1 that, for the recognition principle, May uses the following resolution of X:

$$B(D_2, D_2, X) \simeq X.$$

The map α defined above induces a map

$$B(D_2, D_2, X) \overset{B(\alpha,1,1)}{\longrightarrow} B(\Omega^2\Sigma^2, D_2, X) \overset{\simeq}{\longrightarrow} \Omega^2 B(\Sigma^2, D_2, X),$$

the first map being a (weak) homotopy equivalence whenever X is group like.

On the simplicial level, $B_*(\Omega^2\Sigma^2, D_2, X) = \Omega^2 B_*(\Sigma^2, D_2, X)$. The S^1-action on $\Omega^2 B(\Sigma^2, D_2, X)$, as a double loop space, is simplicial as the boundary maps are of the form $\delta_i(f) = \delta_i \circ f$, for $f \in \Omega^2\Sigma^2 D_2^q X$, while the circle action is of the form $f^s = f \circ r_s$, where r_s denotes the rotation by $s \in S^1$. So we also have an action on the space $B(\Omega^2\Sigma^2, D_2, X)$. We would like to pull back this action on $B(D_2, D_2, X)$. We explain how we fail miserably.

Note that the S^1-actions in our setting are part of a richer structure, an $f\mathcal{D}_2$-algebra structure. So S^1-maps should actually be $f\mathcal{D}_2$-maps.

We have the following commutative diagram:

$$\begin{tikzcd}
 & B(fD_2, D_2, X)^{\circlearrowleft S^1} \arrow[dr, "f\mathcal{D}_2-\mathrm{map}"] & \\
B(D_2, D_2, X) \arrow[ur] \arrow[rr] & & B(\Omega^2\Sigma^2, D_2, X)^{\circlearrowleft S^1}.
\end{tikzcd}$$

This diagram suggests that we won't be able to pull back the S^1-action from $B(\Omega^2\Sigma^2, D_2, X)$ to $B(D_2, D_2, X)$, as the natural pull back of the action is on $B(fD_2, D_2, X)$, not on $B(D_2, D_2, X)$. The action is encoded in the framed discs and the set of unframed discs, the discs with $\underline{s} \in \mathbb{R}^k$, is not stable under this action.

In particular, our attempts to construct a circle action on $B(D_2, D_2, X)$ respecting or not respecting May's map, all failed... The following phenomena happens: we want to construct a simplicial action on $B(D_2, D_2, X)$. We think of $B_q(D_2, D_2, X) = D_2^{q+1}X$ as having $q+2$ "levels", the first being the element of D_2 coming from the first D_2 (counting from the left), the second being composed of the elements coming from the second D_2, up to the last level which is formed of elements of X. If you want to rotate the first component of an element of $B(D_2, D_2, X)$, the boundary maps tell you that you have to rotated the next level as well, and the next, etc. Eventually, the last boundary maps tells you that you have to rotate the elements of X in a way which is compatible with its $\mathcal{D}_2$-structure, which is precisely the sort of rotation we are trying to construct! And we want to rotate the first component because it is largely suggested by what the action is on $B(\Omega^2\Sigma^2, D_2, X)$ or $B(fD_2, D_2, X)$.

The idea is that in the two last mentioned spaces, the rotation is “remembered” by the first factor (fD_2 or $\Omega^2\Sigma^2$), so that it does some sort of rotation to what follows when you do the first boundary map. If the rotation is not recorded, as it is the case with $\mathcal{D}_2$, then you have to rotate what follows “in advance”.

2.3.3 S^1-action form D_2X

We would want to think that it should be possible to use the S^1-action on D_2X to get one on X. More precisely, for $x = \theta(c, y_1, \dots, y_k)$ and $s \in S^1$, we would want to define x^s as being $\theta(c^s, y_1, \dots, y_k)$. This would become the problem of finding a section $X \longrightarrow D_2X$. We know a section $(x \mapsto (1, x))$, which induces the trivial action of S^1 on X, which is not compatible with the D_2-structure. This is unfortunately true in general. Whatever the section is –if ever we could find another one–, the action induced would not be compatible. Indeed, we need $x^s = \theta(c^s, y_1^s, \dots, y_k^s)$, not $x^s = \theta(c^s, y_1, \dots, y_k)$. Note also that a tricky sort of induction (deciding some elements as fixed points and then induce the rest) sounds rather hopeless...

2.4 Annex 2: Twisted monads and another recognition principle

In this section, we work in the category of pointed topological spaces. "Twisted monads" generalise the notion of semidirect product for topological operads. Recall that any unital operad has an associated monad which has the same algebras as the operad. Proposition 1.3.7 tells us that the monad associated to a semidirect product operad is of the form $P(G_+ \wedge X)$, where P is the monad associated to an operad and G is a topological monoid. We know that the algebras over this monad are G-spaces with a G-equivariant P-algebra structure. Now not all monads come from operads. We give in this section a larger class of monads having a property of this type for their algebras. We call these twisted monads, rather than semidirect products, for historical reasons.

The motivation for extending semidirect products to more general monads is that we wanted to define a monad "$\Omega^2\Sigma^2 \rtimes S^1$" which would correspond to $fD_2 = D_2 \rtimes S^1$. Clearly, as a functor, we want $(\Omega^2\Sigma^2 \rtimes S^1)X$ to be $\Omega^2\Sigma^2(S^1_+ \wedge X)$. We define its monad structure through a general theory of twisted monads. Studying this particular monad more closely, we obtain a characterisation of its algebras: on double loop spaces, we show that we can "deloop" the S^1-action induced by the $(\Omega^2\Sigma^2 \rtimes S^1)$-structure.

Having constructed this twisted version of $\Omega^2\Sigma^2$, we can now write a "framed recognition principle", which is a variant of the equivariant recognition principle. For the equivariant recognition principle, we constructed group actions on the spaces used in the original recognition principle. In this other version, we change the spaces so that they come directly equipped with a framed discs algebra structure. In the bar constructions, we replace D_2 by fD_2 and $\Omega^2\Sigma^2$ by $\Omega^2\Sigma^2 \rtimes S^1$. As in the equivariant principle, we obtain a double loop space with an explicit S^1-action on the deloop.

We study only the case $n = 2$ for simplicity.

2.4.1 Twisted monads

Let G be a monoid. Note that we can consider G as a monad by defining $G(X) = G_+ \wedge X$, μ_G to be the multiplication of G (multiplication by the added base point gives the base point) and η_G the inclusion $X \hookrightarrow G_+ \wedge X$, mapping x to (e, x). A structure of algebra over this monad $\lambda : G_+ \wedge X \to X$ is then precisely a base point preserving G-action on X.

Let (M, μ_M, η_M) be a monad. Suppose that we are given a continuous homomorphism

$$\rho : G \longrightarrow \mathrm{End}_{Mon}(M),$$

from G to the monad endomorphisms of M. So, for each $g \in G$, the map $\rho(g)$ is a natural transformation from M to M commuting with the structure map η_M and μ_M of M. In the case of μ_M, this means that

$$\begin{tikzcd} MMX \arrow[r, "\rho(g)\rho(g)"] \arrow[d, "\mu_M"'] & MMX \arrow[d, "\mu_M"] \\ MX \arrow[r, "\rho(g)"] & MX \end{tikzcd}$$

commutes.

For any space Y we want an assembly map

$$a : G_+ \wedge MY \longrightarrow M(G_+ \wedge Y).$$

Such a map always exists but might not be continuous: each $g \in G$ defines an inclusion $i_g : Y \hookrightarrow G_+ \wedge Y$. Fitting the $M(i_g)$'s together, this induces a map $G_+ \wedge Y \longrightarrow M(G_+ \wedge Y)$. We ask this map to be continuous. It is the case for example if M is a continuous functor and $M(*) = *$.

The map a is natural in Y in the sense that for $f : Y \longrightarrow Z$, the following diagram commutes:

$$\begin{tikzcd} G_+ \wedge MY \arrow[r, "a"] \arrow[d, "id\wedge Mf"'] & M(G_+ \wedge Y) \arrow[d, "M(id\wedge f)"] \\ G_+ \wedge MZ \arrow[r, "a"] & M(G_+ \wedge Z). \end{tikzcd}$$

The diagram commutes as for each $g \in G$, we have $(id \wedge f) \circ i_g = i_g \circ f$. The assembly map also commutes with η_M and μ_M in the following sense:

$$\begin{tikzcd} G_+ \wedge Y \arrow[r, "id\wedge\eta_M"] \arrow[dr, "\eta_M"'] & G_+ \wedge MY \arrow[d, "a"] \\ & M(G_+ \wedge Y) \end{tikzcd} \qquad \begin{tikzcd} G_+ \wedge MMY \arrow[r, "a^2"] \arrow[d, "id\wedge\mu_M"'] & MM(G_+ \wedge Y) \arrow[d, "\mu_M"] \\ G_+ \wedge MY \arrow[r, "a"] & M(G_+ \wedge Y) \end{tikzcd}$$

commute. This is obtained, for each $g \in G$, using the naturality of η_M and μ_M respectively.

Define the functor

$$(M \rtimes G)(X) := M(G_+ \wedge X),$$

and the natural transformations μ and η via the following diagrams

$$\begin{tikzcd}
M(G_+ \wedge M(G_+ \wedge X)) \arrow[rr, "\mu"] \arrow[d, "M(\Delta\wedge id)"'] & & M(G_+ \wedge X) \\
M(G_+ \wedge G_+ \wedge M(G_+ \wedge X)) \arrow[d, "M(id\wedge a)"'] & & \\
M(G_+ \wedge M(G_+ \wedge G_+ \wedge X)) \arrow[d, "M(\rho)"'] & & \\
MM(G_+ \wedge G_+ \wedge X), \arrow[uuurr, "\mu_M\mu_G"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, "\eta"] \arrow[dr, "\eta_G"'] & & M(G_+ \wedge X). \\
& G_+ \wedge X \arrow[ur, "\eta_M"'] &
\end{tikzcd}$$

Note that, in the definition of μ, using the naturality of ρ, we can reverse the order of a and ρ. This is also the case for the multiplication maps μ_M and μ_G, a consequence of the naturality of μ_M, so that the notation $\mu_M\mu_G$ is not ambiguous.

Proposition 2.4.1. *This defines a monad structure on $M \rtimes G$.*

Proof. The fact that η is a left unit for μ is an easy consequence of the same property for η_M and μ_M, and of the naturality of η_M. To prove that it is also a right unit, we need furthermore the commutation of a and ρ with η_M, as well as the commutation of a with ρ.

The associativity of μ follows from the associativity of μ_M, μ_G, the fact that ρ is a group homomorphism and $\rho(g)$ is a morphism of monads (commutation with μ_M), and from the naturality of μ_M, a, and ρ. $\square$

A monad map between twisted monads

$$M \rtimes G \longrightarrow M' \rtimes G'$$

consists of a group map $G \longrightarrow G'$ and a monad map $M \longrightarrow M'$ commuting with the group actions on M and M' and with the assembly maps. This is a straightforward consequence of the definition of μ and η for a twisted monad.

The twisted monad $M \rtimes G$ was constructed so that its algebras are precisely the algebras over M having a G-action which commutes with their M-algebra structure in a sense we make precise in the following proposition.

Proposition 2.4.2. *(X, ξ) is an $M \rtimes G$-algebra if and only if X is equipped with an M-algebra structure $\lambda : MX \to X$ and a G-action $\Psi : G_+ \wedge X \to X$ such that $\xi = \lambda \circ \Psi$ and the following diagram commutes:*

$$\begin{tikzcd}
G_+ \wedge MX \arrow[r, "l"] \arrow[d, "id\wedge\lambda"'] & M(G_+ \wedge X) \arrow[r, "M(\Psi)"] & MX \arrow[d, "\lambda"] \\
G_+ \wedge X \arrow[rr, "\Psi"] & & X,
\end{tikzcd}$$

where $l = \rho \circ (id \wedge a) \circ (\Delta \wedge id)$ (see definition of μ for $M \rtimes G$ above).

Proof. Suppose first that λ and Ψ are given and that the diagram of the proposition commutes. Define ξ to be $\lambda \circ \Psi$. We have to check that ξ defines an $M \rtimes G$-algebra structure on X. This follows easily from the fact that λ is an M-algebra structure on X, Ψ is a group action and the maps are natural.

