Some computations for the exceptional groups relevant to the classification of p-compact groups

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Ph.D. thesis

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Dear reader,

Except for some minor changes the present manuscript constitutes my Ph.D. thesis in Mathematics as submitted to the Faculty of Science at the University of Copenhagen. The thesis is written under the supervision of Jesper Michael Møller, who I would like to thank for his constant optimism, interest and support.

The manuscript consists of three independent parts. The first part forms the core of a forthcoming joint paper with Jesper Grodal, Jesper Møller and Antonio Viruel. The last two parts are papers in preparation.

Kasper Klinkby Sonne Andersen
Copenhagen, June 2001

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Part I

Elementary abelian subgroups of the exceptional simple groups

Chapter 1

Elementary abelian subgroups of the exceptional simple groups

1.1 Introduction and notation

Let G be one of the exceptional simple complex Lie groups, i.e. one of type G_2, F_4, E_6, E_7 or E_8. Let p be a prime number. For reasons explained below, we are interested in determining the conjugacy classes of elementary abelian p-subgroups E of G. These may conveniently be divided in two classes, the *toral* ones and the *nontoral* ones. Here, an abelian subgroup A is called toral if A is contained in a torus of G and nontoral if it is not. If H is a maximal torus of G, then any two subsets of H are conjugate in G if and only if they are conjugate in the normalizer of the maximal torus $N_G(H)$ (see Lemma (1.1.3(4))). Since the action of the Weyl group $W = N_G(H)/H$ of G on H is well known, this makes the determination of the conjugacy classes of toral subgroups routine. In fact, in most cases these are the only elementary abelian p-subgroups:

(**1.1.1**) **Proposition**. Assume that $p \geq 3$ if G has type G_2, $p \geq 5$ if G has type F_4, E_6 or E_7 and $p \geq 7$ if G has type E_8. Then all elementary abelian p-subgroups of G are toral.

(**1.1.2**) **Proof**: By [18, Exercise 5.11, p. 209] the conclusion holds if p is not a torsion prime for G. Thus the result follows by examining the list [18, I.4.4, p. 178–179]. □

Thus we are reduced to describing the conjugacy classes of nontoral elementary abelian p-subgroups in the finitely many cases left by the proposition.

In the present paper we carry out the computations for all odd primes p. For each conjugacy class of nontoral elementary abelian p-subgroups, we give an explicit representative $E \subseteq G$ and compute its *Weyl group*, which by definition is the group

$W(E) = N_G(E)/C_G(E)$. We also compute the *class distribution* of E, i.e. the number of elements in E from each conjugacy class of elements of order p in G. Moreover we determine the *oversize* of E, which by definition is the codimension of the largest toral subgroup of E. In almost all cases we also compute the centralizer $C_G(E)$ and some auxiliary information which we need for our applications.

Briefly outlined, the strategy of our proof is as follows. The important work of Griess [13] gives the maximal nontoral elementary abelian p-subgroups and some further information as well. Unfortunately this paper is not very accessible or explicit and we have therefore tried to keep our treatment as independent as possible. We have collected some corrections to [13] in the last section.

Following Griess, we first find representatives for the maximal nontoral elementary abelian p-subgroups. We then get lower bounds for their Weyl groups by producing explicit elements in these. From this we are able to identify the nonmaximal nontoral elementary abelian p-subgroups and get lower bounds for their Weyl groups. Finally we get exact results on the Weyl groups by computing centralizers.

The reason for our interest in the problem is that the computations form the core of the classification of p-compact groups [10] for odd primes p. This work, which is joint with Jesper Grodal, Jesper Møller and Antonio Viruel, is the subject of the forthcoming paper [1]. I am happy to take this opportunity to thank all three of them deeply for their collaboration.

Our notation is fairly standard. For elements x, y in a group G we let $y^x = x^{-1}yx$ denote the conjugate of y by x, and $[x, y] = x^{-1}y^{-1}xy$ be the commutator of x and y.

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{F}_p$ the finite field with p elements. Now let K be an arbitrary field. We let $K^\times$ denote the multiplicative group of K. The algebra of $n \times n$-matrices over K is denoted $M_n(K)$ and we let $I_n \in M_n(K)$ be the identity matrix. The diagonal matrix with entries $a_1, \dots, a_n \in K$ is denoted $\mathrm{diag}(a_1, \dots, a_n)$. We let $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ be the general linear group and the special linear group respectively. For an elementary abelian p-subgroup $E \subseteq G$ of rank r, the Weyl group $W(E)$ is a subgroup of $\mathrm{GL}(E)$. Fixing a basis of E, we may consider $W(E)$ as a subgroup of $\mathrm{GL}_r(\mathbb{F}_p)$. We make the convention that all matrices acts on columns. For a permutation σ in the symmetric group Σ_n we also let σ denote the associated permutation matrix $A = [a_{ij}]$, where

$$a_{ij} = \delta_{i,\sigma(j)} = \begin{cases} 1 & \text{for } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

We shall on occasions also use the following special notation. For matrices $A_1 \in M_{n_1}(K)$, $\dots$, $A_m \in M_{n_m}(K)$ we let

$$A_1 \oplus \dots \oplus A_m = \left[\begin{array}{c|c|c} A_1 & \dots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \dots & A_m \end{array}\right]$$

denote the $n \times n$-block matrix with the A_i's in the diagonal ($n = n_1 + \ldots + n_m$). We also need the "blowup" homomorphism $\Delta_{n,m} : M_n(K) \longrightarrow M_{mn}(K)$ defined by

$$\Delta_{n,m}\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = \left[\begin{array}{c|c|c} a_{11}I_m & \cdots & a_{1n}I_m \\ \hline \vdots & \ddots & \vdots \\ \hline a_{n1}I_m & \cdots & a_{nn}I_m \end{array}\right],$$

i.e. the $mn \times mn$-block matrix obtained by replacing each entry a_{ij} by $a_{ij}I_m$.

As we have $p = 3$ in almost all the cases we consider we use some special notation. We let $*$ denote an arbitrary element of $\mathbb{F}_3$ and ε denote an element of the multiplicative group $\mathbb{F}_3^\times$. Moreover we let $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/9}$. We also consider the elements

$$\beta = \operatorname{diag}(1, \omega, \omega^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad \gamma = (1,2,3) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\tau_1 = \frac{e^{-\pi i/18}}{\sqrt{3}} \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{bmatrix}, \qquad \tau_2 = \operatorname{diag}(\eta, \eta^{-2}, \eta) = \begin{bmatrix} \eta & 0 & 0 \\ 0 & \eta^{-2} & 0 \\ 0 & 0 & \eta \end{bmatrix}$$

in $\mathrm{SL}_3(\mathbb{C})$ and note that $\beta^{\tau_1} = \beta\gamma$, $\gamma^{\tau_1} = \gamma$, $\beta^{\tau_2} = \beta$ and $\gamma^{\tau_2} = \beta\gamma$.

For complex Lie groups we use standard notation taken from the literature on algebraic groups. Some basic references are [15] and [17]. For the root systems we use the conventions used in [3]. We shall need the following well-known result repeatedly.

(1.1.3) Lemma. *Let G be an algebraic group over an algebraically closed field K.*

(1) If A is an abelian subgroup of G and $x \in A$, then A is toral in G if and only if A is toral in $C_G(x)$.

(2) If G is simply connected and reductive, then the centralizer of a semisimple element is connected.

(3) If G is simply connected and reductive and A is an abelian subgroup of G consisting of semisimple elements generated by at most two elements, then A is toral.

(4) If H is a maximal torus, then two subsets of H are conjugate in G if and only if they are conjugate in $N_G(H)$. If A is toral in G, then $W(A) = N_G(A)/C_G(A)$ is isomorphic to a subquotient of the Weyl group $W = N_G(H)/H$ of G.

(5) Assume that G is reductive, let $Z \subseteq G$ be a central subgroup, and let $\pi : G \to G/Z$ be the quotient homomorphism. Then $A \subseteq G/Z$ is toral in G/Z if and only if $\pi^{-1}(A)$ is toral in G.

(6) Assume $\operatorname{char} K = 0$*, let* G *be connected and let* $\mathfrak{g}$ *be the Lie algebra of* G*. If* $S \subseteq G$ *is a finite subset of* G*, then the Lie algebra of* $C_G(S)$ *is given by*

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \,|\, \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\}.$$

In particular, if $S \subseteq G$ *is a finite subgroup, then*

$$\dim C_G(S) = \frac{1}{|S|} \sum_{s \in S} \mathrm{tr}\ \mathrm{Ad}(s)\,|_{\mathfrak{g}}.$$

(**1.1.4**) **Proof**: (1): Obviously, if A is toral in $C_G(x)$ then A is toral in G. Conversely, if A is toral in G, then $A \subseteq H$ for a torus H in G. Since $x \in A$ we get $H \subseteq C_G(x)$ and thus A is toral in $C_G(x)$.

(2): Our assumptions imply that G is semisimple, and thus the result follows from [18, Theorem II.3.9, p. 197].

(3): This follows directly from part (2), cf. [18, II.5.1, p. 206].

(4): The first part follows by a Frattini argument. Let $A, A^g \subseteq H$ be conjugate. Then H and $H^{g^{-1}}$ are maximal tori of $C_G(A)$. Thus we may write $H = H^{g^{-1}c}$ with $c \in C_G(A)$ and we conclude that $n = g^{-1}c \in N_G(H)$. Then $A^{n^{-1}} = A^{c^{-1}g} = A^g$, which proves the first part. The second part follows similarly, cf. [16, Proposition 1.1(i)].

(5): By [15, Corollary 21.3.C] we know that if H is a maximal torus of G, then $\pi(H)$ is a maximal torus of G/Z, and all maximal tori of G/Z are of this form. Since G is reductive we have $Z \subseteq H$ for all maximal tori H of G by [15, Corollary 26.2.A(b)]. From the above we then see that if H' is a maximal torus of G/Z, then $\pi^{-1}(H')$ is a maximal torus of G and all maximal tori of G are obtained in this way. From this relation between the maximal tori of G and G/Z, the claim follows immediately.

(6): The first part follows by combining [15, Theorem 13.4(a)] and [15, Theorem 12.5]. Now assume that $S \subseteq G$ is a finite group, and let χ be the character of the adjoint representation of G restricted to S. Then the dimension of

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \,|\, \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\}.$$

equals the multiplicity of the trivial character in χ. By the orthogonality relations this is given by

$$(\chi \,|\, 1) = \frac{1}{|S|} \sum_{s \in S} \chi(s),$$

and we are done. □

For some of our computations we have used the computer algebra system MAGMA [2]. More precisely we use this for our computations in $3E_6(\mathbb{C})$ and $E_8(\mathbb{C})$. For both of these, we represent the group by the action given by its smallest faithful complex representation. We also use the computer to make some computations of minimal overgroups in $\mathrm{GL}_r(\mathbb{F}_p)$ relevant to the exact determination of the Weyl groups of the elementary abelian p-group in consideration.

The remainder of this part of thesis is divided into 6 sections. The first one, which is extremely short, deals with $E_8(\mathbb{C})$ in the case $p = 5$ and $F_4(\mathbb{C})$ in the case $p = 3$. The next sections deals in succession with the groups $3E_6(\mathbb{C})$ and $E_6(\mathbb{C})$, $2E_7(\mathbb{C})$ and $E_8(\mathbb{C})$. In section 1.6 we compute some of the centralizers $C_G(E)$. The final section collects some corrections to the papers [7; 8; 13].

1.2 The groups $E_8(\mathbb{C})$, $p = 5$ and $F_4(\mathbb{C})$, $p = 3$

(**1.2.1**) **Theorem** (E_8). *Up to conjugacy, $E_8(\mathbb{C})$ contains a unique nontoral elementary abelian 5-subgroup E. This group has rank 3, is 1-oversize and has Weyl group $W(E) = \mathrm{SL}_3(\mathbb{F}_5)$.*

(**1.2.2**) **Proof**: This follows from [13, Lemma 10.3], except for the result on the oversize. However since E is the only nontoral elementary abelian subgroup up to conjugacy, it must be exactly 1-oversize. □

(**1.2.3**) **Theorem** (F_4). *Up to conjugacy, $F_4(\mathbb{C})$ contains a unique nontoral elementary abelian 3-subgroup E. This group has rank 3, is 1-oversize and has Weyl group $W(E) = \mathrm{SL}_3(\mathbb{F}_3)$.*

(**1.2.4**) **Proof**: This follows from [13, Theorem 7.4], except for the result on the oversize. However since E is the only nontoral elementary abelian subgroup up to conjugacy, it must be exactly 1-oversize. □

1.3 The groups $E_6(\mathbb{C})$ and $3E_6(\mathbb{C})$, $p = 3$

(**1.3.1**) **Setup.** In this section we consider the elementary abelian 3-subgroups of the groups of type E_6 over $\mathbb{C}$. The adjoint group is denoted $E_6(\mathbb{C})$, and the simply connected group is denoted by $3E_6(\mathbb{C})$. The group $3E_6(\mathbb{C})$ has two nonisomorphic faithful irreducible 27-dimensional representations, these have highest weight λ_1 and λ_6 respectively and are dual to each other. An explicit construction of $3E_6(\mathbb{C})$ based on one of these representations was originally given by Freudenthal [11]. This construction is described in more detail in [8, section 2] from which we take most of our notation. In particular we let $\mathbb{K}$ be the 27-dimensional complex vector space consisting of triples $m = (m_1, m_2, m_3)$ of complex 3×3-matrices m_i, $1 \le i \le 3$ where addition and scalar multiplication is defined coordinatewise. We define a cubic form $\langle\cdot\rangle$ on $\mathbb{K}$ by

$$\langle m\rangle = \det(m_1) + \det(m_2) + \det(m_3) - \operatorname{tr}(m_1m_2m_3).$$

Then $3E_6(\mathbb{C})$ is the subgroup of $\mathrm{GL}(\mathbb{K})$ preserving the form $\langle\cdot\rangle$. Moreover the stabilizer in $3E_6(\mathbb{C})$ of the element $(I_3, 0, 0) \in \mathbb{K}$ is the group $F_4(\mathbb{C})$. For $g_1, g_2, g_3 \in \mathrm{SL}_3(\mathbb{C})$ we have the element s_{g_1,g_2,g_3} of $3E_6(\mathbb{C})$ given by

$$s_{g_1,g_2,g_3}(m_1, m_2, m_3) = \left(g_1m_1g_2^{-1}, g_2m_2g_3^{-1}, g_3m_3g_1^{-1}\right)$$

for $m = (m_1, m_2, m_3) \in \mathbb{K}$. This gives a representation of $\mathrm{SL}_3(\mathbb{C})^3$ which has kernel C_3 generated by $(\omega I_3, \omega I_3, \omega I_3)$, and we thus get an embedding of $\mathrm{SL}_3(\mathbb{C})^3/C_3$ in $3E_6(\mathbb{C})$. We will denote the element s_{g_1,g_2,g_3} by $[g_1, g_2, g_3]$.