For the converse, we suppose that ξ is an $M \rtimes G$-algebra structure map for X. Define

$$\begin{tikzcd}
\lambda :\ MX \arrow[rr] \arrow[dr, "M(\eta_G)"'] & & X. \\
& M(G_+ \wedge X) \arrow[ur, "\xi"'] &
\end{tikzcd}$$

Using the map $M(\eta_G) : MX \longrightarrow M(G_+ \wedge X)$, the required properties of λ are obtained as a consequence of the same properties for ξ with respect to $M \rtimes G$, the fact that $\rho(e)$ is the identity and the naturality of η_M and μ_M.

Define

$$\begin{tikzcd}
\Psi :\ G_+ \wedge X \arrow[rr] \arrow[dr, "\eta_M"'] & & X. \\
& M(G_+ \wedge X) \arrow[ur, "\xi"'] &
\end{tikzcd}$$

The fact that Ψ is a G-action follows from the fact that ξ is an algebra structure map for X and from the naturality and unit property of η_M.

Mapping $M(G_+ \wedge X) \xrightarrow{\eta} M(G_+ \wedge M(G_+ \wedge X))$ and using the properties of ξ (diagram involving μ and ξ), and the fact that $\mu \circ \eta = id$, one obtains that $\xi = \lambda \circ \Psi$. $\square$

2.4.2 Some examples and their properties

From the operad case, we know already a few examples of twisted monads: $fD_n = D_n \rtimes SO(n)$, $BC^{R\beta} = BC^{\beta} \rtimes B\mathcal{Z}$. We are interested in the following —non-operadic— examples: $\Omega^2\Sigma^2 \rtimes S^1$, with the endomorphisms of $\Omega^2\Sigma^2$ defined in the first case as being trivial (i.e. $\rho_1(s)$ is the identity for each $s \in S^1$), and in the second case as

$$\begin{array}{rlcl} \rho_2 : & S^1_+ \wedge \Omega^2\Sigma^2 X & \longrightarrow & \Omega^2\Sigma^2 X \\ & (s, [\sigma(t), x(t)]) & \mapsto & [s.\sigma(t.s^{-1}), x(t.s^{-1})], \end{array}$$

where $t \in D^2$, $s \in S^1$, considering $S^1 \subset D^2 \subset \mathbb{C}$, and $[\sigma(t), x(t)]$ denote the element of $\Omega^2\Sigma^2 X$ which maps t to $[\sigma(t), x(t)] \in \Sigma^2 X$. The S^1-action on $\Omega^2\Sigma^2 X$ given by ρ_2 is actually defined the make the map $fD_2X = (D_2 \rtimes S^1)X \longrightarrow (\Omega^2\Sigma^2 \rtimes S^1)X$ induced by $\alpha : D_2X \longrightarrow \Omega^2\Sigma^2 X$ into a monad map. In particular, α is an S^1-map with this action.

If $X = \Omega^2 Y$ is a double loop space, it is an algebra over $\Omega^2\Sigma^2$ with a canonical map

$$\lambda : \Omega^2\Sigma^2\Omega^2 Y \longrightarrow \Omega^2 Y; \quad [\sigma(t), [y(t,u)]] \mapsto [y(t, \sigma(t)],$$

where $[y(t,u)]$ denotes for each $t \in D^2$ the loop $u \mapsto [y(t,u)]$. If this structure extends to an $\Omega^2\Sigma^2 \rtimes S^1$-algebra structure, the circle action on X has to be of a specific form, as stated below.

Proposition 2.4.3. *If $X = \Omega^2 Y$ is a double loop space whose canonical $\Omega^2\Sigma^2$-algebra structure extends to an $\Omega^2\Sigma^2 \rtimes S^1$-algebra structure, then the induced S^1-action Ψ on $\Omega^2 Y$ factors in an action on D^2 followed by an action on Y. In the first case (trivial twisting of $\Omega^2\Sigma^2$ and S^1), the action on D^2 is trivial:*

$$\Psi(s, [y(t)]) = [sy(t)];$$

and in the second case (monad structure obtained with ρ_2), Ψ rotates the disc before acting on Y:

$$\Psi(s, [y(t)]) = [sy(t.s^{-1})].$$

In particular, the action on Y can be trivial, so any double loop space is an algebra over $\Omega^2\Sigma^2 \rtimes S^1$ in both cases.

Remark 2.4.4. We can formulate a similar statement for $\Omega^2\Sigma^2 \rtimes G$, where G is any group acting on the disk D^2 for which $\Omega^2\Sigma^2 \rtimes G$ is well-defined. We could also consider $\Omega^n\Sigma^n$.

Proof. The commutativity of the diagram of proposition 2.4.2 provides, in the first case, the following equality : $\Psi(s, [y(t, \sigma(t))]) = \Psi(s, [y(t, u)])(\sigma(t))$, where the right hand side is, for a fixed t, the image of the action of s on the double loop $u \mapsto y(t, u)$, evaluated at $\sigma(t)$. In particular, taking $[y(t, u)] = [y(u)]$, and $\sigma(t) = t_0$, we have

$$\Psi(s, [y(t_0)]) = \Psi(s, [y(u)])(t_0). \qquad (*)$$

It follows that, if $[y(u)]$ is a constant loop, $\Psi(s, [y(u)])$ is also constant. We then have a well-define S^1-action on Y given by $sy = \Psi(s, [y])(0)$, where $[y]$ denotes the constant loop at y. Clearly, 1 acts as the identity. Moreover, $s_1 s_2 y = \Psi(s_1, [\Psi(s_2, [y])(0)])(0) = \Psi(s_1, \Psi(s_2, [y]))(0)$ as $\Psi(s_2, [y])$ is a constant loop. Now this is $\Psi(s_1 s_2, [y])(0)$ as Ψ is an action. Using $(*)$ again, we obtain that $\Psi(s, [y(u)])(t_0) = \Psi(s, [y(t_0)])$ which is $[sy(t_0)]$ by definition of the action on Y.

In the second case, the diagram of proposition 2.4.2 gives $\Psi(s, [y(t, \sigma(t))]) = \Psi(s, [y(t.s^{-1}, u)])(s.\sigma(t.s^{-1}))$. Again, taking $[y(t, u)] = [y(u)]$, and $\sigma(t) = t_0$, we have

$$\Psi(s, [y(t_0)]) = \Psi(s, [y(u)])(s.t_0). \qquad (**).$$

For the same reason as above, this allows us the define the same action on Y. The equation $(**)$ gives then that $\Psi(s, [y(u)])(t_o) = \Psi(s, [y(t_0.s^{-1})])$, which is $[sy(t_0.s^{-1})]$ by definition of the action on Y. □

We have encountered the monads D_2, $fD_2 = D_2 \rtimes S^1$, $\Omega^2\Sigma^2$, and $\Omega^2\Sigma^2 \rtimes S^1$ (considered here with the non-trivial twisted structure). They are related as shown in the following commutative diagram :

$$\begin{tikzcd}
fD_2 X \arrow[r, "\beta"] & \Omega^2\Sigma^2 X \\
D_2(S^1_+ \wedge X) \arrow[u, "\cong"] \arrow[r, "\alpha"] & \Omega^2\Sigma^2(S^1_+ \wedge X) \arrow[u, "f"'] \\
D_2 X \arrow[u, "i"] \arrow[r, "\alpha"] & \Omega^2\Sigma^2 X, \arrow[u, "i"']
\end{tikzcd}$$

where α and β were defined in 2.3.2 and f is given by

$$f([\sigma(t), s(t), x(t)]) = [\sigma(t).s(t)^{-1}, x(t)].$$

All maps are monad maps but only the top diagram is composed of S^1-maps.

Remark 2.4.5. Except for D_2, all the monads involved have "their" circle action as part of their monad structure, i.e of the form $\mu(m(s),\_)$, for $m(s)$ "in the monad". For example, in the case of $\Omega^2\Sigma^2$, the action $\Psi(s, [\sigma(t), x(t)]) = \mu_{\Omega^2\Sigma^2}([u.s, [\sigma(t), x(t)]])$. Thus, as we have monad maps, those circle actions are sent to circle actions. However, there might be several circle actions of that sort for a monad and so a monad map is not always an S^1-map. This is the case for $i : \Omega^2\Sigma^2 \longrightarrow \Omega^2\Sigma^2 \rtimes S^1$.

2.4.3 Framed recognition principle

If X is an $f\mathcal{D}_2$-algebra, we can consider two forms of "recognition principle". One form is by taking the recognition principle for D_2-algebras and define appropriate circle actions on the spaces used in that case. This is what we did for our equivariant recognition principle (theorem 2.1.1). The other possibility is to use more appropriate monads to construct the spaces involved. We explain this other approach here.

Instead of using the monads D_2 and $\Omega^2\Sigma^2$, we can rewrite the recognition principle for $f\mathcal{D}_2$-algebras using the monads fD_2 and $\Omega^2\Sigma^2 \rtimes S^1$. The point is that the equivalence $\alpha : D_2X \longrightarrow \Omega^2\Sigma^2 X$ for X connected, provides an equivalence

$$\alpha : D_2(S^1_+ \wedge X) \longrightarrow \Omega^2\Sigma^2(S^1_+ \wedge X),$$

i.e. $fD_2X \xrightarrow{\simeq} (\Omega^2\Sigma^2 \rtimes S^1)X$ for X connected. Hence we have

$$X \xleftarrow{\simeq} B(fD_2, fD_2, X) \xrightarrow{\simeq} B(\Omega^2\Sigma^2 \rtimes S^1, fD_2, X),$$

where both maps are fD_2-algebra maps and weak equivalences for X connected. This time, the fD_2-actions are obtained by acting on the left factor of the bar constructions.

This provides the following double deloop of X:

$$B(\Omega^2\Sigma^2 \rtimes S^1, fD_2, X) \xrightarrow{\simeq} \Omega^2(B(\Sigma^2 \rtimes S^1, fD_2, X)),$$

where $\Sigma^2 \rtimes S^1$ is the functor sending X to $\Sigma^2(S^1_+ \wedge X)$. Again, the circle action on this double loop space is both by rotating the disc and by rotating the deloop $Y = B(\Sigma^2 \rtimes S^1, fD_2, X)$.

The inclusion $B(\Sigma^2, D_2, X) \longrightarrow B(\Sigma^2 \rtimes S^1, fD_2, X)$ is an S^1-map which provides an equivalence of deloops. So we obtain a deloop equivalent as S^1-space to the one obtained in theorem 2.1.1.

Chapter 3

Infinite loop space structures on the stable mapping class groups

Let $F_{g,1}$ be a surface of genus g with one boundary component. The mapping class group $\Gamma_{g,1}$ is the group of components of diffeomorphisms of $F_{g,1}$ which fix the boundary pointwise. The groups $\Gamma_{g,1}$ are of special interest as, when $g > 0$, the classifying space of $\Gamma_{g,1}$ is homotopy equivalent to the moduli space of Riemann surfaces of topological type $F_{g,1}$.

By attaching a torus with two boundary components to $F_{g,1}$, one can define a homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$. A theorem of Harer [12] improved by Ivanov [13] says that this map induces an isomorphism on homology groups in dimension less than $g/2$. Tillmann used this fact to show the existence of an infinite loop space structure on $B\Gamma_\infty^+$, the group completion of the classifying space of the stable mapping class group $\Gamma_\infty = \lim_{g\to\infty}\Gamma_{g,1}$.

She has two different proofs of this fact. The first one uses a cobordism 2-category $\mathcal{S}$: the objects are one dimensional manifolds, the *1-morphisms* are cobordisms between them and *2-morphisms* are diffeomorphisms of the cobordisms. This category has a natural symmetric monoidal structure given by disjoint union. The result follows from the fact that $\Omega B\mathcal{S}$ is equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$. The second proof uses an operad $\mathcal{M}$ associated to this category, considering only the surfaces with n "incoming" and one "outgoing" boundary components. This operad is an infinite loop space operad.

We show that the two infinite loop space structures are equivalent by producing an equivalence of spectra.

3.1 The mapping class groups operads

We give in this section a summary of Tillmann's construction of the mapping class group operad $\mathcal{M}$, as given in [40], spelling out some of the details we will need in section 3.3.

Let $F_{g,n+1}$ denote an oriented surface of genus g with $n+1$ boundary components. One of the boundary components is marked; we call the n other components *free*. Each free boundary component ∂_i comes equipped with a collar, a map from $[0,\epsilon) \times S^1$ to a neighbourhood of ∂_i; for the marked boundary component, there is a map from $(\epsilon, 0] \times S^1$ to a neighbourhood of the boundary. Let $\mathrm{Diff}^+(F_{g,n+1};\partial)$ be the group of orientation preserving diffeomorphisms which fix the collars, and let

$$\Gamma_{g,n+1} = \pi_0(\mathrm{Diff}^+(F_{g,n+1};\partial))$$

be its group of components, the associated *mapping class group*. There is a normal extension

$$\Gamma_{g,n+1} \hookrightarrow \Gamma_{g,(n),1} \twoheadrightarrow \Sigma_n.$$

The extension $\Gamma_{g,(n),1}$ is the group of components of the orientation preserving diffeomorphisms that may permute the n free boundary components, preserving the collars.

Consider the groupoids

$$G_n = \coprod_{g\geq 0} \Gamma_{g,(n),1} \quad \text{and} \quad H_n = \coprod_{g \geq 0} \Gamma_{g,n+1}$$

for $n \geq 0$.

We want to construct an operad with those groupoids, using the techniques given in section 1.2. There are maps

$$\omega : \Gamma_{g,(k),1} \times \Gamma_{h_1,(n_1),1} \times \ldots \times \Gamma_{h_k,(n_k),1} \to \Gamma_{g+h_1+\cdots+h_k,(n_1+\cdots+n_k),1}$$

induced by gluing the marked boundary of the k surfaces F_{h_i,n_i+1} to the k free boundaries of the first surface. However, these maps cannot be made associative on the level of the mapping class groups. In order to rectify this, we need to enlarge the groupoids and so consider larger categories than the categories $\mathcal{C}_{\Gamma_{g,n+1}}^{\Gamma_{g,(n),1}}$ (in the notation of section 1.2).

3.1.1 Construction of the categories $\mathcal{E}_{g,n,1}$

Pick three atomic surfaces with fixed collars of the boundary components: a disc $D = F_{0,1}$, a pair of pants surfaces $P = F_{0,3}$ and a torus $T = F_{1,2}$ with two boundary components (see figure 3.1).

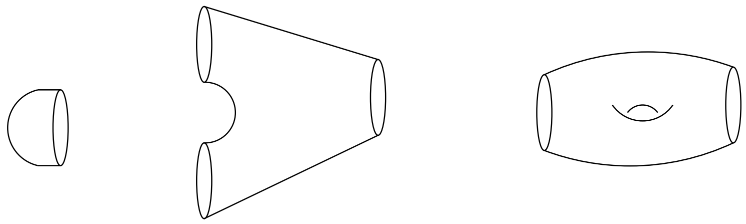


Figure 3.1: Basic surfaces

Define a connected groupoid $\mathcal{E}_{g,n,1}$ with objects (F,σ), where F is a surface of type $F_{g,n+1}$ constructed from D, P and T by gluing the marked boundary of one surface to one of the free boundaries of another using the given parametrisation, and σ is an ordering of the n free boundary components. Note that each boundary component of F comes equipped with a collar. The set of morphisms from (F,σ) to (F',σ') are the homotopy classes $\Gamma(F,F') = \pi_0 Diff^+(F,F';\partial)$ of orientation preserving diffeomorphisms preserving the collars and the ordering of the boundaries. The group Σ_n acts freely on $\mathcal{E}_{g,n,1}$ and there is a homotopy equivalence $B\mathcal{E}_{g,n,1} \simeq B\Gamma_{g,n+1}$.

By construction, gluing of surfaces induces now an associative and Σ-equivariant structure map on the categories. Thus on the classifying spaces, we have maps

$$\gamma : B\mathcal{E}_{g,k,1} \times B\mathcal{E}_{h_1,n_1,1} \times \ldots \times B\mathcal{E}_{h_k,n_k,1} \to B\mathcal{E}_{g+h_1+\cdots+h_k,n_1+\cdots+n_k,1},$$

which are associative and Σ-equivariant. However, $\{\coprod_{g\geq 0} B\mathcal{E}_{g,n,1}\}_{n\in\mathbb{N}}$ does not precisely form an operad yet as there is no unit.

The operad $\mathcal{M}$ is obtained by applying two quotient constructions on the categories $\mathcal{E}_{g,n,1}$ and taking the classifying spaces of the modified categories. Those constructions provide $\mathcal{M}$ with a unit and make the product induced by the gluing of a pair of pants associative and unital. This ensures compatibility with the operads $\Gamma, B\mathcal{C}^{\beta}$ and $B\mathcal{C}^{R\beta}$ constructed in section 1.2 and gives an actual monoid structure to $\mathcal{M}$-algebras.

For the first quotient construction, fix isotopies $\phi_1 : \gamma(P; D, \_) \xrightarrow{\simeq} \gamma(P; \_, D)$, and $\phi_2 : \gamma(P; \_, P) \xrightarrow{\simeq} \gamma(P; P, \_)$. Define two surfaces F and F' to be equivalent if F can be isotoped to F' by applying ϕ_1, ϕ_2 and their inverses to its subsurfaces. Let F^0 be the unique representative in each equivalent class having no subsurfaces of the type $\gamma(P; D, \_)$ or $\gamma(P; \_, P)$ and consider the full subcategory generated by these surfaces. The isotopies ϕ_1 and ϕ_2 define a isotopies $\phi_F : F \to F^0$ for every F to the representative of its equivalent class. Define the new structure maps γ^0 on objects by taking the unique representative of the image of γ on the surfaces. On morphisms, $\gamma^0(\psi, \psi_1, \ldots, \psi_k) = \phi_{F'}\gamma(\psi, \psi_1, \ldots, \psi_k)\phi_F^{-1}$, where F and F' are the images by γ of the sources and target surfaces of the maps ψ. The map γ^0 is again associative.

For the second quotient construction, choose an isotopy $\phi : \gamma(P; \_, D) \to S^1 \times [0,1]$ and denote by $\overline{\phi}$ its composition with the projection map $S^1 \times [0,1] \to S^1$. Now replace every occurrence of $\gamma(P; \_, D)$ by S^1, yielding surfaces $\overline{F}$, and identify the mapping class groups $\Gamma(F, F')$ and $\Gamma(\overline{F}, \overline{F'})$ via the homotopies $\phi_F : F \to \overline{F}$ and $\phi_{F'} : F' \to \overline{F'}$ generated by $\overline{\phi}$. The structure maps γ^0 induce maps $\overline{\gamma^0}$ on this new category as we are working on the level of the mapping class groups. The map $\overline{\gamma^0}$ has now a unit, the only object S^1 in $\mathcal{E}_{0,1,1}$ (with automorphism group $\mathbb{Z}$ thought of as generated by the Dehn twist around S^1).

There are operad maps

$$B\mathcal{C}^{\beta} \longrightarrow B\mathcal{C}^{R\beta} \longrightarrow \mathcal{M} \xrightarrow{\pi} \Gamma,$$

where π is given on objects by the projection on the labels.

3.1.2 Infinite loop space structure

U. Tillmann showed in [40] that $\mathcal{M}$ is an infinite loop space operad, i.e. that the group completion of any $\mathcal{M}$-algebra is weakly homotopy equivalent to an infinite loop space. She first identified the free $\mathcal{M}$-algebra on any space X.

Let $\mathcal{G}$ denote the group completion functor, let M denote the monad associated to the operad $\mathcal{M}$ (definition 1.1.9) and let $\mathcal{Q}(X) = lim_{n\to\infty}\Omega^n\Sigma^n$. Barrat and Eccles show in [1] that $\mathcal{G}\Gamma X$ is naturally homotopy equivalent to $\mathcal{Q}(X)$.

Theorem 3.1.1. *[40] The maps $M(X) \to M(*)$ and $M(X) \to \Gamma(X)$ induce a natural homotopy equivalence on group completion:*

$$\mathcal{G}M(X) \simeq (\mathbb{Z} \times B\Gamma_\infty^+) \times \mathcal{Q}(X).$$

In particular,

$$\mathcal{G}M(*) \simeq \mathbb{Z} \times B\Gamma_\infty^+.$$

The map $\tau : \mathcal{G}M(X) \to \mathcal{G}M(*)$, induced by $X \to *$, is a map of simplicial groups and hence a Kan fibration. Let $\mathcal{GF}(X)$ be its fibre. The inclusion $* \to X$ induces a splitting $\mathcal{G}M(*) \to \mathcal{G}M(X)$ of τ which is also a map of simplicial groups. Thus $\mathcal{G}M(X)$ splits as a simplicial set.

Corollary 3.1.2. *[40] $\mathcal{G}M(X) = \mathcal{G}M(*) \times \mathcal{GF}(X)$ and $\mathcal{GF}(X) \xrightarrow{\simeq} \mathcal{G}\Gamma(X)$.*

Theorem 3.1.3. *[40] If X is an $\mathcal{M}$-algebra, $\mathcal{G}(X)$ is weakly homotopy equivalent to an infinite loop space.*

The theorem is proved by constructing a space equivalent to $\mathcal{G}(X)$ for which one can give explicit deloops. We give here a precise description of the deloops we will use for the comparison.

Consider the space $|\mathcal{G}M(M^*(Y))|$, which is equivalent to Y. On each simplicial level, we have $\mathcal{G}M(M^n(Y)) = \mathcal{G}M(*) \times \mathcal{GF}(M^n(Y))$. The space $|\mathcal{G}M(M^*(Y))|$ carries a free $\mathcal{G}M(*)$-action which gives rise to the principal fibration

$$\mathcal{G}M(*) \longrightarrow |\mathcal{G}M(M^*(Y))| \longrightarrow |\mathcal{GF}(M^*(Y))|,$$

where the simplicial space $\mathcal{GF}(M^*(Y))$ is defined to be $\mathcal{G}M(M^*(Y))/\mathcal{G}M(*)$. We know that on each level, $(\mathcal{GF}(M^*(Y)))_n = \mathcal{GF}(M^n(Y))$. It inherits a simplicial structure from $\mathcal{G}M(M^*(Y))$, with δ_0 induced by $\mathcal{GF}MX \xrightarrow{\mu_M} \mathcal{G}MX \xrightarrow{p} \mathcal{GF}X$. Now consider $Y = M(*) \times X$ and compare the above fibration to

$$\mathcal{G}M(*) \longrightarrow \mathcal{G}M(*) \times \mathcal{G}X \longrightarrow (\mathcal{G}M(*) \times \mathcal{G}X)/\mathcal{G}M(*),$$

where $\mathcal{G}M(*)$ acts diagonally on $\mathcal{G}M(*) \times \mathcal{G}X$, acting on $\mathcal{G}X$ via the map $\mathcal{G}M(*) \to \mathcal{G}M(X) \to \mathcal{G}X$. The equivalence $|\mathcal{G}M(M^*(Y))| \xrightarrow{\simeq} Y$, induces a map of principal fibrations:

$$\begin{tikzcd}
\mathcal{G}M(*) \arrow[r] \arrow[d, "="'] & {|\mathcal{G}M(M^*(M(*) \times X))|} \arrow[r, "p"] \arrow[d, "\simeq"] & {|\mathcal{GF}(M^*(M(*) \times X))|} \arrow[d, "\lambda"] \\
\mathcal{G}M(*) \arrow[r] & \mathcal{G}M(*) \times \mathcal{G}X \arrow[r] & (\mathcal{G}M(*) \times \mathcal{G}X)/\mathcal{G}M(*) \arrow[d, "\simeq"] \\
 & & \mathcal{G}X \arrow[ul, "i"]
\end{tikzcd}$$

We deduce that λ is an equivalence. Furthermore, the map $\mathcal{G}X \xrightarrow{i} \mathcal{G}M(*) \times \mathcal{G}X \xrightarrow{j} |\mathcal{G}M(M^*(M(*) \times X))| \xrightarrow{p} |\mathcal{GF}(M^*(M(*) \times X))| \xrightarrow{\simeq} (\mathcal{G}M(*) \times \mathcal{G}X)/\mathcal{G}M(*) \xrightarrow{\simeq}$

$\mathcal{G}X$ is the identity. It follows that the map $p \circ j \circ i$, which sends x to $(1, D, x) \in |\mathcal{GF}(M^*(M(*) \times X))|$, where D is the disc, induces the equivalence

$$\mathcal{G}X \xrightarrow{\simeq} |\mathcal{GF}(M^*(M(*) \times X))|.$$

The ith deloop of $\mathcal{G}X$ is then given by $|\mathcal{GF}(S^i \wedge M^*(M(*) \times X))|$. We give below in details the simplicial structure of the case $X = M(*)$. This is the case we are interested in as the group completion of $M(*)$ is equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$. We will in fact consider a slight variant of the deloops because of base point problem in the map we construct in the next section. We will also work with $\mathcal{G}\Gamma$ rather than $\mathcal{GF}$ to simplify the comparison map.