We let $\{e^i_{j,k}\}$, $1 \le i, j, k \le 3$ be the natural basis of $\mathbb{K}$ consisting of the elements $e^i_{j,k}$ whose entries are all 0 except for the (j, k)-entry of the i'th matrix which equals 1. The elements of $3E_6(\mathbb{C})$ which acts diagonally with respect to this basis of $\mathbb{K}$ form a maximal torus H in $3E_6(\mathbb{C})$. Let $e_{j,k}$ be the 3×3-matrix whose only nonzero element is 1 occurring at the (j, k)-entry, and $m_i^{j,k}$ be the (j, k)-entry of the matrix m_i. We then have H-invariant subgroups

$$u_{\alpha_1}(t) = [I_3, I_3 + te_{1,3}, I_3], \qquad u_{-\alpha_1}(t) = [I_3, I_3 + te_{3,1}, I_3],$$

$$u_{\alpha_2}(t) = [I_3 + te_{2,1}, I_3, I_3], \qquad u_{-\alpha_2}(t) = [I_3 + te_{1,2}, I_3, I_3],$$

$$u_{\alpha_3}(t) = [I_3, I_3 + te_{2,1}, I_3], \qquad u_{-\alpha_3}(t) = [I_3, I_3 + te_{1,2}, I_3],$$

$$u_{\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t\cdot\begin{bmatrix} 0 & -m^{2,3}_{i+2} & 0\\ 0 & 0 & 0\\ 0 & m^{2,1}_{i+2} & 0\end{bmatrix}\right)_{i=1,2,3}$$

$$u_{-\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t\cdot\begin{bmatrix} 0 & 0 & 0\\ m^{3,2}_{i+1} & 0 & -m^{1,2}_{i+1}\\ 0 & 0 & 0\end{bmatrix}\right)_{i=1,2,3}$$

$$u_{\alpha_5}(t) = [I_3, I_3, I_3 + te_{2,1}]\,, \qquad u_{-\alpha_5}(t) = [I_3, I_3, I_3 + te_{1,2}]\,,$$
$$u_{\alpha_6}(t) = [I_3, I_3, I_3 + te_{1,3}]\,, \qquad u_{-\alpha_6}(t) = [I_3, I_3, I_3 + te_{3,1}]\,.$$

Here, in the description of $u_{\pm\alpha_4}(t)$, the m_i's should be counted cyclicly mod 3, e.g. $m_{i+2} = m_1$ for $i = 2$.

The associated roots α_i, $1 \leq i \leq 6$, of these root subgroups form a base for the root system $\Phi(E_6)$ of $3E_6(\mathbb{C})$ (our numbering agrees with [3, p. 260–262]). For this base of $\Phi(E_6)$, the highest weight of $\mathbb{K}$ is λ_1. Furthermore the root subgroups $u_{\pm\alpha_i}$, $1 \leq i \leq 6$, have been chosen so that they satisfy [17, 8.1.1(i) and 8.1.4(i)], i.e. they form part of a *realization* ([17, p. 133]) of $\Phi(E_6)$ in $3E_6(\mathbb{C})$. For $\alpha = \pm\alpha_i$, $1 \leq i \leq 6$, and $t \in \mathbb{C}^\times$, we may then define the elements

$$n_\alpha(t) = u_\alpha(t)u_{-\alpha}(-1/t)u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements $h(t_1, t_2, t_3, t_4, t_5, t_6) = \prod_{i=1}^6 h_{\alpha_i}(t_i)$ and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \leq i \leq 6$. It should be noted that this notation differs from the one used in [8]. More precisely, the element $h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ in [8] is $h(\delta, \alpha^{-1}, \gamma^{-1}, \beta, \varepsilon^{-1}, \zeta)$ in our notation, and the elements n_1, n_2, n_3, n_4, n_5 and n_6 in [8] equals respectively $n_1 h_{\alpha_1}(-1)h_{\alpha_3}(-1)$, $n_2 h(-1, 1, 1, -1, 1, -1)$, $n_3 h_{\alpha_1}(-1)$, n_4, $n_5 h_{\alpha_6}(-1)$ and $n_6 h_{\alpha_5}(-1)h_{\alpha_6}(-1)$ in our notation.

From the description of the root system of type E_6 in [3, p. 260–262] we see that the center Z of $3E_6(\mathbb{C})$ is cyclic of order 3 and is generated by the element $z = [I_3, \omega^2 I_3, \omega I_3]$. We consider also the element $a = [\omega I_3, I_3, I_3]$. A straightforward computation shows that the roots of the centralizer $C_{3E_6(\mathbb{C})}(a)$ are

$$\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_6, \pm\widetilde{\alpha}, \pm(\alpha_1 + \alpha_3), \pm(\alpha_5 + \alpha_6), \pm(\alpha_2 - \widetilde{\alpha})\},$$

where $\widetilde{\alpha}$ is the longest root. The Dynkin diagram for this centralizer is the same as the extended Dynkin diagram for E_6 with the node α_4 removed. In particular it has type $A_2A_2A_2$ and basis $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$. Since $3E_6(\mathbb{C})$ is simply connected, Lemma (1.1.3(2)) implies that the centralizer $C_{3E_6(\mathbb{C})}(a)$ is connected, and thus it is generated by the maximal torus H and the root subgroups $u_{\pm\alpha}(t)$ where α runs through the simple roots in the basis $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$ of the root system of centralizer. Now note that $u_{\widetilde{\alpha}}(t) = [I_3 + te_{3,1}, I_3, I_3]$ and $u_{-\widetilde{\alpha}}(t) = [I_3 + te_{1,3}, I_3, I_3]$ are root subgroups with associated roots $\widetilde{\alpha}$ and $-\widetilde{\alpha}$ respectively. Since these along with H and the root subgroups $u_{\pm\alpha_1}$, $u_{\pm\alpha_2}$, $u_{\pm\alpha_3}$, $u_{\pm\alpha_5}$ and $u_{\pm\alpha_6}$ generate the subgroup $\mathrm{SL}_3(\mathbb{C})^3/C_3$ of $3E_6(\mathbb{C})$ from above, we conclude that $C_{3E_6(\mathbb{C})}(a) = \mathrm{SL}_3(\mathbb{C})^3/C_3$.

To describe the conjugacy classes of elementary abelian 3-subgroups we need to introduce some more elements. Consider the following elements in $\mathrm{SL}_3(\mathbb{C})^3/C_3 \subseteq 3E_6(\mathbb{C})$:

$$x_1 = [I_3, \beta, \beta]\,, \qquad x_2 = [\beta, \beta, \beta]\,, \qquad y_1 = \left[I_3, \gamma, \gamma^2\right], \qquad y_2 = [\gamma, \gamma, \gamma]\,.$$

We also need the following elements in $N(H)$:

$$\begin{aligned} s_1 &= n_1n_3n_4n_2n_5n_4n_3n_1n_6n_5n_4n_2n_3n_4n_5n_6 \\ s_2 &= n_1n_2n_3n_1n_4n_2n_3n_1n_4n_3n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_2n_3n_1n_4\cdot \\ &\quad n_3n_5n_4n_2n_6n_5n_4n_3n_1 \end{aligned}$$

The action of these elements are as follows:

$$s_1(m_1,m_2,m_3)=(m_3,m_1,m_2), \qquad s_2(m_1,m_2,m_3)=(m_3^{\mathtt{T}},m_2^{\mathtt{T}},m_1^{\mathtt{T}}),$$

where $m_i^{\mathtt{T}}$ denotes the transpose of m_i. From the description of the subgroup $\mathrm{SL}_3(\mathbb{C})^3/C_3$ we see that conjugation by these elements acts as follows

$$[g_1,g_2,g_3]^{s_1}=[g_2,g_3,g_1], \qquad [g_1,g_2,g_3]^{s_2}=\left[\left(g_1^{-1}\right)^{\mathtt{T}},\left(g_3^{-1}\right)^{\mathtt{T}},\left(g_2^{-1}\right)^{\mathtt{T}}\right].$$

It may be checked that the image of s_2 in the Weyl group $W=W(E_6)$ equals the longest element in W. Moreover the element $s_1s_2^{-1}$ lies in the subgroup $F_4(\mathbb{C})$ and its image in $W(F_4)$ equals the longest element in $W(F_4)$. We will not need these facts however.

(**1.3.2**) **Lemma**. *We have*

$$\begin{aligned} z &= h(\omega,1,\omega^2,1,\omega,\omega^2), \quad a=h(\omega,1,\omega^2,1,\omega^2,\omega), \quad x_1=h(\omega,1,\omega,1,\omega,\omega), \\ x_2 &= h(1,\omega^2,\omega^2,1,\omega^2,1), \quad y_1=n_1n_3n_5n_6h_{\alpha_5}(-1), \\ y_2 &= n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5h_{\alpha_2}(-1). \end{aligned}$$

Moreover conjugation by the element

$$n_1n_4n_2n_3n_1n_4n_5n_4n_6n_5n_4n_2n_3n_1n_4\cdot h_{\alpha_2}(-1)h_{\alpha_4}(-1)$$

acts as follows:

$$a\mapsto x_2, \qquad x_2\mapsto a, \qquad y_1\mapsto s_1, \qquad y_2\mapsto y_2^2, \qquad x_2x_1^{-1}\mapsto h_{\alpha_4}(\omega)=[\tau_2,\tau_2,\tau_2].$$

(**1.3.3**) **Proof**: Both parts of the lemma may be checked by direct computation. The second part also follows from the first by using the following relations in $N(H)$: The element n_i has image s_{α_i} in W ([17, 8.1.4(i)]), we have $n_i^2=h_{\alpha_i}(-1)$ ([17, 8.1.4(ii)]) and

$$n_in_jn_i\ldots=n_jn_in_j\ldots$$

for $1\le i,j\le 6$, where the number of factors on both sides equals the order of $s_{\alpha_i}s_{\alpha_j}$ in W ([17, 9.3.2]). □

(**1.3.4**) **Conjugacy classes.** For our calculations, we need some information on the conjugacy classes of elements of order 3 in $3E_6(\mathbb{C})$. These are given in [8, Table 2]: There are 7 such conjugacy classes, which we label **3A**, **3B**, **3B′**, **3C**, **3D**, **3E** and **3E′**, where **3B′** and **3E′** denotes the inverses of the classes **3B** and **3E**. This notation is almost identical to the notation in [8], but differs from [13]. We will need the following, which follows quickly from [8, Table 2] using the action of W on H: We have $z \in \mathbf{3E}$, $a, x_2, y_2 \in \mathbf{3C}$, $x_1, y_1 \in \mathbf{3D}$ and $x_2x_1^{-1} \in \mathbf{3A}$. Multiplication by z acts as follows on the conjugacy classes:

$$\mathbf{3A} \mapsto \mathbf{3B},\ \mathbf{3B} \mapsto \mathbf{3B}',\ \mathbf{3B}' \mapsto \mathbf{3A},\ \mathbf{3C} \mapsto \mathbf{3C},\ \mathbf{3D} \mapsto \mathbf{3D},\ \mathbf{3E} \mapsto \mathbf{3E}',\ \mathbf{3E}' \mapsto \mathbf{1},$$

where $\mathbf{1}$ denotes the conjugacy class consisting of the identity element.

(**1.3.5**) **Theorem** ($3E_6$). *The conjugacy classes of nontoral elementary abelian 3-subgroups of $3E_6(\mathbb{C})$ are given by the following table.*

rank	*name*	*ordered basis*	$3E_6(\mathbb{C})$*-class distribution*	$\dim C_{3E_6(\mathbb{C})}(E)$
3	$E^3_{3E_6}$	$\langle a, x_2, y_2\rangle$	$\mathbf{3C^{26}}$	0
4	$E^4_{3E_6}$	$\langle z, a, x_2, y_2\rangle$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	0

Their Weyl groups with respect to the given ordered bases satisfies the following:

$$W(E^3_{3E_6}) \supseteq \mathrm{SL}_3(\mathbb{F}_3), W(E^4_{3E_6}) \supseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

(**1.3.6**) **Remark**. In section 1.6 we will prove that the Weyl groups of $E^3_{3E_6}$ and $E^4_{3E_6}$ are exactly the lower bounds given above.

(**1.3.7**) **Proof**:

NONTORAL SUBGROUPS: By [13, Theorem (11.13)], there are two conjugacy classes of nontoral elementary abelian 3-subgroups in $3E_6(\mathbb{C})$, one nonmaximal of rank 3 and one maximal of rank 4. We may concretely realize these as follows. Consider the subgroups

$$E^3_{3E_6} = \langle a, x_2, y_2\rangle \text{ and } E^4_{3E_6} = \langle z, a, x_2, y_2\rangle\,,$$

which are readily seen to be elementary abelian 3-subgroups of rank 3 and 4 respectively. In particular both groups are subsets of $C_{3E_6(\mathbb{C})}(a) = \mathrm{SL}_3(\mathbb{C})^3/C_3$, and since $\beta, \gamma \in \mathrm{SL}_3(\mathbb{C})$ does not commute, we see that the preimages of $E^3_{3E_6}$ and $E^4_{3E_6}$ under the projection $\mathrm{SL}_3(\mathbb{C})^3 \to \mathrm{SL}_3(\mathbb{C})^3/C_3$ are nonabelian. Thus by Lemma (1.1.3(5)) $E^3_{3E_6}$ and $E^4_{3E_6}$ are nontoral in $\mathrm{SL}_3(\mathbb{C})^3/C_3 = C_{3E_6(\mathbb{C})}(a)$ and hence also nontoral in $3E_6(\mathbb{C})$ by Lemma (1.1.3(1)). Thus by the above these two groups represent the conjugacy classes of nontoral elementary abelian 3-subgroups in $3E_6(\mathbb{C})$.

WEYL GROUPS: Note that we have an inclusion $F_4(\mathbb{C}) \subseteq 3E_6(\mathbb{C})$. Thus the unique nontoral subgroup E of $F_4(\mathbb{C})$ of rank 3 from Theorem (1.2.3) may also be considered as a subgroup of $3E_6(\mathbb{C})$. As its Weyl group in $F_4(\mathbb{C})$ is $\mathrm{SL}_3(\mathbb{F}_3)$, its Weyl group in $3E_6(\mathbb{C})$ must contain $\mathrm{SL}_3(\mathbb{F}_3)$. In particular it has order divisible by 13 and since the order of $W(E_6)$ is $2^7 \cdot 3^4 \cdot 5$ which is not divisible by 13, we conclude by Lemma (1.1.3(4)) that E is nontoral in $3E_6(\mathbb{C})$ as well. Thus by the above E must be conjugate to $E^3_{3E_6}$, and we get that $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbb{F}_3)$. From this we immediately see that $W(E^4_{3E_6})$ contains the group

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

Note that the element $[I_3, \beta, \beta^2]$ commutes with z, a and x_2 and conjugates y_2 to $y_2 z$. Thus it normalizes $E^4_{3E_6}$ and produces the element

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in $W(E^4_{3E_6})$. As a result we see that $W(E^4_{3E_6})$ contains

$$\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

CLASS DISTRIBUTIONS: Since $a \in \mathbf{3C}$ by (1.3.4) and $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbb{F}_3)$ which acts transitively on $E^3_{3E_6} \setminus \{1\}$, the class distribution of $E^3_{3E_6}$ follows immediately. Using this and the information given in (1.3.4) about the action of z on the conjugacy classes, the class distribution of $E^4_{3E_6}$ is easily found.

CENTRALIZER DIMENSIONS: By Lemma (1.1.3(6)) we see that the dimension of $C_{3E_6(\mathbb{C})}(E)$ may be computed from the class distribution of E and the action of E on the Lie algebra $\mathfrak{e}_6$. The eigenvalue multiplicities on $\mathfrak{e}_6$ for the different conjugacy classes in $3E_6(\mathbb{C})$ are given in [8, Table 2], and using this we easily get the dimensions given above. As an example consider $E^3_{3E_6}$. We find:

$$\dim C_{3E_6(\mathbb{C})}(E^3_{3E_6}) = \frac{1}{|E^3_{3E_6}|} \sum_{x \in E^3_{3E_6}} \operatorname{tr} x|_{\mathfrak{e}_6} = \frac{1}{27} \cdot \big(78 + 26 \cdot (24 + 27\omega + 27\omega^2)\big) = 0.$$

A similar computation for $E^4_{3E_6}$ finishes the proof. □

(1.3.8) Notation. We now turn to the group $E_6(\mathbb{C})$. As above we let Z be the center of $3E_6(\mathbb{C})$ and we let $\pi : 3E_6(\mathbb{C}) \to E_6(\mathbb{C}) = 3E_6(\mathbb{C})/Z$ denote the projection. For $g \in 3E_6(\mathbb{C})$ we write simply $\overline{g}$ instead of $\pi(g)$ and similarly we let $\overline{S} = \pi(S)$ for a subset $S \subseteq 3E_6(\mathbb{C})$.

(1.3.9) Lemma. *Let E be a rank 2 nontoral elementary abelian 3-subgroup of $E_6(\mathbb{C})$. Then the Weyl group $W(E)$ is a subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$.*

(1.3.10) Proof: Let $E = \langle \overline{g_1}, \overline{g_2} \rangle$. By Lemma (1.1.3) part (5) and (3) the group $\langle g_1, g_2 \rangle \subseteq 3E_6(\mathbb{C})$ is nonabelian. Thus setting $z' = [g_1, g_2] \in Z$ we have $z' \neq 1$. Assume that $\sigma \in W(E)$ is represented by the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, i.e. we have $\sigma(\overline{g_1}) = (\overline{g_1})^{a_{11}}(\overline{g_2})^{a_{21}}$ and $\sigma(\overline{g_2}) = (\overline{g_1})^{a_{12}}(\overline{g_2})^{a_{22}}$. Since σ is given by a conjugation in $E_6(\mathbb{C})$, it lifts to a conjugation in $3E_6(\mathbb{C})$. Now the relation $[g_1, g_2] = z' \in Z$ shows $(z')^{a_{11}\cdot a_{22} - a_{12}\cdot a_{21}} = z'$, so since $z' \neq 1$ we have $\sigma \in \mathrm{SL}_2(\mathbb{F}_3)$. □

(1.3.11) Theorem (E_6). *The conjugacy classes of nontoral elementary abelian 3-subgroups of $E_6(\mathbb{C})$ are given by the following table,*

rank	*name*	*ordered basis*	$3E_6(\mathbb{C})$*-class distribution*	$\dim C_{E_6(\mathbb{C})}(E)$
2	$E^{2a}_{E_6}$	$\langle \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{18}3D^{6}3E^{1}3E'^{1}}$	8
2	$E^{2b}_{E_6}$	$\langle \overline{y_1}, \overline{x_1} \rangle$	$\mathbf{3D^{24}3E^{1}3E'^{1}}$	14
3	$E^{3a}_{E_6}$	$\langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{60}3D^{18}3E^{1}3E'^{1}}$	2
3	$E^{3b}_{E_6}$	$\langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	0
3	$E^{3c}_{E_6}$	$\langle \overline{a}, \overline{y_1}, \overline{x_1} \rangle$	$\mathbf{3C^{6}3D^{72}3E^{1}3E'^{1}}$	8
3	$E^{3d}_{E_6}$	$\left\langle \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle$	$\mathbf{3A^{2}3B^{2}3B'^{2}3C^{48}3D^{24}3E^{1}3E'^{1}}$	4
4	$E^{4a}_{E_6}$	$\langle \overline{a}, \overline{y_2}, \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{186}3D^{54}3E^{1}3E'^{1}}$	0
4	$E^{4b}_{E_6}$	$\left\langle \overline{a}, \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle$	$\mathbf{3A^{6}3B^{6}3B'^{6}3C^{150}3D^{72}3E^{1}3E'^{1}}$	2

where the last entry in the row corresponding to $E \subseteq E_6(\mathbb{C})$ is the class distribution of $\pi^{-1}(E) \subseteq 3E_6(\mathbb{C})$.

These groups are all 1-oversize and their Weyl groups with respect to the given ordered bases satisfies the following:

$$W(E^{2a}_{E_6}) = \begin{bmatrix} \varepsilon & * \\ 0 & \varepsilon \end{bmatrix}, W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbb{F}_3), W(E^{3a}_{E_6}) = \begin{bmatrix} \varepsilon_1 & * & * \\ 0 & \varepsilon_2 & * \\ 0 & 0 & \varepsilon_2 \end{bmatrix},$$

$$W(E^{3b}_{E_6}) \supseteq \mathrm{SL}_3(\mathbb{F}_3), W(E^{3c}_{E_6}) = \left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right], W(E^{3d}_{E_6}) = \left[\begin{array}{c|cc} \varepsilon & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right],$$

$$W(E^{4a}_{E_6}) \supseteq \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & det & * \\ 0 & 0 & 0 & det \end{array}\right], W(E^{4b}_{E_6}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & 0 & & \end{array}\right],$$

where det denotes the determinant of the matrix from $\mathrm{GL}_2(\mathbb{F}_3)$ in the description of $W(E^{4a}_{E_6})$.

(1.3.12) Remark. In section 1.6 we will prove that the Weyl groups of $E^{3b}_{E_6}$ and $E^{4a}_{E_6}$ are exactly the lower bound given above.

(1.3.13) Proof:

MAXIMAL NONTORAL SUBGROUPS: By [13, Theorem (11.14)], there are two conjugacy classes of maximal nontoral elementary abelian 3-subgroups in $E_6(\mathbb{C})$, both of which have rank 4. We may concretely realize these as follows. Consider the subgroups

$$E_a = \langle z, a, y_1, y_2, x_2 \rangle \text{ and } E_b = \left\langle z, a, x_2 x_1^{-1}, y_1, x_1 \right\rangle$$

of $C_{3E_6(\mathbb{C})}(a) = \mathrm{SL}_3(\mathbb{C})^3/C_3$. Since the commutator subgroup of both of these is Z, we see that $E_{E_6}^{4a} = \pi(E_a)$ and $E_{E_6}^{4b} = \pi(E_b)$ are elementary abelian subgroups of rank 4 in $E_6(\mathbb{C})$. It follows from Lemma (1.1.3(5)) that both $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ are nontoral in $E_6(\mathbb{C})$. We will see below that their class distributions are as given in the table. From this it follows that they are not conjugate and thus represents the two conjugacy classes of maximal elementary abelian 3-subgroups in $E_6(\mathbb{C})$.

LOWER BOUNDS FOR WEYL GROUPS OF MAXIMAL NONTORAL SUBGROUPS: We can find lower bounds for the Weyl groups of the maximal nontoral elementary abelian 3-subgroups by conjugating with elements coming from the centralizer $C_{3E_6(\mathbb{C})}(a) = \mathrm{SL}_3(\mathbb{C})^3/C_3$ and the normalizer $N(H)$ of the maximal torus.

The elements $\overline{[\beta^2, I_3, I_3]}$, $\overline{[I_3, \tau_1, \tau_1^2]}$, $\overline{s_1}$ and $\overline{s_2}$ normalize $E_{E_6}^{4a}$ and conjugation by these elements induce the automorphisms given by the matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

on $E_{E_6}^{4a}$. Moreover, by Lemma (1.3.2) we may conjugate the ordered basis of $E_{E_6}^{4a}$ into the ordered basis $\left\langle \overline{x_2}, \overline{y_2^2}, \overline{s_1}, \overline{a} \right\rangle$. Noting that the element $\overline{[\tau_1, \tau_1, \tau_1]}$ commutes with $\overline{y_2}$, $\overline{s_1}$ and $\overline{a}$ and conjugates $\overline{x_2}$ into $\overline{x_2 y_2}$, we see that $W(E_{E_6}^{4a})$ contains the automorphism given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The above matrices are easily seen to generate the group

$$W'(E_{E_6}^{4a}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbb{F}_3)} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array} \right],$$

and thus $W_{E_6}^{4a}$ contains this group.

Now consider $E_{E_6}^{4b}$ and let $\sigma = -(2,3) \in \mathrm{SL}_3(\mathbb{C})$. We then see that the elements $\overline{[I_3, \tau_1, \tau_1^2]}$, $\overline{[I_3, \tau_2\beta, \tau_2^2]}$, $\overline{[\sigma, I_3, I_3]}$, $\overline{[\gamma, I_3, I_3]}$, $\overline{[I_3, \beta^2, I_3]}$ and $\overline{s_2}$ normalize $E_{E_6}^{4b}$, and conjugation by these elements induce the automorphisms given by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

on $E_{E_6}^{4b}$. We see that these matrices generate the group

$$W'(E_{E_6}^{4b}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{\mathrm{SL}_2(\mathbb{F}_3)}} \\ 0 & 0 & & \end{array}\right]$$

and thus $W_{E_6}^{4b}$ contains this group.

ORBIT COMPUTATION: Any elementary abelian 3-subgroup of rank 1 is toral since $E_6(\mathbb{C})$ is connected. As we already know that $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ are representatives of the maximal nontoral elementary abelian 3-subgroups, we may find the conjugacy classes of nontoral elementary abelian 3-subgroups of rank 2 and 3 by studying subgroups of these.

Under the action of $W'(E_{E_6}^{4a})$, the set of rank 2 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{2a} = \langle \overline{y_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_2} \rangle, \langle \overline{a}, \overline{y_1} \rangle \text{ and } \langle \overline{a}, \overline{y_2} \rangle,$$

and under the action of $W'(E_{E_6}^{4b})$, the set of rank 2 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{2a} = \langle \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{2b} = \langle \overline{y_1}, \overline{x_1} \rangle, \langle \overline{a}, \overline{x_2} \rangle, \langle \overline{a}, \overline{y_1} \rangle, \left\langle \overline{a}, \overline{x_2 x_1^{-1}} \right\rangle \text{ and } \left\langle \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle.$$

Similarly we find that under the action of $W'(E_{E_6}^{4a})$, the set of rank 3 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{3a} = \langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{3b} = \langle \overline{a}, \overline{y_2}, \overline{x_2} \rangle \text{ and } \langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle,$$

and that under the action of $W'(E_{E_6}^{4b})$, the set of rank 3 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{3a} = \langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{3c} = \langle \overline{a}, \overline{y_1}, \overline{x_1} \rangle, E_{E_6}^{3d} = \left\langle \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle \text{ and } \left\langle \overline{a}, \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle.$$

OTHER NONTORAL SUBGROUPS: We see directly that the subgroups $\langle \overline{a}, \overline{x_2} \rangle$, $\left\langle \overline{a}, \overline{x_2x_1^{-1}} \right\rangle$, $\left\langle \overline{x_2x_1^{-1}}, \overline{x_1} \right\rangle$ and $\left\langle \overline{a}, \overline{x_2x_1^{-1}}, \overline{x_1} \right\rangle$ are toral. Noting that the elements β and γ are conjugate in $\mathrm{SL}_3(\mathbb{C})$ we see that the group $\langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle$ is conjugate to the group $\left\langle \overline{a}, \overline{[I_3, \beta, \beta^2]}, \overline{x_2} \right\rangle$ which is obviously toral. Thus we see that the groups $\langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle$, $\langle \overline{a}, \overline{y_1} \rangle$ and $\langle \overline{a}, \overline{y_2} \rangle$ are also toral. Using the fact that $[y_1, x_1] = [y_1, x_2] = z$ we see from Lemma (1.1.3(5)) that both $E^{2a}_{E_6}$ and $E^{2b}_{E_6}$ are nontoral in $E_6(\mathbb{C})$. Since the groups $E^{3a}_{E_6}$, $E^{3c}_{E_6}$ and $E^{3d}_{E_6}$ all contain either $E^{2a}_{E_6}$ or $E^{2b}_{E_6}$ they are also nontoral. Using Lemma (1.1.3(5)) we see that the group $E^{3b}_{E_6}$ is nontoral in $E_6(\mathbb{C})$, since we know that $\pi^{-1}(E^{3b}_{E_6}) = E^4_{3E_6}$ is nontoral in $3E_6(\mathbb{C})$ by Theorem (1.3.5).

OVERSIZE: Since the groups $\langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle$ and $\left\langle \overline{a}, \overline{x_2x_1^{-1}}, \overline{x_1} \right\rangle$ are toral and are codimension 1 subgroups of $E^{4a}_{E_6}$ and $E^{4b}_{E_6}$ respectively, we see that $E^{4a}_{E_6}$ and $E^{4b}_{E_6}$ are at most 1-oversize. Since they are nontoral they are both exactly 1-oversize. Since the other nontoral elementary abelian 3-subgroups are contained in one of these, they are also 1-oversize.

CLASS DISTRIBUTIONS: Using (1.3.4) and the action of the groups $W'(E^{4a}_{E_6})$ and $W'(E^{4b}_{E_6})$ it is not hard to verify the class distributions in the table. As an example consider the group $E^{4b}_{E_6}$. From the action of $W'(E^{4b}_{E_6})$ we see that $E^{4b}_{E_6} \setminus \{1\}$ contains 2 elements conjugate to $\overline{a}$, 6 elements conjugate to $\overline{x_2x_1^{-1}}$, 24 elements conjugate to $\overline{x_1}$ and 48 elements conjugate to $\overline{x_2}$. Thus by (1.3.4), the set $\pi^{-1}(E^{4b}_{E_6} \setminus \{1\})$ contains 6 elements from each of the classes **3A**, **3B** and **3B′**, $3 \cdot (2+48) = 150$ elements from the class **3C** and $3 \cdot 24 = 72$ elements from the class **3D**. Including the elements z and z^2 from the classes **3E** and **3E′** respectively, we get the class distribution of $\pi^{-1}(E^{4b}_{E_6}) \setminus \{1\}$ given in the table. Similar computations give the remaining entries in the table. Since these distributions are different we see that the groups in the table are not conjugate and thus they provide a set of representatives for the conjugacy classes of nontoral elementary abelian 3-subgroups of $E_6(\mathbb{C})$.

CENTRALIZER DIMENSIONS: Noting that Z acts trivially on the Lie algebra $\mathfrak{e}_6$, we see from Lemma (1.1.3(6)) that for any finite subgroup $E \subseteq E_6(\mathbb{C})$ we have $\dim C_{E_6(\mathbb{C})}(E) = \dim C_{3E_6(\mathbb{C})}(\pi^{-1}(E))$. Thus we can use the class distributions above and [8, Table 2] to find the dimensions of the centralizers. As an example consider $E^{4b}_{E_6}$:

$$\dim C_{E_6(\mathbb{C})}(E^{4b}_{E_6}) = \dim C_{3E_6(\mathbb{C})}(\pi^{-1}(E^{4b}_{E_6})) = \frac{1}{|\pi^{-1}(E^{4b}_{E_6})|} \sum_{x \in \pi^{-1}(E^{4b}_{E_6})} \operatorname{tr} \, x|_{\mathfrak{e}_6} =$$

$$\frac{1}{3^5} \cdot \big(3 \cdot 78 + 18 \cdot (36 + 21\omega + 21\omega^2) + 150 \cdot (24 + 27\omega + 27\omega^2) + 72 \cdot (30 + 24\omega + 24\omega^2)\big) = 2.$$

Similar computations give the results for the other groups in the table.