For $i \geq 0$, define

$$E^i := |\mathcal{G}\Gamma(S^i \wedge M^*((M(*) \times M(*))_+))|.$$

Proposition 3.1.4. *The map $|\mathcal{GF}(S^i \wedge M^*(M(*) \times M(*)))| \longrightarrow E^i$ induced by $\mathcal{GF}(X) \xrightarrow{\simeq} \mathcal{G}\Gamma(X)$ and $M(*) \times M(*) \hookrightarrow (M(*) \times M(*))_+$, gives an equivalence of spectra.*

Proof. Consider first the group completion of $(M(*) \times M(*))_+$.
As $M(*) \simeq \coprod_{g \geq 0} B\Gamma_{g,1}$, the monoid of components of $(M(*) \times M(*))_+$ is $(\mathbb{N} \times \mathbb{N})_+$. Its group completion is $\mathbb{Z} \times \mathbb{Z}$. Also, since the limits on the components of $(M(*) \times M(*))$ and $(M(*) \times M(*))_+$ are the same, a calculation similar to the calculation of $\mathcal{G}M(*)$ yields

$$\mathcal{G}((M(*) \times M(*))_+) \simeq \mathbb{Z} \times \mathbb{Z} \times B\Gamma_\infty^+ \times B\Gamma_\infty^+ \simeq \mathcal{G}(M(*) \times M(*)).$$

We have again a principal fibration

$$\mathcal{G}M(*) \longrightarrow |\mathcal{G}M(M^*((M(*) \times M(*))_+))| \longrightarrow |\mathcal{GF}(M^*((M(*) \times M(*))_+))|.$$

Comparing this fibration to the fibration $\mathcal{G}M(*) \to \mathcal{G}M(*) \times \mathcal{G}M(*) \to \mathcal{G}M(*)$, we deduce that $|\mathcal{GF}(M^*((M(*) \times M(*))_+))| \simeq \mathcal{G}M(*)$. Comparing it in addition to the fibration for $|\mathcal{G}M(M^*(M(*) \times M(*)))|$, we deduce that the map $|\mathcal{GF}(M^*(M(*) \times M(*)))| \to |\mathcal{GF}(M^*((M(*) \times M(*))_+))|$, induced by the inclusion, gives the equivalence.

Now $|\mathcal{GF}(S^i \wedge M^*(X))| \to |\mathcal{G}\Gamma(S^i \wedge M^*(X))|$ is a map of spectra. So the map given in the proposition is a map of connective spectra which induces an equivalence on the first spaces. Hence it is an equivalence of spectra.

$\square$

Let μ_M, η_M and μ_Γ, η_Γ denote the monad structure maps of M and Γ (induced by the operad structure) and let θ denote the M-algebra structure map of $(M(*) \times M(*))_+$. Let $\pi : \mathcal{M} \to \Gamma$ be the projection of operads.

For any operad $\mathcal{P}$, there is a map $\phi : A \times P(X) \to P(A \times X)$, sending an element $(a, p, x_1, \ldots, x_k)$ to $(p, (a, x_1), \ldots, (a, x_k))$. As $\Gamma(*) = \{*\}$, the sequence of maps

$$\Gamma(A \times M(X)) \xrightarrow{\phi} \Gamma(M(A \times X)) \xrightarrow{\pi} \Gamma\Gamma(A \times X) \xrightarrow{\mu_\Gamma} \Gamma(A \times X)$$

induces a map on smash products

$$\lambda : \Gamma(A \wedge M(X)) \longrightarrow \Gamma(A \wedge X).$$

The simplicial structure on E^i, $i \geq 0$, is defined as follows.

$$E^i_p = \mathcal{G}\Gamma(S^i \wedge M^p((M(*) \times M(*))_+)$$

and

$$\begin{array}{ll}
\delta_0 = \mathcal{G}(\lambda) : E^i_p \to E^i_{p-1}; & \\
\delta_i = \mathcal{G}\Gamma(S^i \wedge M^{i-1}(\mu_M)) : E^i_p \to E^i_{p-1} & \text{for } 1 \leq i < p; \\
\delta_p = \mathcal{G}\Gamma(S^i \wedge M^{p-1}(\theta)) : E^i_p \to E^i_{p-1}; & \\
s_i = \mathcal{G}\Gamma(S^i \wedge M^i(\eta_M) : E^i_p \to E^i_{p+1} & \text{for } 0 \leq i \leq p.
\end{array}$$

3.2 Infinite loop space structure via disjoint union

U. Tillmann's first proof of the existence of an infinite loop space structure on $B\Gamma^+_\infty$ was through the construction of a symmetric monoidal 2-category. A 2-category is a category whose sets of morphisms are again categories and composition is given by functors. Taking the classifying spaces of those morphism categories yields a category enriched over Top. We denote by $\mathcal{BS}$ the resulting category for $\mathcal{S}$. As we will not use the structure of 2-category of $\mathcal{S}$, we will only give here a precise description of $\mathcal{BS}$. We will actually construct a variant of the one given in [39], which was itself a variant of [38]. The 2-category $\mathcal{S}$ has the natural numbers as set of objects, thought of as the set of closed one-dimensional manifolds. The "1-morphisms" in $\mathcal{S}$ are cobordisms between them, and the "2-morphisms" are diffeomorphisms of the cobordisms which preserve the boundaries (in [39]) or elements of the corresponding mapping class group (in [38]). We use here the mapping class groups as 2-morphisms. Also, we will apply to the morphism categories the same quotient constructions we applied to the categories $\mathcal{E}_{g,n,1}$ when constructing the operad $\mathcal{M}$ in 3.1.1. This is to help the comparison of the two infinite loop space structures.

Tillmann showed that $\Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma_{\infty}^{+}$, where $B\mathcal{S}$ denotes the classifying space of $\mathcal{BS}$. As disjoint union induces a symmetric monoidal structure on $\mathcal{BS}$, $B\mathcal{S}$ is an infinite loop space, which yields the second infinite loop space structure on $B\Gamma_{\infty}^{+}$.

The monoidal category $\mathcal{BS}$ is defined as follows. The objects of $\mathcal{S}$ are the natural numbers $0, 1, 2, \ldots$, with monoidal structure given by addition.

To define the morphism spaces, we first construct categories $\mathcal{E}(n, m)$ for each $n, m \in \mathbb{N}$ with $m \neq 0$. An object of $\mathcal{E}(n, m)$ is a couple (F, σ) where F is a surface with n 'in-coming' and m 'out-going' boundary component, obtained from the basic surfaces D, P and T (see figure 3.1), as in 3.1.1, by gluing the marked boundary of one surface to a free boundary of another, with the additional possibility here to take disjoint union, and σ is a labelling of the n free boundary components and of the m marked components (see figure 3.2). So $F = F_1 \coprod \cdots \coprod F_m$ where F_i is an object of $\mathcal{E}_{g_i,k_i,1}$ for some g_i, k_i with $\Sigma k_i = n$.

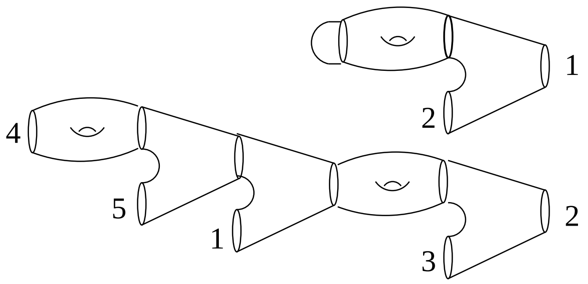


Figure 3.2: Object of $\mathcal{E}(5, 2)$

The set of morphisms between F and G in $\mathcal{E}(n, m)$ is empty unless F and G are diffeomorphic. In that case, it is the group of components $\pi_0 \mathrm{Diff}^+(F, G, \partial)$ of orientation preserving diffeomorphisms which fix the boundaries preserving the labels. In particular, $\mathcal{E}(n, 1) = \coprod_{g \geq 0} \mathcal{E}_{g,n,1}$.

Now apply to $\mathcal{E}(n, m)$ the quotient constructions defined for the categories $\mathcal{E}_{g,n,1}$ in 3.1.1. So we apply the quotient on each component of the objects and take the full subcategory containing the chosen reprensentatives. Note that the modified version of $\mathcal{E}(n, n)$ now contains the symmetic group, represented by disjoint copies of the circle with labels "on each side".

The morphism space $\mathcal{BS}(n, m)$, for $m \neq 0$, is defined to be the classifying space of the resulting quotient category. $\mathcal{BS}(n, 0)$ is empty unless $n = 0$, in which case it is just the identity. Composition in $\mathcal{BS}$ is induced by gluing the surfaces according to the labels, which can be done using the operad map γ on each component. This is well-defined as γ is well-defined on the quotient categories.

Disjoint union of surfaces induces a symmetric monoidal structure on $\mathcal{BS}$.

Let $B\mathcal{S}$ denote the classifying space of $\mathcal{BS}$:

$$B\mathcal{S} := B(\mathcal{BS}).$$

Theorem 3.2.1. *[39] $\Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$.*

The proof of [39] can be applied to our modified version of $\mathcal{BS}$. Indeed, our morphism spaces are equivalent to the ones constructed in [39], except for $\mathcal{BS}(n,n)$, which has an added base point in [39]. However, the telescopes of $\mathcal{BS}(1,1)$ and $\mathcal{BS}(1,1)_+$ are equivalent and this is enough to adapt the proof.

As $\mathcal{BS}$ is a symmetric monoidal category and $B\mathcal{S}$ is connected, we know that $B\mathcal{S}$ is an infinite loop space [34]. Thus $\Omega B\mathcal{S}$ is an infinite loop space. So we get another proof of the existence of an infinite loop space structure on the stable mapping class group:

Corollary 3.2.2. *$\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space.*

In this case, we use Barrat and Eccles machinery to get the deloops of $\mathbb{Z} \times B\Gamma_\infty^+$ [1]. They are given, in our notations, by the following formula:

$$F^i = |\mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^*(\mathcal{BS}))| \quad \text{for } i \geq 1,$$

where the simplicial structure is defined similarly to the one of E^i, which is given in detail at the end of section 3.1.

3.3 Comparison of the two structures

The existence of an infinite loop space structure on $\mathbb{Z} \times B\Gamma_\infty^+$ is a surprising fact. The existence of two such structures would be even more surprising! U. Tillmann showed in [38] that the first deloops are the same. We construct here a map of spectra from the "pair of pants" spectrum to the "disjoint union" spectrum, which is an equivalence. The map we construct is different from the map constructed in [38].

The deloops are given as realisation of simplicial spaces E^i_* and F^i_*. We will first construct maps $f_p : E^i_p \to F^i_p$ on the p-simplices. The maps f_p almost form a simplicial map. All commutations required for f_* to be a simplicial map are satisfied except that $\delta_p f_p$ is homotopic to $f_{p-1}\delta_p$ rather than equal. Thanks to the existence of higher homotopies, we will show how this map can be rectified. Once the map rectified, we show that it gives an equivalence of spectra.

3.3.1 Construction of the maps f^i_p

We want to construct a map from $E^i_p = \mathcal{G}\Gamma(S^i \wedge M^p((M(*) \times M(*))_+))$ to $F^i_p = \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathcal{S}))$ for $i \geq 1$. By construction, to an element of $M(*) = M(0)$ precisely corresponds a morphism from 0 to 1 in the category $\mathcal{BS}$. In particular, two such elements define a loop in $B\mathcal{S}$. This is what we will use fact to construct our map.