LOWER BOUNDS FOR OTHER WEYL GROUPS: We now show that the other matrix groups in the table are all lower bounds for the remaining Weyl groups. To do this consider one of the nonmaximal groups E from the table. We then have $E \subseteq E^{4a}_{E_6}$ or $E \subseteq E^{4b}_{E_6}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E^{4a}_{E_6})$ or $W'(E^{4b}_{E_6})$ fixing E. As an example we see that $E^{2a}_{E_6} \subseteq E^{4a}_{E_6}$ and that the stabilizer of $E^{2a}_{E_6}$ inside $W'(E^{4a}_{E_6})$ is

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \det & x \\ 0 & 0 & 0 & \det \end{array}\right],$$

where det is the determinant of the matrix from $\mathrm{GL}_2(\mathbb{F}_3)$. The action of such a matrix on $E^{2a}_{E_6}$ is given by

$$\overline{y_1} \mapsto (\overline{y_1})^{\det}, \qquad \overline{x_2} \mapsto (\overline{y_1})^{x}(\overline{x_2})^{\det}.$$

Thus $W(E^{2a}_{E_6})$ contains the group

$$W'(E^{2a}_{E_6}) = \begin{bmatrix} \varepsilon & * \\ 0 & \varepsilon \end{bmatrix}$$

as claimed. Similar computations show that for the groups $E = E^{2b}_{E_6}$, $E^{3a}_{E_6}$, $E^{3c}_{E_6}$ and $E^{3d}_{E_6}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

For the group $E^{3b}_{E_6} = \langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$ we note that by Theorem (1.3.5) the group $E^{3}_{3E_6} = \langle a, x_2, y_2 \rangle$ has Weyl group at least $\mathrm{SL}_3(\mathbb{F}_3)$. Thus we see that $W(E^{3b}_{E_6})$ also contains $\mathrm{SL}_3(\mathbb{F}_3)$.

EXACT WEYL GROUPS: By now we know that the groups in the table is a set of representatives for the conjugacy classes of nontoral elementary abelian 3-subgroups of $E_6(\mathbb{C})$, that they are all 1-oversize and that the matrix groups given in the theorem are lower bounds for the Weyl groups. We thus only need to verify that for all the groups except $E^{3b}_{E_6}$ and $E^{4a}_{E_6}$ the Weyl group is the one given above.

By Lemma (1.3.9) the Weyl groups $W(E^{2a}_{E_6})$ and $W(E^{2b}_{E_6})$ are subgroups of $\mathrm{SL}_2(\mathbb{F}_3)$. From this we see that $W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbb{F}_3)$ and that $W(E^{2a}_{E_6})$ is equal to either $W'(E^{2a}_{E_6})$ or $\mathrm{SL}_2(\mathbb{F}_3)$, since these are the only subgroups of $\mathrm{SL}_2(\mathbb{F}_3)$ containing $W'(E^{2a}_{E_6})$. We have $E^{2a}_{E_6} = \langle \overline{y_1}, \overline{x_2} \rangle$, and by (1.3.4) we see that the elements $\overline{y_1}$ and $\overline{x_2}$ are not conjugate in $E_6(\mathbb{C})$. In particular we see that $W(E^{2a}_{E_6})$ cannot act transitively on the nontrivial elements of $E^{2a}_{E_6}$, and we conclude that $W(E^{2a}_{E_6}) = W'(E^{2a}_{E_6})$ is the group from above.

Note that the groups $E^{3a}_{E_6}$, $E^{3d}_{E_6}$, and $E^{4b}_{E_6}$ all contain $E^{2a}_{E_6}$. By using for example the computer program mentioned in the introduction, we find that the minimal overgroups of $W'(E)$ in $\mathrm{GL}(E)$ are:

$$E^{3a}_{E_6}: \left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right], \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$E^{3d}_{E_6}: \left[\begin{array}{c|cc} \varepsilon & 0 & 0 \\ \hline * & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ * & & \end{array}\right], \left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right], \left[\begin{array}{c|cc} \varepsilon & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right]$$

$$E^{4b}_{E_6}: \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} \\ 0 & 0 & & \end{array}\right], \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & * & * \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & 0 & & \end{array}\right], \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & * & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbb{F}_3)$}} \\ 0 & * & & \end{array}\right].$$

From this we see that if $W(E^{3a}_{E_6}) \neq W'(E^{3a}_{E_6})$ then $W(E^{3a}_{E_6})$ will contain one of the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

both fixing the subgroup $E^{2a}_{E_6}$ of $E^{3a}_{E_6}$. However the automorphisms of $E^{2a}_{E_6}$ induced by these elements are given by the matrices

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since none of these are contained in $W(E^{2a}_{E_6})$ we get $W(E^{3a}_{E_6}) = W'(E^{3a}_{E_6})$ as claimed. Similar computations using the above list of minimal overgroups show that $W(E) = W'(E)$ for $E = E^{3d}_{E_6}$ and $E^{4b}_{E_6}$. For the group $E^{3c}_{E_6}$ we get that $W'(E^{3c}_{E_6})$ has only one minimal overgroup, namely:

$$\left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} \\ 0 & & \end{array}\right]$$

If $W(E^{3c}_{E_6})$ contained this group, then it would contain the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

fixing the subgroup $E^{2b}_{E_6}$. However the action induced on $E^{2b}_{E_6}$ is given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is not contained in $W(E^{2b}_{E_6})$. Thus we get $W(E^{3c}_{E_6}) = W'(E^{3c}_{E_6})$ as well. $\square$

1.4 The group $2E_7(\mathbb{C})$, $p = 3$

(**1.4.1**) **Setup.** In this section we consider the elementary abelian 3-subgroups of $2E_7(\mathbb{C})$, the simply connected group of type E_7 over $\mathbb{C}$. We let H be a maximal torus of $2E_7(\mathbb{C})$, $\Phi(E_7)$ be the root system relative to H, and choose a *realization* ([17, p. 133]) $(u_\alpha)_{\alpha\in\Phi(E_7)}$ of $\Phi(E_7)$ in $2E_7(\mathbb{C})$.

Note that we have an inclusion $\Phi(E_6) \subseteq \Phi(E_7)$ of root systems (we use the notation of [3, p. 260–266] for the root systems $\Phi(E_6)$ and $\Phi(E_7)$). The subgroup G of $2E_7(\mathbb{C})$ generated by the root subgroups $(u_\alpha)_{\alpha\in\Phi(E_6)}$ is isomorphic to $3E_6(\mathbb{C})$ and these root subgroups form a realization of $\Phi(E_6)$ in G. Fixing an isomorphism $G \cong 3E_6(\mathbb{C})$ and using the choice of root subgroups for $3E_6(\mathbb{C})$ from section 1.3 we get the root subgroups $u'_{\pm\alpha_i}$, $1 \le i \le 6$ in $G \subseteq 2E_7(\mathbb{C})$. Now using [17, 8.1.4(iv)] we may actually suppose that $u_{\pm\alpha_i} = u'_{\pm\alpha_i}$ for $1 \le i \le 6$. For $\alpha = \alpha_i$, $1 \le i \le 7$, and $t \in \mathbb{C}^\times$ we define the elements

$$n_\alpha(t) = u_\alpha(t)u_{-\alpha}(-1/t)u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements $\prod_{i=1}^7 h_{\alpha_i}(t_i)$ and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 7$.

As in section 1.3 we define the following elements in $2E_7(\mathbb{C})$:

$$\begin{aligned}
z &= h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega)h_{\alpha_6}(\omega^2), \qquad a = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega),\\
x_2 &= h_{\alpha_2}(\omega^2)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2),\\
y_2 &= n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5h_{\alpha_2}(-1).
\end{aligned}$$

(**1.4.2**) **Conjugacy classes.** The conjugacy classes of elements of order 3 in $2E_7(\mathbb{C})$ are given in [13, Table VI] and [7, Table 6] from which we take our notation. In particular, there are 5 such conjugacy classes, which we label **3A**, **3B**, **3C**, **3D** and **3E**. Moreover these classes may be distinguished by their traces on $\mathfrak{e}_7$, except for the classes **3A** and **3D** which have the same trace. Since the trace of the element $h \in H$ is given by $7 + \sum_{\alpha\in\Phi(E_7)} \alpha(h)$ we easily obtain the inclusions

$$\mathbf{3C}[3E_6(\mathbb{C})] \subseteq \mathbf{3C}[2E_7(\mathbb{C})], \quad \mathbf{3E}[3E_6(\mathbb{C})] \subseteq \mathbf{3B}[2E_7(\mathbb{C})], \quad \mathbf{3E'}[3E_6(\mathbb{C})] \subseteq \mathbf{3B}[2E_7(\mathbb{C})],$$

corresponding to the inclusion $3E_6(\mathbb{C}) \cong G \subseteq 2E_7(\mathbb{C})$.

(**1.4.3**) **Theorem** ($2E_7$). *The conjugacy classes of nontoral elementary abelian 3-subgroups of $2E_7(\mathbb{C})$ are given by the following table.*

rank	*name*	*ordered basis*	*$2E_7(\mathbb{C})$-class distribution*	$\dim C_{2E_7(\mathbb{C})}(E)$
3	$E^3_{2E_7}$	$\langle a, x_2, y_2\rangle$	$\mathbf{3C^{26}}$	3
4	$E^4_{2E_7}$	$\langle z, a, x_2, y_2\rangle$	$\mathbf{3B^2 3C^{78}}$	1

These groups are all 1-oversize and their Weyl groups with respect to the given ordered bases satisfies the following:

$$W(E^3_{2E_7}) \supseteq \mathrm{SL}_3(\mathbb{F}_3), W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

(1.4.4) Remark. In section 1.6 we will prove that the Weyl groups of $E^3_{2E_7}$ and $E^4_{2E_7}$ are exactly the lower bounds given above.

(1.4.5) Proof:

NONTORAL SUBGROUPS: From the way the realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ is chosen above, it follows from Theorem (1.3.5) that $E^3_{2E_7}$ and $E^4_{2E_7}$ are elementary abelian 3-subgroups of $2E_7(\mathbb{C})$ and that we have

$$W(E^3_{2E_7}) \supseteq \mathrm{SL}_3(\mathbb{F}_3), W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

In particular we see that both $W(E^3_{2E_7})$ and $W(E^4_{2E_7})$ have orders divisible by 13 and since the order of $W(E_7)$ is $2^{10}\cdot 3^4\cdot 5\cdot 7$ which is not divisible by 13, we conclude by Lemma (1.1.3(4)) that $E^3_{2E_7}$ and $E^4_{2E_7}$ are nontoral in $2E_7(\mathbb{C})$. However by [13, Theorem (11.16)] we know that there are precisely two conjugacy classes of nontoral elementary abelian 3-subgroups in $2E_7(\mathbb{C})$, and thus $E^3_{2E_7}$ and $E^4_{2E_7}$ represent these two conjugacy classes.

CLASS DISTRIBUTIONS: The class distributions follows directly from the class distributions of the groups $E^3_{3E_6}$ and $E^4_{3E_6}$ given in Theorem (1.3.5) and the information in (1.4.2) about the behavior of conjugacy classes in $3E_6(\mathbb{C})$ under the inclusion $3E_6(\mathbb{C}) \cong G \subseteq 2E_7(\mathbb{C})$.

CENTRALIZER DIMENSIONS: By using Lemma (1.1.3(6)), the class distributions just found and the traces given in [13, Table VI], the dimensions of the centralizers are verified easily.

OVERSIZE: Since the group $\langle \overline{z}, \overline{a}, \overline{x_2}\rangle$ is toral and has codimension 1 in $E^4_{2E_7}$ we see that $E^4_{2E_7}$ is at most 1-oversize. Since $E^3_{2E_7}$ is contained in $E^4_{2E_7}$ it is also at most 1-oversize. Now both $E^3_{2E_7}$ and $E^4_{2E_7}$ are nontoral, so they are both exactly 1-oversize.

LOWER BOUNDS FOR WEYL GROUPS: Using our realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ we may define a canonical map $\varphi : W \to N(H)$ as follows ([17, 9.3.3]): If $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_r}}$ is a reduced expression for $w \in W$ we let $\varphi(w) = n_{i_1} \dots n_{i_r}$ (by [17, 8.3.3 and 9.3.2] this does not depend on the reduced expression for w). Note that the element $\varphi(w)$ is a representative in $N(H)$ for $w \in W$. Now let $w_0 \in W$ be the longest element in W, and let $n_0 = \varphi(w_0)$. From [3, p. 264–266] it follows that w_0 equals the scalar transformation -1, and so conjugation by n_0 acts as inversion on H. Now let $w \in W$ and define w' by $ww' = w_0$. Since w_0 is central in our case, we have $(ww')\,w^{-1} = w^{-1}\,(ww') = w'$ so we conclude that $w'w = ww' = w_0$. Now let ℓ be the length function on W. By [14, p. 16] we have $\ell(w) + \ell(w') = \ell(w_0)$. In general the map φ is not a homomorphism, but we do have $\varphi(w_1w_2) = \varphi(w_1)\varphi(w_2)$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ by [17, 9.3.4(i)]. From this it follows that $\varphi(w)\varphi(w') = \varphi(w')\varphi(w) = \varphi(w_0) = n_0$, and we conclude that n_0 commutes with $\varphi(w)$ for all $w \in W$.

Now consider the element

$$w = s_1s_2s_3s_4s_3s_1s_5s_4s_2s_3s_4s_5s_6s_5s_4s_2s_3s_1s_4s_3s_5s_4s_6s_5.$$

Using the fact that the length of an element is given by the number of positive roots it sends to negative roots ([14, Corollary 1.7]), we see that the above product is a reduced expression for w. Thus we have $y_2 = \varphi(w)h_{\alpha_2}(-1)$. From the above we then conclude that conjugation by n_0 acts as follows:

$$z \mapsto z^2, \qquad a \mapsto a^2, \qquad x_2 \mapsto x_2^2, \qquad y_2 \mapsto y_2.$$

Thus n_0 normalizes $E^4_{2E_7}$ and induces the element

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in $W(E^4_{2E_7})$. Combined with the above we conclude that

$$W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|c} \varepsilon & * \quad * \quad * \\ \hline 0 & \\ 0 & \mathrm{SL}_3(\mathbb{F}_3) \\ 0 & \end{array}\right]$$

as claimed. $\square$

1.5 The group $E_8(\mathbb{C})$, $p = 3$

(**1.5.1**) **Setup.** In this section we consider the elementary abelian 3-subgroups of the group $E_8(\mathbb{C})$, which is both the simply connected and adjoint group of type E_8 over $\mathbb{C}$. By using [4, Table 2, p. 214] we see that the smallest faithful representation of $E_8(\mathbb{C})$ is the adjoint representation, i.e. the representation given by the action of $E_8(\mathbb{C})$ on its Lie algebra $\mathfrak{e}_8$, which has dimension 248. We may explicitly construct this representation by following the recipe in [6, Chapter 4]. Letting $\Phi(E_8)$ denote the root system of type E_8 (we use the notation of [3, p. 268–270]), we have in particular a maximal torus H generated by the elements $h_{\alpha_i}(t)$, $1 \le i \le 8$, $t \in \mathbb{C}^\times$ ([6, p. 92, p. 97]) and root subgroups $u_\alpha(t)$, $\alpha \in \Phi(E_8)$, $t \in \mathbb{C}$. The normalizer $N(H)$ of the maximal torus, is generated by H and the elements $n_i = n_{\alpha_i}$, $1 \le i \le 8$ ([6, p. 93, p. 101]). We let

$$h(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) = \prod_{i=1}^{8} h_{\alpha_i}(t_i).$$

Note that by [6, p. 100 and Lemma 6.4.4] the root subgroups u_α form a *realization* ([17, p. 133]) of $\Phi(E_8)$ in $E_8(\mathbb{C})$. In particular we have the following relations. The element n_i has image s_{α_i} in $W = W(E_8)$ ([17, 8.1.4(i)]), we have $n_i^2 = h_{\alpha_i}(-1)$ ([17, 8.1.4(ii)]) and

$$n_i n_j n_i \ldots = n_j n_i n_j \ldots$$

for $1 \le i, j \le 8$, where the number of factors on both sides equals the order of $s_{\alpha_i} s_{\alpha_j}$ in W ([17, 9.3.2]).

Now let $\overline{a} = h_{\alpha_1}(\omega) h_{\alpha_2}(\omega) h_{\alpha_3}(\omega^2) \in E_8(\mathbb{C})$. Direct computation shows that for any root $\alpha \in \Phi(E_8)$ we have $\alpha(\overline{a}) = \omega^{2\langle \alpha, \lambda_2 \rangle}$. From this we see that Dynkin diagram of the centralizer $C_{E_8(\mathbb{C})}(\overline{a})$ is the same as the extended Dynkin diagram of E_8 with the node α_2 removed. Thus it has type A_8 and basis

$$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, -\widetilde{\alpha}\},$$

where $\widetilde{\alpha}$ is the longest root. As in [3, p. 250–251] we identify $\Phi(A_8)$ with the set of elements in $\mathbb{R}^9$ of the form $e_i - e_j$ with $i \ne j$ and $1 \le i, j \le 9$, where e_i denotes the i'th canonical basis vector in $\mathbb{R}^9$. We now consider $\mathrm{SL}_9(\mathbb{C})$, which is the simply connected group of type E_8 over $\mathbb{C}$. For any $1 \le i, j \le 9$ we let $e_{i,j}$ be the 9×9-matrix over $\mathbb{C}$ whose only nonzero element is 1 occurring at the (i, j)-entry. Given a root $\alpha' = e_i - e_j \in \Phi(A_8)$ we let $u'_{\alpha'}(t) = I_9 + t e_{i,j}$ for $t \in \mathbb{C}$. With respect to the maximal torus consisting of the diagonal matrices, this is a root subgroup of $\mathrm{SL}_9(\mathbb{C})$ corresponding to the root α'. The roots $\alpha'_i = e_i - e_{i+1}$, $1 \le i \le 8$, form a basis of $\Phi(A_8)$. From the above we then see that

$$\begin{array}{llll} u'_{\alpha'_1}(t) \mapsto u_{\alpha_1}(t), & u'_{\alpha'_2}(t) \mapsto u_{\alpha_3}(t), & u'_{\alpha'_3}(t) \mapsto u_{\alpha_4}(t), & u'_{\alpha'_4}(t) \mapsto u_{\alpha_5}(t), \\ u'_{\alpha'_5}(t) \mapsto u_{\alpha_6}(t), & u'_{\alpha'_6}(t) \mapsto u_{\alpha_7}(t), & u'_{\alpha'_7}(t) \mapsto u_{\alpha_8}(t), & u'_{\alpha'_8}(t) \mapsto u_{-\widetilde{\alpha}}(t) \end{array}$$

defines a homomorphism $\mathrm{SL}_9(\mathbb{C}) \to E_8(\mathbb{C})$ onto the centralizer $C_{E_8(\mathbb{C})}(\overline{a})$. It is easy to check that this map has kernel $C_3 = \langle \omega I_9 \rangle$ and thus we may make the identification $C_{E_8(\mathbb{C})}(\overline{a}) = \mathrm{SL}_9(\mathbb{C})/C_3$. For any $g \in \mathrm{SL}_9(\mathbb{C})$ we denote by $\overline{g}$ its image in $\mathrm{SL}_9(\mathbb{C})/C_3 = C_{E_8(\mathbb{C})}(\overline{a}) \subseteq E_8(\mathbb{C})$. In particular we see that $a = \eta I_9$ corresponds to the element $\overline{a}$ from above. We also define the following elements in $\mathrm{SL}_9(\mathbb{C})$:

$$x_1 = \mathrm{diag}(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2), \qquad x_2 = \mathrm{diag}(1,1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2),$$
$$x_3 = \mathrm{diag}(1,1,1,1,1,1,\omega,\omega,\omega), \qquad y_1 = (1,2,3)(4,5,6)(7,8,9),$$
$$y_2 = (1,4,7)(2,5,8)(3,6,9).$$

From the explicit homomorphism above we easily find

$$\overline{a} = h_{\alpha_1}(\omega)h_{\alpha_2}(\omega)h_{\alpha_3}(\omega^2), \qquad \overline{x_1} = h_{\alpha_1}(\omega)h_{\alpha_5}(\omega)h_{\alpha_8}(\omega),$$
$$\overline{x_2} = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega), \qquad \overline{x_3} = h_{\alpha_1}(\omega^2)h_{\alpha_3}(\omega)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega),$$

and a direct computation in $E_8(\mathbb{C})$ shows that

$$\begin{aligned} n_{-\widetilde{\alpha}} = {} & n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8\cdot \\ & n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8. \end{aligned}$$

From this and the explicit homomorphism above we find, either by direct computation or by using the relations in $N(H)$, that

$$\begin{aligned} y_1 = {} & n_1n_3n_5n_6n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5\cdot \\ & n_4n_2n_3n_4n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3\cdot \\ & n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8 \cdot h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_7}(-1), \end{aligned}$$

$$\begin{aligned} y_2 = {} & n_2n_3n_1n_4n_2n_3n_4n_5n_4n_2n_3n_4n_6n_5n_4n_2n_3n_1n_4n_7n_6n_5n_4\cdot \\ & n_2n_3n_1n_4n_3n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5\cdot \\ & n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_8n_7n_6 \cdot h_{\alpha_2}(-1)h_{\alpha_5}(-1). \end{aligned}$$

(**1.5.2**) **Conjugacy classes.** To distinguish subgroups of $E_8(\mathbb{C})$, we need some information on the conjugacy classes of elements of order 3. These are given in [13, Table VI] (which is taken from [7, Table 4]): There are 4 such conjugacy classes, which we label **3A**, **3B**, **3C** and **3D**. Moreover these classes may be distinguished by their traces on $\mathfrak{e}_8$. Since the trace of the element $h \in H$ is given by $8+\sum_{\alpha\in\Phi(E_8)} \alpha(h)$ we easily obtain that $\overline{a} \in$ **3A**, $\overline{x_1}, \overline{x_2}, \overline{x_3} \in$ **3B** and $\overline{x_3a^{-1}} \in$ **3D**.

(**1.5.3**) **Notation.** If K is a field and n is a natural number, we define *the general symplectic group* $\mathrm{GSp}_{2n}(K)$ as the group of $2n \times 2n$ matrices over K which leaves the skew-symmetric scalar product given by the matrix

$$B = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ copies}},$$

invariant up to a nonzero scalar. Thus we have

$$\mathrm{GSp}_{2n}(K) = \{X \in \mathrm{GL}_{2n}(K) | X^t BX = cB, c \in K^\times\}.$$

We define $\chi : \mathrm{GSp}_{2n}(K) \to K^\times$ as follows. If $X \in \mathrm{GSp}_{2n}(K)$ with $X^tBX = cB$, $c \in K^\times$, we let $\chi(X) = c$. This is easily seen to be a group homomorphism. The kernel of χ is *the symplectic group* $\mathrm{Sp}_{2n}(K)$ which is the subgroup leaving the above skew-symmetric scalar product invariant, i.e.

$$\mathrm{Sp}_{2n}(K) = \{X \in \mathrm{GL}_{2n}(K) | X^t BX = B\}.$$

These groups were first investigated by Jordan and Dickson. In [9, §110] the groups $\mathrm{GSp}_{2n}(K)$ and $\mathrm{Sp}_{2n}(K)$ are called the *general Abelian linear group* and *the special Abelian linear group* and denoted by respectively $\mathrm{GA}(2n, K)$ and $\mathrm{SA}(2n, K)$. It should be noted that our notation $\mathrm{GSp}_{2n}(K)$ is not standard. Indeed no standard notation seems to be available since this group does not occur as often in applications as the symplectic group $\mathrm{Sp}_{2n}(K)$.

(1.5.4) Theorem (E_8). *The conjugacy classes of nontoral elementary abelian 3-subgroups of $E_8(\mathbb{C})$ are given by the following table.*

rank	*name*	*ordered basis*	*$E_8(\mathbb{C})$-class distribution*	$\dim C_{E_8(\mathbb{C})}(E)$
3	$E^{3a}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{a}\rangle$	$\mathbf{3A^{18}3B^{8}}$	8
3	$E^{3b}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{x_3}\rangle$	$\mathbf{3B^{26}}$	14
4	$E^{4a}_{E_8}$	$\left\langle \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3a^{-1}}\right\rangle$	$\mathbf{3A^{52}3B^{26}3D^{2}}$	4
4	$E^{4b}_{E_8}$	$\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a}\rangle$	$\mathbf{3A^{54}3B^{26}}$	2
4	$E^{4c}_{E_8}$	$\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}\rangle$	$\mathbf{3B^{80}}$	8
5	$E^{5a}_{E_8}$	$\left\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3a^{-1}}\right\rangle$	$\mathbf{3A^{156}3B^{80}3D^{6}}$	2
5	$E^{5b}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a}\rangle$	$\mathbf{3A^{162}3B^{80}}$	0

These groups are all 1-oversize and their Weyl groups with respect to the given ordered bases satisfies the following:

$$W(E^{3a}_{E_8}) = \left[\begin{array}{c|c} \mathrm{GL}_2(\mathbb{F}_3) & \begin{matrix} * \\ * \end{matrix} \\ \hline 0 \;\; 0 & det \end{array}\right], W(E^{3b}_{E_8}) \supseteq \mathrm{SL}_3(\mathbb{F}_3), W(E^{4a}_{E_8}) = \left[\begin{array}{c|c} \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 \;\; 0 \;\; 0 & \varepsilon \end{array}\right],$$

$$W(E^{4b}_{E_8}) = \left[\begin{array}{c|c|c} \varepsilon & * \;\; * & * \\ \hline \begin{matrix} 0 \\ 0 \end{matrix} & \mathrm{GL}_2(\mathbb{F}_3) & \begin{matrix} * \\ * \end{matrix} \\ \hline 0 & 0 \;\; 0 & det \end{array}\right], W(E^{4c}_{E_8}) \supseteq \left[\begin{array}{c|c} \varepsilon & * \;\; * \;\; * \\ \hline \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \mathrm{SL}_3(\mathbb{F}_3) \end{array}\right],$$

$$W(E^{5a}_{E_8}) = \left[\begin{array}{c|c|c} \varepsilon_1 & * \;\; * \;\; * & * \\ \hline \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 \;\; 0 \;\; 0 & \varepsilon_2 \end{array}\right], W(E^{5b}_{E_8}) = \left[\begin{array}{c|c} GSp_4(\mathbb{F}_3) & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \\ \hline 0 \;\; 0 \;\; 0 \;\; 0 & \chi \end{array}\right]$$

where det is the determinant of the matrix from $\mathrm{GL}_2(\mathbb{F}_3)$ *in the description of* $W(E^{3a}_{E_8})$ *and* $W(E^{4b}_{E_8})$*. In the description of* $W(E^{5b}_{E_8})$*,* χ *denotes the value of the homomorphism* $\chi : GSp_4(\mathbb{F}_3) \to \mathbb{F}_3^\times$ *defined in (1.5.3) evaluated on the matrix from* $GSp_4(\mathbb{F}_3)$*.*

(**1.5.5**) **Remark**. In section 1.6 we will prove that the Weyl groups of $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ are exactly the lower bounds given above.

(**1.5.6**) **Proof**:

MAXIMAL NONTORAL SUBGROUPS: By [13, Lemma (11.7) and Lemma (11.9)], any maximal nontoral elementary abelian 3-subgroup of $E_8(\mathbb{C})$ contains an element of type **3A**. We may thus find representatives in $C_{E_8(\mathbb{C})}(\overline{a}) = \mathrm{SL}_9(\mathbb{C})/C_3$. From [13, Corollary 11.10], it follows that there are two conjugacy classes of these maximal nontoral elementary abelian 3-subgroups: $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$, both of which have rank 5. Moreover, by [13, Lemma 11.5], their preimages in $\mathrm{SL}_9(\mathbb{C})$ may be chosen to have the shape $3^{1+2} \circ 9 \times 3 \times 3$ and $3^{1+4} \circ 9$, where $\circ$ denotes a central product over $C_3 = \langle \omega I_3 \rangle$. Using the representation theory of extraspecial 3-groups ([12, Chapter 5.5]) we find that $E^{5a}_{E_8}$ is represented by $\left\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle$ and that $E^{5b}_{E_8}$ is represented by $\langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a} \rangle$.

LOWER BOUNDS FOR WEYL GROUPS OF MAXIMAL NONTORAL SUBGROUPS: We can find lower bounds for the Weyl groups of $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ by conjugating with elements in the centralizer $C_{E_8(\mathbb{C})}(\overline{a}) = \mathrm{SL}_9(\mathbb{C})/C_3$ and the normalizer $N(H)$ of the maximal torus.