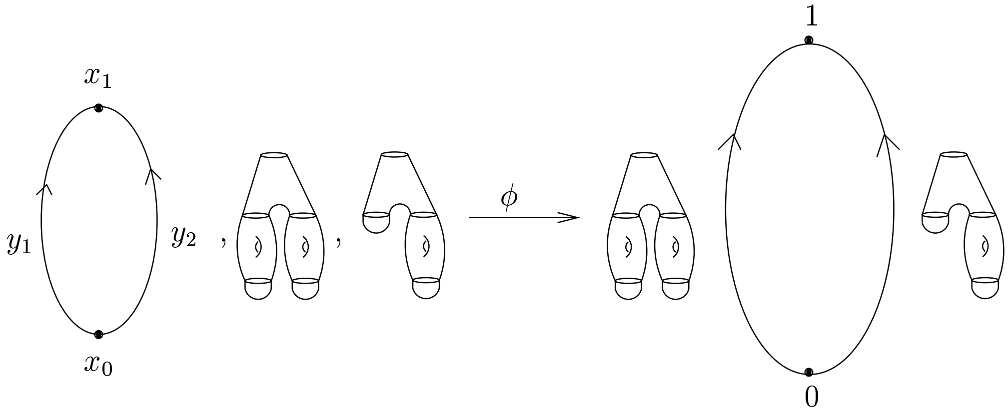


Figure 3.3: Map from $S^1 \wedge (M(*) \times M(*))_+$ to $B\mathcal{S}$

We construct first a bisimplicial map

$$\tilde{\phi}_{p,q} : S^1_p \times (M_q(*) \times M_q(*)) \longrightarrow B_{p,q}\mathcal{S},$$

where S^1 is viewed as a simplicial space with two 0-simplices x_0, x_1 and two non-degenerate 1-simplices y_1, y_2 (see figure 3.3), and $B\mathcal{S}$ is viewed as the bisimplicial space $B_{p,q}\mathcal{S} = B_p(\mathcal{B}_q\mathcal{S})$, where the second simplicial dimension is from the simplicial structure of the morphism sets. Let $\underline{F} = F_0 \xleftarrow{f_1} \dots \xleftarrow{f_q} F_q$ and $\underline{G} = G_0 \xleftarrow{g_1} \dots \xleftarrow{g_q} G_q$ be q-simplices of $M(*)$.
Define

$$\begin{aligned}
\tilde{\phi}_{0,q}(x_i, \underline{F}, \underline{G}) = &\quad i \qquad\qquad \text{for } i = 0, 1;\\
\tilde{\phi}_{1,q}(y_1, \underline{F}, \underline{G}) = &\quad 0 \xrightarrow{F} 1;\\
\tilde{\phi}_{1,q}(y_2, \underline{F}, \underline{G}) = &\quad 0 \xrightarrow{G} 1.
\end{aligned}$$

This induces a map

$$\phi : S^1 \wedge (M(*) \times M(*))_+ \longrightarrow B\mathcal{S},$$

where $\phi_q = \tilde{\phi}_{q,q}$ is given explicitly by

$$\phi_q(s_{j_{q-i}} \dots s_{j_1} z, \underline{F}, \underline{G}) = s_{j_{q-i}} \dots s_{j_1} \tilde{\phi}_{i,q}(z, \underline{F}, \underline{G}),$$

for any i-simplex z of S^1 with $i = 0, 1$, and $s_j : S^1_p \to S^1_{p+1}$ on the left hand side and $s_j : B_p(B_q\mathcal{S}) \to B_{p+1}(B_q\mathcal{S})$ on the right, with $1 \le p \le q - i$.

The basepoint of $M(*) \times M(*)$ is (D, D) and its image under $\tilde{\phi}$ is a contractible loop, but it is not actually the trivial loop. This is why we work with $(M(*) \times M(*))_+$ rather than $M(*) \times M(*)$.

Recall from [1] that for Γ there is an assembly map

$$a : A \wedge \Gamma(X) \to \Gamma(A \wedge X).$$

Combining a, ϕ and the projection of operads π, we get a map

$$\begin{tikzcd}
\mathcal{G}\Gamma(S^i \wedge M^p((M(*) \times M(*))_+)) \arrow[r, "f^i_p"] \arrow[d, "\mathcal{G}\Gamma(S^i \wedge \pi^p)"'] & \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathcal{S})) \\
\mathcal{G}\Gamma(S^i \wedge \Gamma^p((M(*) \times M(*))_+)) \arrow[r, "\mathcal{G}\Gamma(S^{i-1} \wedge a)"'] & \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(S^1 \wedge (M(*) \times M(*))_+)). \arrow[u, "\mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(\phi))"']
\end{tikzcd}$$

Proposition 3.3.1. *Let δ_j and s_j denote the boundary and degeneracy maps of both simplicial spaces E^i_* and F^i_*. Then the maps $f^i_p : E^i_p \to F^i_p$ satisfy*

$$\delta_j f^i_p = f^i_{p-1} \delta_j \qquad \text{for } \ 0 \le j < p$$

and

$$s_j f^i_p = f^i_{p+1} s_j \qquad \text{for } \ 0 \le j \le p.$$

Proof. The boundary maps δ_i, for $0 \le i < p$, and the degeneracies for E^i_* and F^i_* are defined in terms of the operad structure maps of $\mathcal{M}$ and Γ. The map f_* commutes with all of those maps because f maps $\mathcal{M}$ to Γ via an operad map. $\square$

The last commutation relation necessary to have a simplicial map, $\delta_p f_p = f_{p-1}\delta_p$ is not satisfied in our case, but it is satisfied up to homotopy. What happens is more easily understood by looking at figure 3.4. The last boundary map uses the $\mathcal{M}$-algebra structure of $M(*)\times M(*)$ in E^i_* and the Γ-structure of $B\mathcal{S}$ in F^i_*. When doing first δ_p, the elements of each copy of $M(*)$ are glued together by a surface F (the same in each component, shown as a pair of pants in the figure). If one starts by applying f^i_p, the surface F is sent to an element of Γ which will act by disjoint union. So we get two different loops in $B\mathcal{S}$, one going along morphisms defined by disjoint unions of the surfaces, the other going along the morphisms where the disjoint surfaces are glued to F. Now F also defines a morphism in $\mathcal{BS}$ which provides a homotopy between the two loops in $B\mathcal{S}$. Indeed, the loops are seen to be the boundaries of a couple of 2-simplices.

The other reason why this map is promising is that it would be a map of spectra if it was a map. We explain this in more details.

The spaces E_i and F_i for $i \geq 1$ form two spectra. This means that there are equivalences $\epsilon^i_E : E^i \xrightarrow{\simeq} \Omega E^{i+1}$ and $\epsilon^i_F : F^i \xrightarrow{\simeq} \Omega F^{i+1}$. We will denote both maps simply by ϵ^i. Both adjoints $\overline{\epsilon}^i$ can be expressed simplicially by maps

$$\overline{\epsilon}^i_p : \Sigma\mathcal{G}\Gamma(S^i \wedge M^p((M(*)\times M(*))_+)) \longrightarrow \mathcal{G}\Gamma(S^{i+1} \wedge M^p((M(*)\times M(*))_+))$$

$$\overline{\epsilon}^i_p : \Sigma\mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathcal{S})) \longrightarrow \mathcal{G}\Gamma(S^i \wedge \Gamma^p(B\mathcal{S})),$$

where both maps are induced by Barrat and Eccles assembly map $A \wedge \mathcal{G}\Gamma(X) \to \mathcal{G}\Gamma(A\wedge X)$. Note that this is possible because for any simplicial space X_*, we have $|\Sigma X_*| = \Sigma|X_*|$.

The following proposition is a direct consequence of the definitions of f^i_p and of the spectrum structure maps.

Proposition 3.3.2. *For $p \geq 0$ and $i \geq 1$, the following diagram commutes:*

$$\begin{tikzcd}
\Sigma\mathcal{G}\Gamma(S^i \wedge M^p((M(*)\times M(*))_+)) \arrow[r, "\Sigma f^i_p"] \arrow[d, "\overline{\epsilon}^i_p"] & \Sigma\mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(B\mathcal{S})) \arrow[d, "\overline{\epsilon}^i_p"] \\
\mathcal{G}\Gamma(S^{i+1} \wedge M^p((M(*)\times M(*))_+)) \arrow[r, "f^{i+1}_p"] & \mathcal{G}\Gamma(S^i \wedge \Gamma^p(B\mathcal{S}))
\end{tikzcd}$$

3.3.2 Rectification of the maps f_p^i

We show in this section how one can rectify the maps f_p^i to get simplicial maps $(f')_*^i$. For this, we use a special case of the theory of rectification of diagrams of Dwyer and Kan [6]. The idea is to look at a commutative diagram as a functor from a discrete category $\mathcal{D}$ to Top, and a not quite commutative diagram as a functor from a category $\tilde{\mathcal{D}}$ to Top, where the maps in $\tilde{\mathcal{D}}$ are "thicker" than in $\mathcal{D}$. The construction they use to rectify the functor is actually a form of double-sided bar construction. The same construction was used by Segal in [34] to compare his Γ-spaces to the operadic approach to infinite loop spaces. We give here a proof of the result we use (proposition 3.3.3) as we need to know more about it than what is spelled out in [6] or [34].

We then construct the categories $\mathcal{D}$ and $\tilde{\mathcal{D}}$ relevant to our situation and show in theorem 3.3.4 that we do have a functor from $\tilde{\mathcal{D}}$ to Top. The existence of a rectification follows immediately.

Formally, let $\mathcal{D}$ be a discrete category and let $\tilde{\mathcal{D}}$ be a category enriched over Top, which has the same objects as $\mathcal{D}$ and such that $\tilde{\mathcal{D}}(x,y) \simeq \mathcal{D}(x,y)$ for any pair of objects x, y.

The projection functor $p : \tilde{\mathcal{D}} \to \mathcal{D}$, which projects $\tilde{\mathcal{D}}(x,y)$ to $\mathcal{D}(x,y)$, induces a functor

$$\mathcal{D}^{\mathsf{Top}} \xrightarrow{p^*} \tilde{\mathcal{D}}^{\mathsf{Top}},$$

from $\mathcal{D}$-*diagrams* to $\tilde{\mathcal{D}}$-*diagrams*. The functor p^* has a left adjoint:

$$\mathcal{D}^{\mathsf{Top}} \xleftarrow{p_*} \tilde{\mathcal{D}}^{\mathsf{Top}},$$

where $(p_*F)(x)$ is defined as the realisation of a simplicial space whose nth space is

$$(p_*F)(x)_n = \coprod_{(y_0,\ldots,y_n)} F(y_0) \times \tilde{\mathcal{D}}(y_0,y_1) \times \ldots \times \tilde{\mathcal{D}}(y_{n-1},y_n) \times \mathcal{D}(p(y_n),x),$$

where the y_i, $0 \leq i \leq n$, are objects of $\tilde{\mathcal{D}}$ and x is in $\mathcal{D}$.

Two functors F and G are said to be *equivalent*, denoted $F \simeq G$, if there is a chain of natural transformations $F \longleftarrow \ldots \longrightarrow G$ inducing an equivalence $F(x) \simeq G(x)$ on each object x.

There is an equivalence of functors

$$p^* p_* F \simeq F$$

for any F in $\tilde{\mathcal{D}}^{\mathsf{Top}}$. As $\mathcal{D}$ and $\tilde{\mathcal{D}}$ have the same objects, it means in particular that $p_*F(x) \simeq F(x)$ for any object x. The functor p_*F is thus a "rectification" of F.

The equivalence above can be given by the following maps:

Proposition 3.3.3. *For a functor $F : \tilde{\mathcal{D}} \to \mathsf{Top}$, define the functor $\overline{p^*p_*F}$ from $\tilde{\mathcal{D}}$ to Top simplicially by*

$$(\overline{p^*p_*F})(y)_n = \coprod_{(y_0,\dots,y_n)} F(y_0) \times \tilde{\mathcal{D}}(y_0, y_1) \times \ldots \times \tilde{\mathcal{D}}(y_{n-1}, y_n) \times \tilde{\mathcal{D}}(y_n, y),$$

where the y_i's and y are objects of $\tilde{\mathcal{D}}$. Then there are natural transformations

$$p^*p_*F \longleftarrow \overline{p^*p_*F} \longrightarrow F$$

inducing equivalences $p^*p_*F(y) \simeq \overline{p^*p_*F}(y) \simeq F(y)$, for all y in $\tilde{\mathcal{D}}$.

Moreover, the natural transformations are natural in F.

Proof. The equivalence $\overline{p^*p_*F} \to p^*p_*F$ is clear because it is induced by the projection functor $\tilde{\mathcal{D}} \to \mathcal{D}$ which is a homotopy equivalence on the space of morphisms.

Now $\overline{p^*p_*F}$ is of the form of a two-sided bar construction $B(F, D, DX)$, which gives a resolution of FX. May gives an explicit simplicial homotopy for the equivalence $B(F, D, DX) \simeq FX$ in [27]. It can be adapted to our case. Indeed, consider the inclusion $i : F(y) \hookrightarrow \overline{p^*p_*F}(y)$ defined by $y \mapsto (y, id_y) \in F(y) \times \tilde{\mathcal{D}}(y, y)$, and the map $d : \overline{p^*p_*F}(y) \to F(y)$ defined by "doing all the boundary maps" and evaluating $F(y_0) \times \tilde{\mathcal{D}}(y_0, y) \to F(y)$, using the fact that F is a functor from $\tilde{\mathcal{D}}$ to Top. Clearly, $d \circ i = id$. We are left to show that $i \circ d$ is homotopic to the identity. This can been done by giving simplicial homotopies. Explicitly on q-simplices, the homotopies h_i for $i = 0, \dots q$ are given by

$$h_i = s_q \dots s_{i+1} \circ \eta \circ \delta_{i+1} \dots \delta_q,$$

where $\eta : (\overline{p^*p_*F})_i \to (\overline{p^*p_*F})_{i+1}$ is defined by adding $id_x \in \tilde{\mathcal{D}}(x, x)$ on the right of the simplex.