Note that

$$\begin{aligned} a &= \eta I_3 \oplus \eta I_3 \oplus \eta I_3, & x_1 &= \beta \oplus \beta \oplus \beta, & x_2 &= I_3 \oplus \omega I_3 \oplus \omega^2 I_3, \\ x_3 &= I_3 \oplus I_3 \oplus \omega I_3, & y_1 &= \gamma \oplus \gamma \oplus \gamma \end{aligned}$$

and $(A \oplus B \oplus C)^{y_2} = B \oplus C \oplus A$. Conjugation by $\tau_1 \oplus \tau_1 \oplus \tau_1$, $\tau_2 \oplus \tau_2 \oplus \tau_2$ and $I_3 \oplus \beta^2 \oplus \beta$ gives

$$\begin{aligned} &\tau_1 \oplus \tau_1 \oplus \tau_1 : a \mapsto a, \quad x_1 \mapsto x_1 y_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2. && \text{(i)} \\ &\tau_2 \oplus \tau_2 \oplus \tau_2 : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_1 y_1, \quad y_2 \mapsto y_2. && \text{(ii)} \\ &I_3 \oplus \beta^2 \oplus \beta : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_2 y_1, \quad y_2 \mapsto x_1 y_2. && \text{(iii)} \end{aligned}$$

Now consider the group $E^{5a}_{E_8}$. From (i)—(iii) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E^{5a}_{E_8}$ and that conjugation by these elements induces the automorphisms on $E^{5a}_{E_8}$ given by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Letting $\sigma = -(1,4)(2,5)(3,6) \in \mathrm{SL}_9(\mathbb{C})$ we see that $(A \oplus B \oplus C)^\sigma = B \oplus A \oplus C$. Using this and the above we obtain that $\overline{\sigma}$, $\overline{y_2}$ and $\overline{I_3 \oplus I_3 \oplus \beta^2}$ normalize $E^{5a}_{E_8}$ and that conjugation by these elements induces the automorphisms on $E^{5a}_{E_8}$ given by the matrices

$$\begin{bmatrix} 2&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&0&0&1 \end{bmatrix}.$$

By using the relations in $N(H)$ given above or by direct computation, it may be checked that conjugation by the element

$$\begin{aligned} n = {} & n_1n_2n_4n_2n_3n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5n_4n_2n_3n_4n_7n_6n_5n_4n_8n_7n_6\cdot \\ & n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_7 \cdot h(1,1,-1,-1,-1,1,-1,-1) \end{aligned}$$

induces the automorphism on $E^{5a}_{E_8}$ represented by the matrix

$$\begin{bmatrix} 1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&2 \end{bmatrix}.$$

It is easy to see that the above matrices generate the group

$$W'(E^{5a}_{E_8}) = \left[\begin{array}{c|ccc|c} \varepsilon_1 & * & * & * & * \\ \hline 0 & & & & 0 \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon_2 \end{array}\right],$$

and thus $W(E^{5a}_{E_8})$ contains this group.

Next consider the group $E^{5b}_{E_8}$. From (i)—(iii) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E^{5b}_{E_8}$ and that conjugation by these elements induces the automorphisms on $E^{5b}_{E_8}$ given by the matrices

$$\begin{bmatrix} 1&0&0&0&0\\1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1&0\\0&1&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}.$$

Now note that $a = \Delta_{3,3}(\eta I_3)$, $x_2 = \Delta_{3,3}(\beta)$ and $y_2 = \Delta_{3,3}(\gamma)$. Noting also that $\Delta_{3,3}(M_1)$ commutes with $M_2 \oplus M_2 \oplus M_2$ for any $M_1, M_2 \in M_3(\mathbb{C})$ we see that the elements $\overline{\Delta_{3,3}(\tau_1)}$ and $\overline{\Delta_{3,3}(\tau_2)}$ normalize $E^{5b}_{E_8}$. The automorphisms induced on $E^{5b}_{E_8}$ by conjugation with these elements have the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The upper left 4×4-corner of these matrices is easily seen to generate the group

$$\left\langle \left[\begin{array}{c|c} \mathrm{SL}_2(\mathbb{F}_3) & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \mathrm{SL}_2(\mathbb{F}_3) \end{array}\right], \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

Using the coordinates $(\eta_1, \xi_1, \eta_2, \xi_2)$ in $(\mathbb{F}_3)^4$ we see that this is exactly the same group as in [9, §114]. By comparing with [9, §110] and (1.5.3) we see that this is the symplectic group $\mathrm{Sp}_4(\mathbb{F}_3)$ from above.

By using the relations in $N(H)$ given above or by direct computation, we get that conjugation by the element

$$\begin{aligned} n = {} & n_2n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6 \cdot \\ & n_5n_4n_2n_3n_4n_5n_6n_7n_8 \cdot h(1,-1,-1,-1,-1,1,1,1) \end{aligned}$$

of $N(H)$ induces the automorphism on $E^{5b}_{E_8}$ represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

It now follows that $W(E^{5b}_{E_8})$ contains the group

$$W'(E^{5b}_{E_8}) = \left[\begin{array}{c|c} \mathrm{GSp}_4(\mathbb{F}_3) & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \\ \hline 0\ 0\ 0\ 0 & \chi \end{array}\right].$$

LOWER BOUNDS FOR OTHER WEYL GROUPS: We now show that the other Weyl groups in the table are all lower bounds. To do this consider one of the nonmaximal groups E from the table. We then have $E \subseteq E^{5a}_{E_8}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E_{E_8}^{5a})$ fixing E. As an example we find that the stabilizer of $E_{E_8}^{3a}$ inside $W'(E_{E_8}^{5a})$ is

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 & x & x \\ 0 & a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & \det & 0 \\ 0 & 0 & 0 & 0 & \det \end{bmatrix},$$

where $\det = a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \neq 0$. The action of such a matrix on $E_{E_8}^{3a}$ is given by

$$\overline{x_1} \mapsto (\overline{x_1})^{a_{11}} (\overline{y_1})^{a_{21}}, \qquad \overline{y_1} \mapsto (\overline{x_1})^{a_{12}} (\overline{y_1})^{a_{22}}, \qquad \overline{a} \mapsto (\overline{x_1})^{a_{13}} (\overline{y_1})^{a_{23}} (\overline{a})^{\det}.$$

Thus $W(E_{E_8}^{3a})$ contains the group

$$W'(E_{E_8}^{3a}) = \left[\begin{array}{c|c} \mathrm{GL}_2(\mathbb{F}_3) & \begin{array}{c} * \\ * \end{array} \\ \hline 0 \;\; 0 & \det \end{array} \right]$$

as claimed. Similar computations show that for the remaining groups $E = E_{E_8}^{3b}$, $E_{E_8}^{4a}$, $E_{E_8}^{4b}$ and $E_{E_8}^{4c}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

ORBIT COMPUTATION: Note first that all elementary abelian subgroups of rank at most two are toral by Lemma (1.1.3(3)). By using the lower bounds on the Weyl groups of $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ established above, we may find a set of representatives for the conjugacy classes of subgroups of $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ of rank 3 and 4.

Under the action of $W'(E_{E_8}^{5a})$, the set of rank 3 subgroups of $E_{E_8}^{5a}$ has orbit representatives

$$\begin{aligned} E_{E_8}^{3a} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, E_{E_8}^{3b} &= \langle \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle, \\ &\langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle, \end{aligned}$$

and under the action of $W'(E_{E_8}^{5b})$, the set of rank 3 subgroups of $E_{E_8}^{5b}$ has orbit representatives

$$E_{E_8}^{3a} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle.$$

Similarly we find that under the action of $W'(E_{E_8}^{5a})$, the set of rank 4 subgroups of $E_{E_8}^{5a}$ has orbit representatives

$$\begin{aligned} E_{E_8}^{4a} = \left\langle \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle, E_{E_8}^{4b} &= \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle, \\ E_{E_8}^{4c} &= \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle, \end{aligned}$$

and that under the action of $W'(E_{E_8}^{5b})$, the set of rank 4 subgroups of $E_{E_8}^{5b}$ has orbit representatives

$$E_{E_8}^{4b} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle \text{ and } E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle.$$

CLASS DISTRIBUTIONS: Recall that by (1.5.2), $\overline{a}$ is in the conjugacy class **3A**, $\overline{x_1}$ and $\overline{x_2}$ are in the class **3B** and $\overline{x_3a^{-1}}$ belongs to the class **3D**. Using the actions of $W'(E^{5a}_{E_8})$ and $W'(E^{5b}_{E_8})$ it is then straightforward to verify the class distributions given in the table. As an example consider the group $E^{5a}_{E_8}$. Under the action of $W'(E^{5a}_{E_8})$ it contains 156 elements conjugate to $\overline{a}$, 78 elements conjugate to $\overline{x_1}$, 2 elements conjugate to $\overline{x_2}$ and 6 elements conjugate to $\overline{x_3a^{-1}}$, which gives the class distribution in the table. Similar computations give the results for the remaining groups.

We also see that the group $E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle$ has class distribution $\mathbf{3B^{80}}$ and from the class distribution of $E^{5b}_{E_8}$ we get $E_0 = (E^{5b}_{E_8} \cap \mathbf{3B}) \cup \{1\}$. It then follows from [13, Lemma (11.5)] that E_0 is toral.

CENTRALIZER DIMENSIONS: By using Lemma (1.1.3(6)), the class distributions found above and the traces given in [7, Table 4], the dimensions given in the last column of the table in the theorem are easily checked.

OTHER NONTORAL SUBGROUPS: We see directly that the groups

$$\langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle$$

are toral. Since the group $\langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle$ is a subgroup of E_0 it is also toral. Alternatively, from the action of $W'(E^{5a}_{E_8})$ we see that it is conjugate to the group $\langle \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle$, which is visibly toral. Thus any nontoral elementary abelian 3-subgroup of $E_8(\mathbb{C})$ is conjugate to a group in the table. Moreover, since their class distributions differ, none of the groups occurring in the table are conjugate.

To see that the groups in the table are actually nontoral we may proceed as follows. The group $E^{3a}_{E_8}$ contains the element $\overline{a}$, so by Lemma (1.1.3(1)) it is toral if and only if it is toral in $C_{E_8(\mathbb{C})}(\overline{a}) = \mathrm{SL}_9(\mathbb{C})/C_3$. However this is not the case by Lemma (1.1.3(5)), since its lift to $\mathrm{SL}_9(\mathbb{C})$ is nonabelian. The groups $E^{4a}_{E_8}$ and $E^{4b}_{E_8}$ are thus also nontoral since they contain $E^{3a}_{E_8}$. We saw above that the Weyl group of $E^{3b}_{E_8}$ contains $\mathrm{SL}_3(\mathbb{F}_3)$, which has order $2^4 \cdot 3^3 \cdot 13$. Since the Weyl group of $E_8(\mathbb{C})$ has order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ which is not divisible by 13 it follows from Lemma (1.1.3(4)) that $E^{3b}_{E_8}$ is nontoral. Since $E^{4c}_{E_8}$ contains $E^{3b}_{E_8}$ it is also nontoral.

OVERSIZE: Note that both $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ are 1-oversize in $E_8(\mathbb{C})$. For $E^{5a}_{E_8}$ this follows from the fact that it is nontoral and contains the toral subgroup $\langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle$ of codimension 1. For $E^{5b}_{E_8}$ it follows similarly using the toral subgroup E_0 from above. Since the other groups in the table are contained in a maximal nontoral elementary abelian 3-subgroup, these are 1-oversize as well.

EXACT WEYL GROUPS: By now we know that the groups in the table is a set of representatives for the conjugacy classes of nontoral elementary abelian 3-subgroups of $E_8(\mathbb{C})$, that they are all 1-oversize and that the matrix groups given in the theorem are lower bounds for the Weyl groups. We thus only need to verify that for all the groups except $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ the Weyl group is the one given above.

First consider the group $E_{E_8}^{3a} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle$. Let $x_4 = \tau_2^{-1} \oplus \tau_2^{-1} \oplus \tau_2^{-1} \in \mathrm{SL}_9(\mathbb{C})$. Note that conjugation by $(2,7,3,4)(5,8,9,6) \in \mathrm{SL}_9(\mathbb{C})$ acts as follows:

$$a \mapsto a, \qquad x_1 \mapsto x_2, \qquad x_2 \mapsto x_1^2, \qquad x_3 a^{-1} \mapsto x_4, \qquad y_1 \mapsto y_2, \qquad y_2 \mapsto y_1^2.$$

In particular we see that $E_{E_8}^{3a}$ is conjugate to $\langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle$. If $W(E_{E_8}^{3a})$ was larger than the group $W'(E_{E_8}^{3a})$ from above, we then see (using for example the computer program mentioned in the introduction) that $W(E_{E_8}^{3a})$ would have to contain one of the groups

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * \\ & & * \\ \hline 0 & 0 & \varepsilon \end{array}\right] \text{ or } \mathrm{SL}_3(\mathbb{F}_3)$$

which are the minimal overgroups of $W'(E_{E_8}^{3a})$ inside $\mathrm{GL}_3(\mathbb{F}_3)$. Thus $W(E_{E_8}^{3a})$ would have to contain one of the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

This would mean that inside $C_{E_8(\mathbb{C})}(\overline{x_2}, \overline{a})$ there would be an element which conjugates $\overline{y_2}$ into either $\overline{y_2}^2$ or $\overline{y_2 a}$. However, since $C_{E_8(\mathbb{C})}(\overline{a}) = \mathrm{SL}_9(\mathbb{C})/C_3$, we get

$$C_{E_8(\mathbb{C})}(\overline{x_2}, \overline{a}) = C_{\mathrm{SL}_9(\mathbb{C})/C_3}(\overline{x_2}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\} \rangle}.$$

From this it is easily seen that no such element exists, and thus we have $W(E_{E_8}^{3a}) = W'(E_{E_8}^{3a})$ as claimed.

Note that the groups $E_{E_8}^{4a}$, $E_{E_8}^{4b}$, $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ all contain $E_{E_8}^{3a}$. The minimal overgroups of $W'(E)$ in $\mathrm{GL}(E)$ for these groups are found to be:

$$E^{4a}_{E_8}: \left[\begin{array}{c|c} \mathrm{GL}_3(\mathbb{F}_3) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\ 0\ 0 & \varepsilon \end{array}\right], \left[\begin{array}{c|c} \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline *\ *\ * & \varepsilon \end{array}\right], \left[\begin{array}{c|c} \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix}*\\ *\\ *\end{matrix} \\ \hline 0\ 0\ 0 & \varepsilon \end{array}\right]$$

$$E^{4b}_{E_8}: \left[\begin{array}{c|c} \varepsilon & *\ *\ * \\ \hline \begin{matrix}0\\0\\0\end{matrix} & \mathrm{SL}_3(\mathbb{F}_3) \end{array}\right], \left[\begin{array}{c|c|c} \varepsilon_1 & *\ * & * \\ \hline \begin{matrix}0\\0\end{matrix} & \mathrm{GL}_2(\mathbb{F}_3) & \begin{matrix}*\\ *\end{matrix} \\ \hline 0 & 0\ 0 & \varepsilon_2 \end{array}\right]$$

$$E^{5a}_{E_8}: \left[\begin{array}{c|c|c} \varepsilon_1 & *\ *\ * & * \\ \hline \begin{matrix}0\\0\\0\end{matrix} & \mathrm{GL}_3(\mathbb{F}_3) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0\ 0\ 0 & \varepsilon_2 \end{array}\right], \left[\begin{array}{c|c|c} \varepsilon_1 & *\ *\ * & * \\ \hline \begin{matrix}0\\0\\0\end{matrix} & \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & *\ *\ * & \varepsilon_2 \end{array}\right], \left[\begin{array}{c|c|c} \varepsilon_1 & *\ *\ * & * \\ \hline \begin{matrix}0\\0\\0\end{matrix} & \mathrm{SL}_3(\mathbb{F}_3) & \begin{matrix}*\\ *\\ *\end{matrix} \\ \hline 0 & 0\ 0\ 0 & \varepsilon_2 \end{array}\right]$$

$$E^{5b}_{E_8}: \left[\begin{array}{c|c} \mathrm{GSp}_4(\mathbb{F}_3) & \begin{matrix}*\\ *\\ *\\ *\end{matrix} \\ \hline 0\ 0\ 0\ 0 & \varepsilon \end{array}\right].$$

From this we see that if $W(E^{4a}_{E_8}) \neq W'(E^{4a}_{E_8})$ then $W(E^{4a}_{E_8})$ will contain one of the matrices

$$\begin{bmatrix} 1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&2&0&0\\0&0&2&1\\0&0&0&1 \end{bmatrix}$$

all fixing the subgroup $E^{3a}_{E_8}$ of $E^{4a}_{E_8}$. However the automorphisms of $E^{3a}_{E_8}$ induced by these elements are given by the matrices

$$\begin{bmatrix} 1&0&0\\0&2&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&2&0\\0&0&1 \end{bmatrix}.$$

Since none of these are contained in $W(E^{3a}_{E_8})$ we get $W(E^{4a}_{E_8}) = W'(E^{4a}_{E_8})$ as claimed. Similar computations using the above list of minimal overgroups show that $W(E) = W'(E)$ for $E = E^{4b}_{E_8}$, $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$. $\square$

1.6 Centralizers

In this section we compute the centralizers of most of the elementary abelian subgroups given in the previous sections. Our calculations also enables us to finish the computation of Weyl groups begun earlier; we show that in the cases where only a lower bound was given, the Weyl group actually equals this lower bound. The notation for the various elements and subgroups is the same as before. For an abelian group A and a prime p, we let ${}_pA$ denote the elementary abelian p-subgroup of A generated by the elements in A of order p.

The results of this section are used in [1] together with the Weyl group computations to show the vanishing of certain higher limits which occurs in the proof of the classification of p-compact groups at odd primes. The main technical result we need for this application is the following:

(**1.6.1**) **Theorem**. *Let p be an odd prime, let G be one of the exceptional simple algebraic groups over $\mathbb{C}$ considered in the previous sections and let E be a non-toral elementary abelian p-subgroup of G. Assume that $p \neq 3$ or that $(G, E) \neq (3E_6(\mathbb{C}), E^3_{3E_6})$, $(E_6(\mathbb{C}), E^{2b}_{E_6})$ and $(E_8(\mathbb{C}), E^{3b}_{E_8})$ for $p = 3$. Then ${}_pZ(C_G(E)) = E$.* $\square$

(**1.6.2**) **Remark**. Obviously, if E is an elementary abelian p-subgroup of G, then ${}_pZ(C_G(E))$ is an elementary abelian p-subgroup containing E. Thus if E is maximal, then the conclusion of Theorem (1.6.1) holds for E. However we are not going to need this since we can compute most of the centralizers and their centers directly.

We do not know if the conclusion of the theorem holds in the cases $(G, E) = (E_6(\mathbb{C}), E^{2b}_{E_6})$ and $(G, E) = (E_8(\mathbb{C}), E^{3b}_{E_8})$ with $p = 3$, since we do not know the exact structure of these centralizers.

For $p = 3$, $G = 3E_6(\mathbb{C})$ and $E = E^3_{3E_6}$ we have ${}_pZ(C_G(E)) = \langle E, Z(G)\rangle$ due to the fact that $Z(G)$ has p-torsion. However, in general we do not have ${}_pZ(C_G(E)) = E$ for any nontoral elementary abelian p-subgroup E of a simple group G of adjoint type. The following example is inspired by [5, Section 3]. Let $p \geq 5$ be a prime number and choose any m with $3 \leq m \leq p-1$. Let $\xi = e^{2\pi i/p}$ and $G = \mathrm{PSL}_{mp}(\mathbb{C})$. For an element $g \in \mathrm{SL}_{mp}(\mathbb{C})$, the corresponding element in G is denoted by $\overline{g}$. Choose different integers $i_1, \ldots, i_m$ in $\{0, \ldots, p-1\}$ satisfying $i_1 + \ldots + i_m \equiv 0 \pmod p$ and consider the following elements in $\mathrm{SL}_{mp}(\mathbb{C})$:

$$\begin{aligned}
g_1 &= \mathrm{diag}(\underbrace{1, \xi, \ldots, \xi^{p-1}}, \ldots, \underbrace{1, \xi, \ldots, \xi^{p-1}}),\\
g_2 &= (1, 2, \ldots, p)(p+1, \ldots, 2p) \ldots ((m-1)p+1, \ldots, mp)\,,\\
g_3 &= \mathrm{diag}(\underbrace{\xi^{i_1}, \ldots, \xi^{i_1}}_{p}, \underbrace{\xi^{i_2}, \ldots, \xi^{i_2}}_{p}, \ldots, \underbrace{\xi^{i_m}, \ldots, \xi^{i_m}}_{p}).
\end{aligned}$$

Note that $E = \langle \overline{g_1}, \overline{g_2}, \overline{g_3}\rangle$ is an elementary abelian p-subgroup of rank 3. Then we have

$$\begin{aligned}
C_{\mathrm{PSL}_{mp}(\mathbb{C})}(\overline{g_1}) &= \overline{\langle g_2, C_{\mathrm{SL}_{mp}(\mathbb{C})}(g_1)\rangle},\\
C_{\mathrm{PSL}_{mp}(\mathbb{C})}(\overline{g_1}, \overline{g_2}) &= \overline{\langle g_1, g_2, C_{\mathrm{SL}_{mp}(\mathbb{C})}(g_1, g_2)\rangle}.
\end{aligned}$$

We directly get $C_{\mathrm{SL}_{mp}(\mathbb{C})}(g_1, g_2) = \Delta_{m,p}\left(\{A \in \mathrm{GL}_m(\mathbb{C}) | \, (\det A)^p = 1\}\right)$. Since m is coprime to p we actually have

$$C_{\mathrm{PSL}_{mp}(\mathbb{C})}(\overline{g_1}, \overline{g_2}) = \overline{\langle g_1, g_2, \Delta_{m,p}(\mathrm{SL}_m(\mathbb{C}))\rangle} \cong \langle \overline{g_1}, \overline{g_2}\rangle \times \mathrm{PSL}_m(\mathbb{C}).$$

Here $\overline{g_3}$ corresponds to the element $x = \overline{\mathrm{diag}(\xi^{i_1}, \ldots, \xi^{i_m})} \in \mathrm{PSL}_m(\mathbb{C})$. Since $i_1, \ldots, i_m$ are different and $m < p$ we get that $C_{\mathrm{PSL}_m(\mathbb{C})}(x) = \mathbb{T}_{m-1}$ is the maximal torus in $\mathrm{PSL}_m(\mathbb{C})$. Therefore

$$C_{\mathrm{PSL}_{mp}(\mathbb{C})}(E) = \overline{\langle g_1, g_2\rangle} \times \mathbb{T}_{m-1},$$

and hence ${}_pZ(C_{\mathrm{PSL}_{mp}(\mathbb{C})}(E))$ has rank $2 + (m-1) = m+1 > 3$.

(**1.6.3**) **Centralizers in $3E_6(\mathbb{C})$ and $E_6(\mathbb{C})$.** Let $\Theta : \mathrm{SL}_3(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C})^3/C_3 \subseteq 3E_6(\mathbb{C})$ denote the homomorphism given by $\Theta(g) = [g, g, g]$ for $g \in \mathrm{SL}_3(\mathbb{C})$. We have already seen that $C_{3E_6(\mathbb{C})}(a) = \mathrm{SL}_3(\mathbb{C})^3/C_3$. From this we get

$$\begin{aligned} C_{3E_6(\mathbb{C})}(a, x_2) &= C_{\mathrm{SL}_3(\mathbb{C})^3/C_3}(x_2) = \left\langle y_2, \left(\mathbb{T}_2 \times \mathbb{T}_2 \times \mathbb{T}_2\right)/C_3\right\rangle, \\ C_{3E_6(\mathbb{C})}(a, x_2, y_2) &= \left\langle x_2, y_2, \left(\langle \omega I_3\rangle \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle\right)/C_3\right\rangle = E^4_{3E_6}. \end{aligned}$$

Thus $C_{3E_6(\mathbb{C})}(E^3_{3E_6}) = C_{3E_6(\mathbb{C})}(E^4_{3E_6}) = E^4_{3E_6}$. We now turn to the subgroups of $E_6(\mathbb{C})$. By Lemma (1.3.2) the group $E^{2a}_{E_6} = \langle \overline{x_2}, \overline{y_1}\rangle$ is conjugate to the group $\langle \overline{a}, \overline{s_1}\rangle$. Since $a^{s_1} = az^2$ we obtain $C_{E_6(\mathbb{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbb{C})^3/C_3\rangle}$, and from this we get

$$C_{E_6(\mathbb{C})}(\overline{a}, \overline{s_1}) = \overline{\langle a, s_1, z, \Theta(\mathrm{SL}_3(\mathbb{C}))\rangle} = \overline{\langle a, s_1, z\rangle \times \Theta(\mathrm{SL}_3(\mathbb{C}))} \cong \langle \overline{a}, \overline{s_1}\rangle \times \mathrm{PSL}_3(\mathbb{C}).$$

We thus have $C_{E_6(\mathbb{C})}(E^{2a}_{E_6}) = E^{2a}_{E_6} \times \mathrm{PSL}_3(\mathbb{C})$. In particular $Z(C_{E_6(\mathbb{C})}(E^{2a}_{E_6})) = E^{2a}_{E_6}$. By abusing the notation slightly, we let $\overline{g}$ denote the image of $g \in \mathrm{SL}_3(\mathbb{C})$ in the quotient $\mathrm{PSL}_3(\mathbb{C})$. From Lemma (1.3.2) we then see that the elements $\overline{a}$, $\overline{y_2}$ and $\overline{x_2 x_1^{-1}}$ in $C_{E_6(\mathbb{C})}(E^{2a}_{E_6})$ equal the elements $\overline{\beta}$, $\overline{\gamma^2}$ and $\overline{\tau_2}$ in the $\mathrm{PSL}_3(\mathbb{C})$ component of $C_{E_6(\mathbb{C})}(E^{2a}_{E_6})$. From this we now immediately get

$$\begin{aligned} C_{E_6(\mathbb{C})}(E^{3a}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}), & C_{E_6(\mathbb{C})}(E^{3d}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\tau_2}), \\ C_{E_6(\mathbb{C})}(E^{4a}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}, \overline{\gamma^2}), & C_{E_6(\mathbb{C})}(E^{4b}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}, \overline{\tau_2}). \end{aligned}$$

Note that $C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}) = \overline{\langle \mathbb{T}_2, \gamma\rangle}$, from which we get $C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}, \overline{\gamma^2}) = \left\langle \overline{\beta}, \overline{\gamma}\right\rangle$ and $C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\beta}, \overline{\tau_2}) = \overline{\mathbb{T}_2}$. We thus have

$$\begin{aligned} C_{E_6(\mathbb{C})}(E^{3a}_{E_6}) &= E^{2a}_{E_6} \times (\mathbb{T}_2 : 3), & Z(C_{E_6(\mathbb{C})}(E^{3a}_{E_6})) &= E^{3a}_{E_6}, \\ C_{E_6(\mathbb{C})}(E^{4a}_{E_6}) &= E^{4a}_{E_6}, & Z(C_{E_6(\mathbb{C})}(E^{4a}_{E_6})) &= E^{4a}_{E_6}, \\ C_{E_6(\mathbb{C})}(E^{4b}_{E_6}) &= E^{2a}_{E_6} \times \mathbb{T}_2, & Z(C_{E_6(\mathbb{C})}(E^{4b}_{E_6})) &= E^{2a}_{E_6} \times \mathbb{T}_2. \end{aligned}$$

In particular ${}_3Z(C_{E_6(\mathbb{C})}(E^{4b}_{E_6})) = E^{4b}_{E_6}$. For $E^{3d}_{E_6}$ we have $C_{\mathrm{PSL}_3(\mathbb{C})}(\overline{\tau_2}) \cong \mathrm{GL}_2(\mathbb{C})$ so

$$C_{E_6(\mathbb{C})}(E^{3d}_{E_6}) = E^{2a}_{E_6} \times \mathrm{GL}_2(\mathbb{C}), \qquad Z(C_{E_6(\mathbb{C})}(E^{3d}_{E_6})) = E^{2a}_{E_6} \times \mathbb{T}_1$$

and ${}_3Z(C_{E_6(\mathbb{C})}(E^{3d}_{E_6})) = E^{3d}_{E_6}$.

We now consider the group $E_{E_6}^{3b}$. Since $C_{E_6(\mathbb{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbb{C})^3/C_3\rangle}$ we get

$$\begin{aligned} C_{E_6(\mathbb{C})}(\overline{a}, \overline{x_2}) &= \overline{\langle s_1, y_1, y_2, (\mathbb{T}_2 \times \mathbb{T}_2 \times \mathbb{T}_2)/C_3\rangle}, \\ C_{E_6(\mathbb{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) &= \overline{\langle s_1, y_1, y_2, [I_3, \beta, \beta^2], x_2, (\langle \omega I_3\rangle \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle)/C_3, \rangle} \end{aligned}$$

which is a finite group of order 3^6. It is not hard to check that the center is the subgroup $\langle \overline{a}, \overline{x_2}, \overline{y_2}\rangle$ and we conclude that $Z(C_{E_6(\mathbb{C})}(E_{E_6}^{3b})) = E_{E_6}^{3b}$.

Finally consider the group $E_{E_6}^{3c}$. We obtain

$$\begin{aligned} C_{E_6(\mathbb{C})}(\overline{a}, \overline{x_1}) &= \overline{\langle y_1, (\mathrm{SL}_3(\mathbb{C}) \times \mathbb{T}_2 \times \mathbb{T}_2)/C_3\rangle}, \\ C_{E_6(\mathbb{C})}(\overline{a}, \overline{x_1}, \overline{y_1}) &= \overline{\langle y_1, x_1, (\mathrm{SL}_3(\mathbb{C}) \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle)/C_3\rangle}. \end{aligned}$$

Thus $C_{E_6(\mathbb{C})}(E_{E_6}^{3c}) = \mathrm{SL}_3(\mathbb{C}) \circ_{\langle \overline{a}\rangle} E_{E_6}^{3c}$ is a central product, and from this we find $Z(C_{E_6(\mathbb{C})}(E_{E_6}^{3c})) = E_{E_6}^{3c}$.

(**1.6.4**) **Centralizers in** $E_8(\mathbb{C})$**.** In section 1.5 we saw that $E_{E_8}^{3a}$ is conjugate to $\langle \overline{a}, \overline{x_2}, \overline{y_2}\rangle$ and that

$$C_{E_8(\mathbb{C})}(\overline{a}, \overline{x_2}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\}\rangle}.$$

From this we directly get

$$\begin{aligned} C_{E_8(\mathbb{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) &= \overline{\left\langle x_2, y_2, \{A \oplus A \oplus A \mid (\det A)^3 = 1\}\right\rangle} \\ &= \overline{\langle x_2, y_2, a, \{A \oplus A \oplus A \mid \det A = 1\}\rangle} \\ &\cong \langle \overline{x_2}, \overline{y_2}, \overline{a}\rangle \times \mathrm{PSL}_3(\mathbb{C}). \end{aligned}$$

Thus $C_{E_8(\mathbb{C})}(E_{E_8}^{3a}) = E_{E_8}^{3a} \times \mathrm{PSL}_3(\mathbb{C})$ and $Z(C_{E_8(\mathbb{C})}(E_{E_8}^{3a})) = E_{E_8}^{3a}$.