Both natural transformations are natural in F as the first one "doesn't touch F" and the second is by applying F to a morphism of $\tilde{\mathcal{D}}$, which is natural in F by definition of natural transformations. □

Note that the inclusion $i : F(y) \hookrightarrow \overline{p^*p_*F}(y)$ is not natural in y.

Let Δ^{op} denote the simplicial category, i.e. the objects of Δ^{op} are the natural numbers and there are maps $\delta_i : p \to p - 1$ and $s_i : p \to p + 1$ for each $i = 0, \dots, p$, satisfying the simplicial identities. So Δ^{op} is such that $(\Delta^{op})^{\mathsf{Top}}$ is the category of simplicial spaces. Note that any morphism in $\Delta^{op}(p,q)$ can be expressed uniquely as a sequence $s_{j_t}\dots s_{j_1}\delta_{i_s}\dots\delta_{i_1}$ with $0\leq i_s<\dots<i_1\leq p$ and $0\leq j_1<\dots<j_t\leq q$ and $q-t+s=p$ [28].

We will consider the category $\mathcal{D}$ whose $\mathcal{D}$-diagrams are precisely a couple of simplicial spaces with a simplicial map between them. Informally, we write

$$\mathcal{D}=[\Delta^{op}\longrightarrow\Delta^{op}].$$

More formally, $\mathcal{D}$ has two copies of the natural numbers as set of objects. We will denote those objects E_p and F_p for $p\in\mathbb{N}$ (E_p and F_p will be mapped later on to the spaces E_p^i and F_p^i defined earlier, hence the notation). The full subcategory of $\mathcal{D}$ containing E_p for each p is isomorphic to Δ^{op}. So $\mathcal{D}(E_p,E_q)=\Delta^{op}(p,q)$, and similarly for the F_p's. Finally, there is a unique map $f_p\in\mathcal{D}(E_p,F_p)$ which satisfies the simplicial identities $\delta_i f_p=f_{p-1}\delta_i$ and $s_i f_p=f_{p+1}s_i$ for $i=0,\dots,p$. So any morphism in $\mathcal{D}$ from E_p to F_q can be written uniquely as a sequence $s_{j_t}\dots s_{j_1}\delta_{i_s}\dots\delta_{i_1}f_p$ where the indices are as above.

We now define a category $\tilde{\mathcal{D}}$ whose diagrams are a "relaxed" version of $\mathcal{D}$-diagrams. $\tilde{\mathcal{D}}$ is in fact defined in such a way that the data given at the beginning if this section will induce a functor from $\tilde{\mathcal{D}}$ to Top.

Informally,

$$\tilde{\mathcal{D}}=[\Delta^{op}\Longrightarrow\Delta^{op}],$$

by which we mean that we have again two copies of Δ^{op} as full subcategories, but now we have "thicker" maps between the two sides. Formally, $\tilde{\mathcal{D}}$ has the same objects as $\mathcal{D}$ and we will again denote them by E_p and F_p. The spaces of morphisms between the E_p's and the F_p's are as before: $\tilde{\mathcal{D}}(E_p,E_q)=\mathcal{D}(E_p,E_q)$ and $\tilde{\mathcal{D}}(F_p,F_q)=\mathcal{D}(F_p,F_q)$. To define $\tilde{\mathcal{D}}(E_p,F_q)$, we first define the *degeneracy degree* $d(g)$ of a morphism $g\in\mathcal{D}(E_p,F_q)$. Let $g=s_{j_t}\dots s_{j_1}\delta_{i_s}\dots\delta_{i_1}f_p$. Then $d(g)$ is the biggest k such that $i_k=p-k+1$, and $d(g)=0$ if no such k exists. In other words, d counts the number of "bad" maps, i.e. last boundary maps, occurring in g.

Now define

$$\tilde{\mathcal{D}}(E_p,F_q)=\coprod_{g\in\mathcal{D}(E_p,F_q)}\Delta_{d(g)},$$

where Δ_d is the standard d-simplex

$$\Delta_d=\{(t_d,\dots,t_0)|t_i\geq 0,\Sigma t_i=1\},$$

which we write backwards to fit with the indices of the δ_i's. For each $g \in \mathcal{D}(E_p, F_q)$, there is a bijection

$$\tilde{g} : \Delta_{d(g)} \longrightarrow \tilde{\mathcal{D}}(E_p, F_q)$$

whose image is $p^{-1}(g)$, the space of morphisms "sitting over g".

Composition in $\tilde{\mathcal{D}}$ is simplicial and is best understood by describing what happens on the vertices.

Recall that all the simplicial identities between δ_i's and s_j's are satisfied in $\tilde{\mathcal{D}}$. Also, note that there is a unique map (0-simplex) between E_p and F_p sitting over the simplicial map $f_p \in \mathcal{D}(E_p, F_p)$. We denote this map again by f_p and we set the relation $s_i f_p = f_{p+1} s_i$ for $0 \leq i \leq p$ and $\delta_i f_p = f_{p-1}\delta_i$ for $0 \leq i < p$. Because the last relation, when $i = p$, does not hold in $\tilde{\mathcal{D}}$, there are exactly $d(g)+1$ maps formed of compositions of δ_i's, s_j's and an f_k projecting down to $g = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_1} f_p$ in $\mathcal{D}$. We define those compositions to be the vertices of $\Delta_{d(g)}$. More precisely, for $0 \leq k \leq d(g)$, define

$$g_k := s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{k+1}} f_{p-k} \delta_{i_k} \dots \delta_{i_1} := \tilde{g}((0, \dots, 1, \dots, 0)),$$

where 1 is in the kth position counting backwards.

Note that the edges of the simplex then represent the homotopies between those maps, i.e. the relations $\delta_{i_{k+1}} f_{p-k} = \delta_{p-k} f_{p-k} \simeq f_{p-k-1}\delta_{p-k}$. The rest of the simplex give the higher coherence, making the space of maps over g contractible.

Now composition in $\tilde{\mathcal{D}}$ is determined by the vertices of the simplices. Indeed, a map in $\tilde{\mathcal{D}}(E_p, F_q)$ can only be pre-composed by a map in $\tilde{\mathcal{D}}(E_r, E_p)$ or post-composed by a map in $\tilde{\mathcal{D}}(F_q, F_s)$. Using the identities given above, we know how those compositions are defined on the vertices of the simplices of $\tilde{\mathcal{D}}(E_p, F_q)$. We then extend the composition simplicially. We give here more details of what happens, in a case by case description.

Consider $g, \tilde{g}$ and g_k as above and set $d = d(g)$.

Composition with $s_j : E_{p-1} \to E_p$.

By definition,

$g_k s_j = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{k+1}} f_{p-k} \delta_{i_k} \dots \delta_{i_1} s_j,$

$\quad = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{d+1}} \delta_{p-d+1} \delta_{p-d+2} \dots \delta_{p-k} f_{p-k} \delta_{p-k+1} \dots \delta_p s_j.$

We want to write $g_k s_j$ in canonical form to know in which simplex it lies.

CASE 1: $j \geq p - d$.

As $\delta_i s_j = s_j \delta_{i-1}$ for $i > j+1$, $\delta_{j+1} s_j = id$ and $f_k s_j = s_j f_{k-1}$, we can move s_j to the left until it meets δ_{j+1}, which it kills, having changed all the previous δ_l to δ_{l-1}. We thus have

$g_k s_j = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{d+1}} \delta_{p-d+1} \delta_{p-d+2} \dots \delta_{p-k} f_{p-k} \delta_{p-k+1} \dots \delta_{p-1}$

if $j \geq p-k$, and

$g_k s_j = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{d+1}} \delta_{p-d+1} \delta_{p-d+2} \dots \delta_{p-k-1} f_{p-k-1} \delta_{p-k} \dots \delta_{p-1}$

if $p-d \leq j < p-k$.

Hence the map $\tilde{g}(\Delta_d) \xrightarrow{.s_j} g.\tilde{s}_j(\Delta_{d-1})$ is the jth degeneracy $\Delta_d \to \Delta_{d-1}$ (if $j = p-l$, g_l and g_{l-1} have the same image). In our example, pre-composing by s_j will actually give a degenerate d-simplex, so the map will be the obvious one.

CASE 2: $j < p-d$.

In this case, while moving s_j down to the left, either s_j will meet δ_j or δ_{j+1} and kill it, or it will take its place with the other s_l's. Thus

$g_k s_j = s_{j'_{t'}} \dots s_{j'_1} \delta_{i'_{s'}} \dots \delta_{i'_{d+1}} \delta_{p-d} \delta_{p-d+1} \dots \delta_{p-k-1} f_{p-k-1} \delta_{p-k} \dots \delta_{p-1}$,

where either $t' = t$ and $s' = s-1$ or $t' = t+1$ and $s' = s$. Note that $d(gs_j) = d$ as $i'_{d+1} = (i_{d+1} - 1$ or $i_{d+2})$. So the map $\tilde{g}(\Delta_d) \xrightarrow{.s_j} g\tilde{s}_j(\Delta_d)$ is the identity on the d-simplex.

Composition with $\delta_j : E_{p+1} \to E_p$.

CASE 1: $j \geq p-d+1$.

In this case, δ_j brings one more "bad" map, so the degeneracy degree increases. The map $\tilde{g}(\Delta_d) \xrightarrow{.\delta_j} g\tilde{\delta}_j(\Delta_{d+1})$ is the jth face map (if $j = p-l$, the map does not hit $(g\delta_j)_l$).

CASE 2: $j < p-d+1$.

Now $g_k \delta_j = s_{j_t} \dots s_{j_1} \delta_{i'_{s+1}} \dots \delta_{i'_{d+1}} \delta_{p-d+2} \dots \delta_{p-k+1} f_{p-k+1} \delta_{p-k+2} \dots \delta_{p+1}$,

where i'_{d+1} is either j or $i_{d+1}+1$, so the degeneracy degree does not increase. The map $\tilde{g}(\Delta_d) \xrightarrow{.\delta_j} g\tilde{\delta}_j(\Delta_d)$ is the identity on the d-simplex.

Composition with $s_j : F_q \to F_{q+1}$.

One gets $\tilde{g}(\Delta_d) \xrightarrow{s_j.} \tilde{s_j g}(\Delta_d)$ is the identity on the d-simplex.

Composition with $\delta_j : F_q \to F_{q-1}$.

Here, while moving δ_j to the right, two things can happen: either δ_j gets killed by an s_l before reaching its place among the δ_i's, or it reaches its place. In the last situation, either the degeneracy degree go up by one, or it does not. Hence either one gets an identity on the d-simplex, or a face map.

Theorem 3.3.4. *Let $\tilde{\mathcal{D}}$ be the category defined above. For each $i \geq 0$, there is a functor*

$$L_i : \tilde{\mathcal{D}} \longrightarrow \mathsf{Top}$$

such that $L_i(E_*) = E_*^i$, $L_i(F_*) = F_*^i$, where E_* and F_* denote the two subcategories of $\tilde{\mathcal{D}}$ isomorphic to Δ^{op}, and $L_i(f_p) = f_p^i$.

Proof. As E_*^i and F_*^i are simplicial spaces, the restriction of L_i to each copy of Δ^{op} in $\tilde{\mathcal{D}}$ is a well-defined functor. As the map f_p^i satisfies the identities satisfied by f_p with the boundary and degeneracy maps, L_i is also well defined on the vertices of the simplices of the morphism spaces $\tilde{\mathcal{D}}(E_p, F_q)$. We have to show that we can extend the definition of L_i to the whole simplices. To do this, we will exhibit the edges of a generic simplex. It will then be clear from the structure of $B\mathcal{S}$ that we can fill in the simplex: as $B\mathcal{S}$ is the realisation of the nerve of a category, it is made out of simplices. We will just map simplices to simplices.

Let $d = d(g)$ and consider
$g_k = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{k+1}} f_{p-k} \delta_{i_k} \dots \delta_{i_1}$ and
$g_l = s_{j_t} \dots s_{j_1} \delta_{i_s} \dots \delta_{i_{k+1}} \delta_{i_k} \dots \delta_{i_{l+1}} f_{p-l} \delta_{i_l} \dots \delta_{i_1}$.
To simplify the writing, set

$$X = (M(*) \times M(*))_+ \quad \text{and} \quad A(Y) = \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^{p-d}(Y)).$$

Consider the following diagram:

$$\begin{tikzcd}
E^i_p \arrow[d, "\mathcal{G}\Gamma(S^i\wedge\pi^{p-d})"] & & \\
\mathcal{G}\Gamma(S^i\wedge\Gamma^{p-d}M^d(X)) \arrow[d, "a"] & & \\
A(S^1\wedge M^d(X)) \arrow[d, "\delta_{i_l}\ldots\delta_{i_1}"] & & \\
A(S^1\wedge M^{d-l}(X)) \arrow[r, "A(a\circ\pi^d)"] \arrow[d, "\delta_{i_k}\ldots\delta_{i_{l+1}}"] & A(\Gamma^{d-l}(S^1\wedge X)) \arrow[r, "A\Gamma(\phi)"] & A(\Gamma^{d-l}(B\mathcal{S})) \arrow[d, "\delta_{i_k}\ldots\delta_{i_{l+1}}"] \\
A(S^1\wedge M^{d-k}(X)) \arrow[r, "A(a\circ\pi^{d-k})"] & A(\Gamma^{d-k}(S^1\wedge X)) \arrow[r, "A\Gamma(\phi)"] & A(\Gamma^{d-k}(B\mathcal{S})) \arrow[d, "A(\delta_{i_d}\ldots\delta_{i_{k+1}})"] \\
& & A(B\mathcal{S}) = F^i_{p-d} \arrow[d, "s_{j_t}\ldots\delta_{i_{d+1}}"] \\
& & F^i_q.
\end{tikzcd}$$

The maps g_k and g_l are obtained by following the bottom and the top of the square respectively. So for each k, g_k can be factorised as

$$g_k : E^i_p \xrightarrow{\alpha} A(S^1\wedge M^d(X)) \xrightarrow{A(h_k)} A(B\mathcal{S}) = F^i_{p-d} \xrightarrow{\beta} F^i_q,$$

with α and β independent of k.

We first construct a simplex of maps $S^1\wedge M^d(X)\to B\mathcal{S}$ having the maps h_k as vertices. For $0\le l<k\le d$, h_k and h_l are given by the two sides of the following diagram:

$$\begin{tikzcd}
S^1\wedge M^{d-k}M^{k-l}(X) \arrow[r] \arrow[d, "\delta_{i_k}\ldots\delta_{i_{l+1}}"] & \Gamma^{d-k}\Gamma^{k-l}(S^1\wedge X) \arrow[r] & \Gamma^{d-k}\Gamma^{k-l}(B\mathcal{S}) \arrow[d, "\delta_{i_k}\ldots\delta_{i_{l+1}}"] & \\
M^{d-k}(S^1\wedge X) \arrow[r] & \Gamma^{d-k}(S^1\wedge X) \arrow[r] & \Gamma^{d-k}(B\mathcal{S}) \arrow[r] & B\mathcal{S}
\end{tikzcd}$$

Note that in what follows, when we say "surface", we generally mean a chain of surfaces and morphisms between them, which are elements of the mapping class groups. As earlier, we use the notation $\underline{F}$ for such a chain. Gluing surfaces then means "gluing" those elements of the mapping class group as well. This operation induced on the mapping class group by gluing the surfaces gives both the operad structure map of $\mathcal{M}$ and the composition in $\mathcal{BS}$.

We think of $M^d(X)$ as being formed of $d+1$ levels $d, \dots, 0$: level d having the surface coming from the M the most on the left, level $d-1$ being composed of m_{d-1} surfaces coming from the second M, and so on up to level 0 which is composed of m_0 elements of X. In the above diagram, we start with the levels $l, \dots, 0$ glued to X. The map $\delta_{i_k} \dots \delta_{i_{l+1}}$ on the left then glues similarly the levels $k, \dots, l+1$, then, following the bottom of the diagram, h_k takes the disjoint union of the surfaces thus obtained, producing a couple of morphisms $\underline{F}_{k,0}, \underline{G}_{k,0}$ from 0 to m_k in $\mathcal{BS}$. If, on the other hand, we follow the top of the diagram, we obtain the map h_l which, once the levels $l, \dots, 0$ glued to X, sends the surfaces to their disjoint union, producing two morphisms $\underline{F}_{l,0}, \underline{G}_{l,0}$ from 0 to m_l in $\mathcal{BS}$.

Now the surface $\underline{H}_{k,l}$ obtained by gluing together the levels $k, \dots, l+1$ gives a morphism from m_l to m_k in $\mathcal{BS}$ which is such that $\underline{H}_{k,l} \circ \underline{F}_{l,0} = \underline{F}_{k,0}$ and $\underline{H}_{k,l} \circ \underline{G}_{l,0} = \underline{G}_{k,0}$. This produces two 2-simplices in $B\mathcal{S}$ (see figure 3.4).

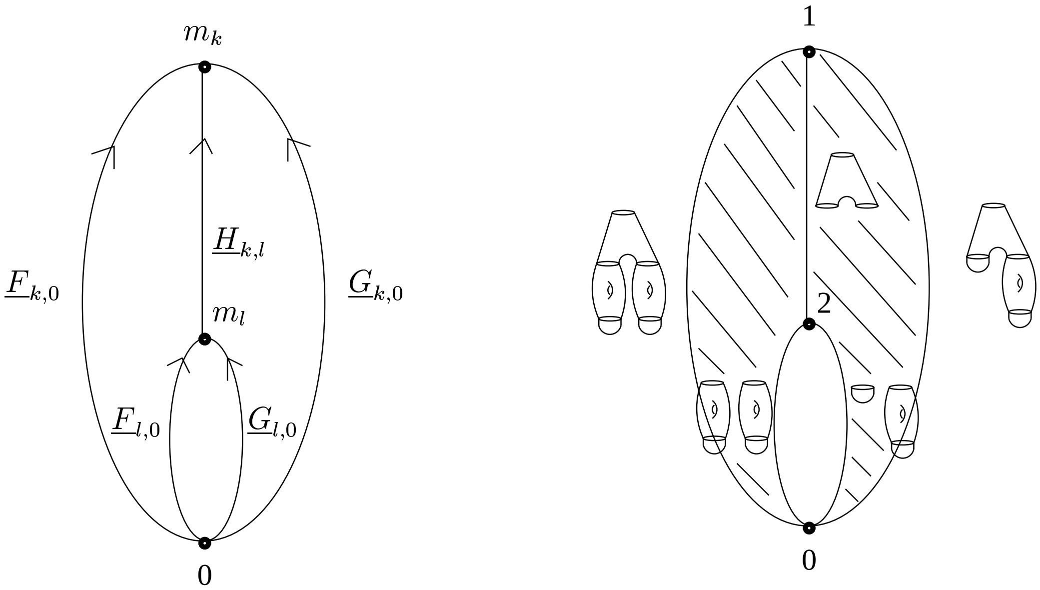


Figure 3.4: Homotopy $H_{k,l}$

Now all the surfaces $\underline{H}_{k,l}$, for $0 \le l < k \le d$, fit together as the edges of the d-simplex in $B\mathcal{S}$

$$m_d \xleftarrow{\underline{F}_d} m_{d-1} \longleftarrow \dots \xleftarrow{\underline{F}_1} m_0,$$

where $\underline{F}_i$ is the disjoint union of the surfaces of level i. Define

$$h_{k,l} : I \times S^1 \wedge M^d(X) \longrightarrow B\mathcal{S}$$

by

$$\begin{array}{rcccl} h_{k,l}(I,x_0,\mathbf{F}) & = & 0, & & \\ h_{k,l}(I,x_1,\mathbf{F}) & = & \underline{H}_{k,l}, & & \\ h_{k,l}(0,S^1,\mathbf{F}) & = & g_k(S^1,\mathbf{F}) & = & \underline{F}_{k,0} \between \underline{G}_{k,0} \\ h_{k,l}(1,S^1,\mathbf{F}) & = & g_l(S^1,\mathbf{F}) & = & \underline{F}_{l,0} \between \underline{G}_{l,0}, \end{array}$$

where $\mathbf{F} = (\underline{F}_d, \dots, \underline{F}_1, (\underline{F}_0, \underline{G}_0))$, and extending simplicially (by filling the two 2-simplices as in figure 3.4).

Now all the images of the maps $h_{k,l}$ are 2-dimensional faces of a couple of $(d+1)$-simplices:

$$m_d \xleftarrow{\underline{F}_d} \dots \xleftarrow{\underline{F}_1} m_0 \xleftarrow{\underline{F}_0} 0$$

and

$$m_d \xleftarrow{\underline{F}_d} \dots \xleftarrow{\underline{F}_1} m_0 \xleftarrow{\underline{G}_0} 0.$$

So we can extend the above to a map

$$\tilde{h} : \Delta_d \times S^1 \wedge M^d(X) \longrightarrow B\mathcal{S}$$

with the 1-skeleton of Δ_d given by the maps $h_{k,l}$ (we are forming a simplex of maps $h_d \xrightarrow{h_{d-1,d}} \dots \xrightarrow{h_{1,0}} h_0$) and with

$$\tilde{h}(\Delta_d, y_1, \mathbf{F}) = m_d \xleftarrow{\underline{F}_d} \dots \xleftarrow{\underline{F}_1} m_0 \xleftarrow{\underline{F}_0} 0$$

$$\tilde{h}(\Delta_d, y_2, \mathbf{F}) = m_d \xleftarrow{\underline{F}_d} \dots \xleftarrow{\underline{F}_1} m_0 \xleftarrow{\underline{G}_0} 0.$$

The is determined by the image of the 1-skeleton of Δ_d.

We finally obtain the map

$$\tilde{g} : \Delta_d \times E^i_p \longrightarrow F^i_q$$

by composing

$$\Delta_d \times E^i_p \xrightarrow{id\times\alpha} \Delta_d \times A(S^1 \wedge M^d(X)) \xrightarrow{a} A(\Delta_d \times S^1 \wedge M^d(X)) \xrightarrow{A(\tilde{h})} A(B\mathcal{S}) \xrightarrow{\beta} F^i_q,$$

where a is the assembly map for A, defined pointwise (each point of Δ_d defines a map $A(S^1 \wedge M^d(X)) \to A(\Delta_d \times S^1 \wedge M^d(X))$). This gives a continuous map as A is a continuous functor. □

Using proposition 3.3.3, we have the following corollary:

Corollary 3.3.5. *There exist simplicial spaces $(E')^i_*$ and $(F')^i_*$ equivalent to the simplicial spaces E^i_* and F^i_* and a simplicial map $(f')^i_* : (E')^i_* \to (F')^i_*$ such that the following diagram commutes:*

$$\begin{tikzcd} E^i_p \arrow[r, "f_p"] & F^i_p \\ (E')^i_p \arrow[u, leftrightarrow, "\simeq"] \arrow[r, "(f')^i_p"] & (F')^i_p \arrow[u, leftrightarrow, "\simeq"'] \end{tikzcd}$$

In particular, $|(E')^i_*| \simeq |E^i_*|$ and $|(F')^i_*| \simeq |F^i_*|$.

Proof. Let $L'_i = p_*(L_i) : \mathcal{D} \to \mathsf{Top}$ be the rectification of the functor L_i defined in theorem 3.3.4. Denote by $(E')^i_p$, $(F')^i_p$ and $(f')^i_p$ the images of E_p, F_p and f^i_p via L'_i. By definition of the category $\mathcal{D}$, $(E')^i_*$ and $(F')^i_*$ are simplicial spaces and $(f')^i_* : (E')^i_* \to (F')^i_*$ is a simplicial map.

We know that $p^*L'_i \simeq L_i : \tilde{\mathcal{D}} \to \mathsf{Top}$. Now note that $p^*(L'_i)\big|_{\Delta^{op}} = (L'_i)\big|_{\Delta^{op}} : \Delta^{op} \to \mathsf{Top}$, for each copy of Δ^{op} in $\tilde{\mathcal{D}}$ and the corresponding one in $\mathcal{D}$, as p is the identity on those subcategories. So $L'_i\big|_{\Delta^{op}} \simeq L_i\big|_{\Delta^{op}}$, which precisely says that the simplicial spaces $(E')^i_*$ and $(F')^i_*$ are equivalent to E^i_* and F^i_* respectively.

Lastly, as $p^*L'_i(f_p) = (f')^i_p$, the diagram given in the corollary commutes by naturality of the equivalence. □

3.3.3 Map of spectra

We have now constructed maps $(f')^i : (E')^i \to (F')^i$ for $i \geq 1$, where the families of spaces $(E')^i$ and $(F')^i$ are equivalent to families of spaces E^i and F^i, both of which form a spectrum. In this section, we will give a spectrum structure on the $(E')^i$'s and $(F')^i$'s, and show that the maps $(f')^i$ give a map of spectra. To obtain this result, we use proposition 3.3.2 and the naturality of the rectification. We need two lemma.

We denote by the same letter ϵ^i the maps $E^i \xrightarrow{\simeq} \Omega E^{i+1}$ and $F^i \xrightarrow{\simeq} \Omega F^{i+1}$ and denote by $\overline{\epsilon}^i$ both adjoints, which are given simplicially by

$$\overline{\epsilon}^i_p : \Sigma E^i_p \longrightarrow E^{i+1}_p \quad \text{and} \quad \overline{\epsilon}^i_p : \Sigma F^i_p \longrightarrow F^{i+1}_p$$

(see definition before proposition 3.3.2).

Lemma 3.3.6. *For each $i \geq 1$, the two sequences of maps $\overline{\epsilon}^i_p$ induce a natural transformation of functors*

$$\overline{\epsilon}^i : \Sigma L_i \longrightarrow L_{i+1}.$$

Proof. Recall that ΣL_i and L_{i+1} are functors from $\tilde{\mathcal{D}}$ to Top, and $\tilde{\mathcal{D}}$ is the category $[\Delta^{op} \Longrightarrow \Delta^{op}]$. We already know that the $\overline{\epsilon}_p^i$ form a couple of simplicial maps and commute with the maps f_p (proposition 3.3.2). So we just need to check that the $\overline{\epsilon}_p^i$ commute with all the homotopies. This is clear because the map $\overline{\epsilon}_p^i$ is induced by an assembly map $\Sigma\mathcal{G}\Gamma X \to \mathcal{G}\Gamma\Sigma X$, which is natural in X [1]. □

The last result will give us a first step towards constructing the spectrum structure on the $(E')^i$'s and $(F')^i$'s, as well as showing that the rectification of f gives a map of spectra. We need however another lemma which will allow us to go from the rectification of ΣL_i to the suspension of the rectification of L_i.

Lemma 3.3.7. *For any functor F, there are natural transformations*

$$\beta : \Sigma(p^* p_* F) \longrightarrow p^* p_*(\Sigma F)$$

$$\overline{\beta} : \Sigma(\overline{p^* p_* F}) \longrightarrow \overline{p^* p_*(\Sigma F)}$$

such that the following diagram is commutative

$$\begin{tikzcd}
\Sigma(p^* p_* F) \arrow[d, "\beta"'] & \Sigma(\overline{p^* p_* F}) \arrow[l, "\Sigma(\simeq)"'] \arrow[d, "\overline{\beta}"'] \arrow[dr, "\Sigma(\simeq)"] & \\
p^* p_*(\Sigma F) & \overline{p^* p_*(\Sigma F)} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \Sigma F,
\end{tikzcd}$$

where $\overline{p^* p_* F}$ was defined in proposition 3.3.2.

In particular, β is an equivalence.

Proof. There is an obvious map $(\Sigma(p_* F)(x))_n \to (p_*(\Sigma F)(x))_n$ inducing β as $(\Sigma(p_* F)(x))_n = \coprod S^1 \wedge (F(y_0) \times \ldots \times \mathcal{D}(y_n, x))$ and $(p_*(\Sigma F)(x))_n = \coprod (S^1 \wedge F(y_0)) \times \ldots \times \mathcal{D}(y_n, x))$. The map collapses a contractible subspace of the first space. A map $\overline{\beta}$ is defined similarly.

One then checks that the diagram commutes. □

We can now prove the following theorem:

Theorem 3.3.8. *There are maps $(\lambda)^i : (E')^i \xrightarrow{\simeq} \Omega(E')^{i+1}$ and $(\lambda)^i : (F')^i \xrightarrow{\simeq} \Omega(F')^{i+1}$ giving a spectrum structure on both $(E')^i$ and $(F')^i$ for $i \geq 1$ such that the maps $(f')^i : (E')^i \to (F')^i$ form a map of spectra.*

Proof. The map p_* is a natural transformation $\mathsf{Top}^{\tilde{\mathcal{D}}} \to \mathsf{Top}^{\mathcal{D}}$. By lemma 3.3.6, we have a natural transformation $\Sigma L_i \xrightarrow{\bar{\epsilon}} L_{i+1}$. We denote by $(\Sigma L_i)' \xrightarrow{\bar{\epsilon}'} L'_{i+1}$ its image under p_*.

Define the maps $\overline{\lambda}$ below to be $\bar{\epsilon}' \circ \beta$ and consider the following diagram:

$$
\begin{tikzcd}
\Sigma(E')^i_p \arrow[rrrrr, "\Sigma(f')^i_p"] \arrow[dr, "\beta"'] \arrow[drr, leftrightarrow, "\Sigma(\simeq)"] \arrow[dddr, dashed, "\overline{\lambda}"'] & & & & & \Sigma(F')^i_p \arrow[dl, "\beta"] \arrow[dll, leftrightarrow, "\Sigma(\simeq)"'] \arrow[dddl, dashed, "\overline{\lambda}"] \\
& (\Sigma E')^i_p \arrow[r, leftrightarrow, "\simeq"] \arrow[dd, "\bar{\epsilon}'"] & \Sigma E^i_p \arrow[r, "\Sigma f^i_p"] \arrow[d, "\bar{\epsilon}"] & \Sigma F^i_p \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\bar{\epsilon}"'] & (\Sigma F')^i_p \arrow[dd, "\bar{\epsilon}'"'] & \\
& & E^{i+1}_p \arrow[r, "f^{i+1}_p"] & F^{i+1} \arrow[dr, leftrightarrow, "\simeq"] & & \\
& (E')^{i+1}_p \arrow[ur, leftrightarrow, "\simeq"] \arrow[rrr, "(f')^{i+1}_p"] & & & (F')^{i+1}_p &
\end{tikzcd}
$$

This diagram commutes by proposition 3.3.2 for commutation of the square in the centre, proposition 3.3.3 (naturality in F of the equivalence $p^*p_*F \simeq F$) for the commutation of the left and right squares, because the equivalence is a natural transformation of functors for the top and bottom squares, and by lemma 3.3.7 for the two triangles.

Now the adjoint of $\overline{\lambda}$ are equivalences as

$$
\begin{tikzcd}
\Sigma(E')^i_p \arrow[r, "\Sigma(\simeq)"] \arrow[d, "\overline{\lambda}^i_p"'] & \Sigma E^i_p \arrow[d, "\bar{\epsilon}^i_p"] \\
(E')^{i+1}_p \arrow[r, "\simeq"] & E^{i+1}_p
\end{tikzcd}
\qquad \Longrightarrow \qquad
\begin{tikzcd}
(E')^i_p \arrow[r, "\simeq"] \arrow[d, "\lambda^i_p"'] & E^i_p \arrow[d, "\epsilon^i_p"] \\
\Omega(E')^{i+1}_p \arrow[r, "\Omega(\simeq)"] & \Omega E^{i+1}_p
\end{tikzcd}
$$

the commutation of the left diagram implies the commutation of the right one, and the maps ϵ^i are equivalences. $\square$

3.3.4 Equivalence

We want to show in this section that the map of spectra f' is an equivalence. As our spectra are connective, it is enough to show that the first $(f')^i$ is an equivalence. We defined f^1, and so $(f')^1$, on the first deloop of $\mathbb{Z} \times B\Gamma^+_\infty$. Define $(f')^0$ to be

$$(f')^0 = \Omega(f')^1 : (E')^0 := \Omega(E')^1 \longrightarrow (F')^0 := \Omega(F')^1.$$

Note that $\Omega(E')^1 \simeq \mathbb{Z} \times B\Gamma^+_\infty \simeq \Omega(F')^1$. We will show that $(f')^0$ induces an equivalence.

Lemma 3.3.9. *There are maps $\phi_p : M(*) \to \mathcal{G}\Gamma(M^p((M(*) \times M(*))_+))$ and $\psi : M(*) \to \Omega B\mathcal{S}$ such that the following diagram commutes.*

$$\begin{tikzcd}
(E')^1_p \arrow[rr, "(f')^1_p"] \arrow[d, leftrightarrow, "\simeq"'] & & (F')^1_p \arrow[d, leftrightarrow, "\simeq"] \\
E^1_p = \mathcal{G}\Gamma(S^1 \wedge M^p((M(*) \times M(*))_+)) \arrow[rr, "f^1_p"] & & \mathcal{G}\Gamma(\Gamma^p(B\mathcal{S})) = F^1_p \arrow[d] \\
\Sigma\mathcal{G}\Gamma(M^p((M(*) \times M(*))_+)) \arrow[u] & & B\mathcal{S} \\
& \Sigma M(*) \arrow[ul, "\Sigma(\phi_p)"] \arrow[ur, "\overline{\psi}"'] &
\end{tikzcd}$$

Proof. Define $\phi_p : M(*) \to \mathcal{G}\Gamma(M^p((M(*) \times M(*))_+))$ by $\underline{F} \mapsto (1, \ldots, 1, (\underline{F}, D))$, where D is the disc. And define ψ to be the map which sends an element $\underline{F}$ of $M(*)$ to the loop in $B\mathcal{S}$ going from 0 to 1 along the morphism defined by $\underline{F}$, and back to 0 along the morphism defined by D, taking it backwards. So

$$\psi : \underline{F} \mapsto \underline{F} \binom{1}{0} D.$$

One then checks that the bottom part of the diagram commutes. The top part commutes by naturality of the equivalence. $\square$

Theorem 3.3.10. *The map of spectra $f' : (E') \to (F')$ is an equivalence.*

Proof. Thinking of $M(*)$ as a constant simplicial space, lemma 3.3.9 yields a commutative diagram of simplicial spaces (with no map from E^1_* to F^1_*). Taking adjoints, we get a commutative diagram

$$\begin{tikzcd}
\Omega(E')^1 \arrow[rr, "\Omega(f')^1_p"] \arrow[d, leftrightarrow, "\Omega(\simeq)"'] & & \Omega(F')^1 \arrow[d, leftrightarrow, "\Omega(\simeq)"] \\
\Omega E^1 = \Omega|\mathcal{G}\Gamma(S^1 \wedge M^p((M(*) \times M(*))_+))| & & \Omega|\mathcal{G}\Gamma(\Gamma^p(B\mathcal{S}))| = \Omega F^1 \arrow[d, "\Omega(\simeq)"] \\
|\mathcal{G}\Gamma(M^*((M(*) \times M(*))_+))| \arrow[u, "\simeq"] & & \Omega B\mathcal{S} \\
& M(*) \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Now the map ϕ induces the equivalence $\mathcal{G}M(*) \xrightarrow{\simeq} \mathcal{G}\Gamma(M^*((M(*) \times M(*))_+))$ (see 3.1.2).

Also, $\psi : M(*) \simeq \coprod_{g\geq 0} B\Gamma_{g,1} \to \Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$ induces the equivalence $\mathcal{G}M(*) \xrightarrow{\simeq} \Omega B\mathcal{S}$. Indeed, ψ sends the gth component to the gth component (this can be calculated by adding up numbers involving genus and boundary components as explained in [37]). And it is the group completion map, generalised to categories.

So we get a commutative diagram

$$\begin{tikzcd}
(E')^0 \arrow[rr, "(f')^0"] \arrow[d, leftrightarrow, "\simeq"'] & & (F')^0 \arrow[d, leftrightarrow, "\simeq"] \\
G & M(*) \arrow[l, "\phi"'] \arrow[r, "\psi"] \arrow[d] & H \\
& \mathcal{G}M(*) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where G and H are group completed spaces. Hence $(f')^0$ is an equivalence. Now the spectra (E') and (F') are connective. Indeed, E^i and F^{i+1} are connected for $i \geq 1$ because $\mathcal{G}\Gamma(X)$ is connected when X is connected and in both case we have a suspension. Also, $F^1 \simeq B\mathcal{S}$ is connected. The result follows from the fact that a map of connective spectra is an equivalence if it is an equivalence on the 0th spaces. $\square$

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