From section 1.5 we see that the elements $\overline{x_2}$, $\overline{x_3 a^{-1}}$ and $\overline{y_2}$ in $C_{E_6(\mathbb{C})}(E_{E_8}^{3a})$ equal the elements $\overline{\beta^2}$, $\overline{\tau_2^{-1}}$ and $\overline{\gamma^2}$ in the $\mathrm{PSL}_3(\mathbb{C})$ component of $C_{E_8(\mathbb{C})}(E_{E_8}^{3a})$. Now the computations in $\mathrm{PSL}_3(\mathbb{C})$ done above in the case of $E_6(\mathbb{C})$ immediately show

$$\begin{aligned} C_{E_8(\mathbb{C})}(E_{E_8}^{4a}) &= E_{E_8}^{3a} \times \mathrm{GL}_2(\mathbb{C}), & Z(C_{E_8(\mathbb{C})}(E_{E_8}^{4a})) &= E_{E_8}^{3a} \times \mathbb{T}_1, \\ C_{E_8(\mathbb{C})}(E_{E_8}^{4b}) &= E_{E_8}^{3a} \times (\mathbb{T}_2 : 3), & Z(C_{E_8(\mathbb{C})}(E_{E_8}^{4b})) &= E_{E_8}^{4b}, \\ C_{E_8(\mathbb{C})}(E_{E_8}^{5a}) &= E_{E_8}^{3a} \times \mathbb{T}_2, & Z(C_{E_8(\mathbb{C})}(E_{E_8}^{5a})) &= E_{E_8}^{3a} \times \mathbb{T}_2, \\ C_{E_8(\mathbb{C})}(E_{E_8}^{5b}) &= E_{E_8}^{5b}, & Z(C_{E_8(\mathbb{C})}(E_{E_8}^{5b})) &= E_{E_8}^{5b}, \end{aligned}$$

In particular ${}_3Z(C_{E_8(\mathbb{C})}(E_{E_8}^{4a})) = E_{E_8}^{4a}$ and ${}_3Z(C_{E_8(\mathbb{C})}(E_{E_8}^{5a})) = E_{E_8}^{5a}$.

We now turn to the nontoral subgroups of $2E_7(\mathbb{C})$ and the two remaining subgroups $E_{E_8}^{3b}$ and $E_{E_8}^{4c}$ in $E_8(\mathbb{C})$.

(**1.6.5**) **Setup.** Consider the description of the root systems $\Phi(E_7) \subseteq \Phi(E_8)$ from [3, p. 264–270]. Let H denote a maximal torus of $E_8(\mathbb{C})$ and fix a realization u_α of $\Phi(E_8)$ in $E_8(\mathbb{C})$ as in section 1.5. The subgroup generated by the root subgroups

u_α, $\alpha \in \Phi(E_7)$ is then isomorphic to $2E_7(\mathbb{C})$ having as maximal torus the subgroup generated by $h_{\alpha_i}(t)$, $1 \le i \le 7$, $t \in \mathbb{C}^\times$. Thus we have an embedding $2E_7(\mathbb{C}) \subseteq E_8(\mathbb{C})$ which is consistent with the embedding $\Phi(E_7) \subseteq \Phi(E_8)$.

In particular define the element $g_1 = h_{\alpha_3}(\omega)h_{\alpha_1}(\omega^2) \in 2E_7(\mathbb{C})$. By using [13, Table VI] and [7, Table 6] we get that g_1 belongs to the conjugacy class $\mathbf{3C}[2E_7(\mathbb{C})] \subseteq \mathbf{3B}[E_8(\mathbb{C})]$ and that the centralizers $C_{2E_7(\mathbb{C})}(g_1)$ and $C_{E_8(\mathbb{C})}(g_1)$ have types A_5A_2 and E_6A_2 respectively. The precise structure of these centralizers may be found as follows. Since $2E_7(\mathbb{C})$ and $E_8(\mathbb{C})$ are both simply connected, Lemma (1.1.3(2)) implies that both $C_{2E_7(\mathbb{C})}(g_1)$ and $C_{E_8(\mathbb{C})}(g_1)$ are connected. Setting

$$\alpha' = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

we see that $\{\alpha_5, \alpha_6, \alpha_7, \alpha', \alpha_2\} \cup \{\alpha_1, \alpha_3\}$ is a basis of the root system of $C_{2E_7(\mathbb{C})}(g_1)$ and that $\{\alpha_5, \alpha_8, \alpha_6, \alpha_7, \alpha', \alpha_2\} \cup \{\alpha_1, \alpha_3\}$ is a basis of the root system of $C_{E_8(\mathbb{C})}(g_1)$. Here the bases of the components of type A_2, A_5 and E_6 have been ordered so that the numbering is consistent with [3, p. 250–251,260–262]. From this we get explicit homomorphisms $\mathrm{SL}_6(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \to 2E_7(\mathbb{C})$ and $3E_6(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \to E_8(\mathbb{C})$ onto the centralizers $C_{2E_7(\mathbb{C})}(g_1)$ and $C_{E_8(\mathbb{C})}(g_1)$. The kernels of these maps are given by $\langle(\omega I_6, \omega^2 I_3)\rangle$ and $\langle(z, \omega^2 I_3)\rangle$ respectively, where $z \in 3E_6(\mathbb{C})$ denotes the central element defined in section 1.3. Thus the centralizers are given by the central products in the following diagram

$$\begin{tikzcd}
\mathrm{SL}_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C}) \cong C_{2E_7(\mathbb{C})}(g_1) \arrow[r] \arrow[d] & 2E_7(\mathbb{C}) \arrow[d] \\
3E_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C}) \cong C_{E_8(\mathbb{C})}(g_1) \arrow[r] & E_8(\mathbb{C}),
\end{tikzcd}$$

where the homomorphisms are defined from the identifications of root systems given above. In particular we have a homomorphism $\Psi : \mathrm{SL}_6(\mathbb{C}) \to 3E_6(\mathbb{C})$.

We write elements in these central products as $A \cdot B$ with $A \in \mathrm{SL}_6(\mathbb{C})$ or $A \in 3E_6(\mathbb{C})$ and $B \in \mathrm{SL}_3(\mathbb{C})$. Then $\omega A \cdot B = A \cdot \omega B$ in $\mathrm{SL}_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C}) \subseteq 2E_7(\mathbb{C})$, and $zA \cdot B = A \cdot \omega B$ in $3E_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C}) \subseteq E_8(\mathbb{C})$. Moreover, under the inclusion $2E_7(\mathbb{C}) \subseteq E_8(\mathbb{C})$, the element $A \cdot B$ in $2E_7(\mathbb{C})$ corresponds to the element $\Psi(A) \cdot B$ in $E_8(\mathbb{C})$. With this notation we have $g_1 = \omega I_6 = \omega I_3$ in $2E_7(\mathbb{C})$ and $g_1 = z = \omega I_3$ in $E_8(\mathbb{C})$. We also introduce the elements

$$g_2 = (\beta \oplus \beta) \cdot \beta, \qquad g_3 = (\gamma \oplus \gamma) \cdot \gamma^2, \qquad g_4 = \omega I_3 \oplus \omega^2 I_3$$

in $2E_7(\mathbb{C}) \subseteq E_8(\mathbb{C})$. Since g_1, g_2, g_3 and g_4 all have order 3 and are pairwise commuting, we see that the groups $E^3 = \langle g_1, g_2, g_3\rangle$ and $E^4 = \langle g_1, g_2, g_3, g_4\rangle$ are elementary abelian 3-groups of rank 3 and 4 respectively.

(1.6.6) Lemma. *We have*

$$C_{2E_7(\mathbb{C})}(E^3) \cong E^3 \times \mathrm{SL}_2(\mathbb{C}), \quad C_{2E_7(\mathbb{C})}(E^4) \cong E^3 \times \mathbb{T}_1, \quad Z(C_{2E_7(\mathbb{C})}(E^4)) = E^4$$

and moreover $W_{E_8(\mathbb{C})}(E^3) \neq \mathrm{GL}_3(\mathbb{F}_3)$.

(**1.6.7**) **Proof**: Since $C_{2E_7(\mathbb{C})}(g_1) = \mathrm{SL}_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C})$ we get

$$C_{2E_7(\mathbb{C})}(g_1, g_2) = \left\langle g_3, C_{\mathrm{SL}_6(\mathbb{C})}(\beta \oplus \beta) \circ_{C_3} \mathbb{T}_2 \right\rangle,$$
$$C_{2E_7(\mathbb{C})}(g_1, g_2, g_3) = \left\langle g_2, g_3, C_{\mathrm{SL}_6(\mathbb{C})}(\beta \oplus \beta, \gamma \oplus \gamma) \circ_{C_3} \langle \omega I_3 \rangle \right\rangle.$$

Then $C_{\mathrm{SL}_6(\mathbb{C})}(\beta \oplus \beta, \gamma \oplus \gamma) = \Delta_{2,3}(\{A \in \mathrm{GL}_2(\mathbb{C}) | (\det A)^3 = 1\})$ is generated by $\Delta_{2,3}(\omega^2 I_2)$ and $\Delta_{2,3}(\mathrm{SL}_2(\mathbb{C}))$. From this we get

$$C_{2E_7(\mathbb{C})}(E^3) = \langle g_1, g_2, g_3, \Delta_{2,3}(\mathrm{SL}_2(\mathbb{C})) \rangle \cong E^3 \times \mathrm{SL}_2(\mathbb{C}).$$

Under this isomorphism g_4 corresponds to the element $\mathrm{diag}(\omega, \omega^2) \in \mathrm{SL}_2(\mathbb{C})$ and thus we get $C_{2E_7(\mathbb{C})}(E^4) \cong E^3 \times \mathbb{T}_1$. As g_4 corresponds to the element $\omega \in \mathbb{T}_1$ we conclude ${}_3Z(C_{2E_7(\mathbb{C})}(E^4)) = E^4$.

To prove the last statement recall that $C_{E_8(\mathbb{C})}(g_1) = 3E_6(\mathbb{C}) \circ_{C_3} \mathrm{SL}_3(\mathbb{C})$ and that we have $g_2 = \Psi(\beta \oplus \beta) \cdot \beta$ and $g_3 = \Psi(\gamma \oplus \gamma) \cdot \gamma^2$ in $E_8(\mathbb{C})$. Thus we get

$$C_{E_8(\mathbb{C})}(g_1, g_2) = \left\langle g_3, C_{3E_6(\mathbb{C})}(\Psi(\beta \oplus \beta)) \circ_{C_3} \mathbb{T}_2 \right\rangle.$$

Note that no element in $\mathbb{T}_2$ conjugates γ^2 into an element of the form $\omega^k \gamma$. Thus g_3 is not conjugate to g_3^{-1} in $C_{E_8(\mathbb{C})}(g_1, g_2)$. Hence $\mathrm{diag}(1, 1, -1) \notin W_{E_8(\mathbb{C})}(E^3)$ and we are done. □

(**1.6.8**) **Proposition**. *The elementary abelian 3-groups E^3 and E^4 are nontoral in $2E_7(\mathbb{C})$ and $E_8(\mathbb{C})$. In particular E^3 is conjugate to $E^3_{2E_7}$ and E^4 is conjugate to $E^4_{2E_7}$ in $2E_7(\mathbb{C})$. In $E_8(\mathbb{C})$ the group E^3 is conjugate to $E^{3b}_{E_8}$ and E^4 is conjugate to $E^{4c}_{E_8}$. Moreover we have*

$$\begin{aligned} C_{2E_7(\mathbb{C})}(E^3_{2E_7}) &= E^3_{2E_7} \times \mathrm{SL}_2(\mathbb{C}), & Z(C_{2E_7(\mathbb{C})}(E^3_{2E_7})) &= E^3_{2E_7} \times 2, \\ C_{2E_7(\mathbb{C})}(E^4_{2E_7}) &= E^3_{2E_7} \times \mathbb{T}_1, & Z(C_{2E_7(\mathbb{C})}(E^4_{2E_7})) &= E^3_{2E_7} \times \mathbb{T}_1, \\ C_{E_8(\mathbb{C})}(E^{4c}_{E_8}) &= E^{4c}_{E_8} \circ_{C_3} \mathrm{SL}_3(\mathbb{C}), & Z(C_{E_8(\mathbb{C})}(E^{4c}_{E_8})) &= E^{4c}_{E_8}. \end{aligned}$$

In particular ${}_3Z(C_{2E_7(\mathbb{C})}(E)) = E$ for $E = E^3_{2E_7}$ and $E^4_{2E_7}$.

(**1.6.9**) **Proof**: Note first that if $E \subseteq G$ is a toral subgroup, then $C_G(E)$ contains a maximal torus of G. From this we see that E^3 is nontoral in $2E_7(\mathbb{C})$ since the rank of the centralizer $C_{2E_7(\mathbb{C})}(E^3)$ is less than 7 by Lemma (1.6.6). As E^4 contains E^3 we see that E^4 is also nontoral in $2E_7(\mathbb{C})$. From Theorem (1.4.3) we then get that E^3 is conjugate to $E^3_{2E_7}$ and E^4 is conjugate to $E^4_{2E_7}$ in $2E_7(\mathbb{C})$. We also see that the Weyl groups $W_{2E_7(\mathbb{C})}(E^3)$ and $W_{2E_7(\mathbb{C})}(E^4)$ have orders divisible by 13. Since $W_{2E_7(\mathbb{C})}(E) \subseteq W_{E_8(\mathbb{C})}(E)$ for any elementary abelian group $E \subseteq 2E_7(\mathbb{C})$ we conclude in particular that the Weyl groups of E^3 and E^4 in $E_8(\mathbb{C})$ also have orders divisible by 13. Since the Weyl group of $E_8(\mathbb{C})$ has order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ which is not divisible by 13, it follows from Lemma (1.1.3(4)) that E^3 and E^4 are nontoral in $E_8(\mathbb{C})$.

From the above and Theorem (1.4.3) we see that the class distributions of E^3 and E^4 are $\mathbf{3C^{26}}$ and $\mathbf{3B^2 3C^{78}}$ respectively in $2E_7(\mathbb{C})$. Now using [7, Table 6] we see that we have the inclusions

$$\mathbf{3B}[2E_7(\mathbb{C})] \subseteq \mathbf{3B}[E_8(\mathbb{C})], \qquad \mathbf{3C}[2E_7(\mathbb{C})] \subseteq \mathbf{3B}[E_8(\mathbb{C})]$$

of conjugacy classes corresponding to the inclusion $2E_7(\mathbb{C}) \subseteq E_8(\mathbb{C})$. Therefore the class distributions of E^3 and E^4 in $E_8(\mathbb{C})$ are $\mathbf{3B^{26}}$ and $\mathbf{3B^{80}}$ respectively. Since E^3 and E^4 are nontoral in $E_8(\mathbb{C})$, Theorem (1.5.4) shows that E^3 is conjugate to $E^{3b}_{E_8}$ and E^4 is conjugate to $E^{4c}_{E_8}$ in $E_8(\mathbb{C})$.

The assertions about the structure of $C_{2E_7(\mathbb{C})}(E)$, $Z(C_{2E_7(\mathbb{C})}(E))$ and ${}_3Z(C_{2E_7(\mathbb{C})}(E))$ for $E = E^3_{2E_7}$ and $E^4_{2E_7}$ follows directly from Lemma (1.6.6) using what we proved above.

Finally consider the inclusion $\mathrm{SL}_3(\mathbb{C}) \circ_{C_3} 3E_6(\mathbb{C}) \subseteq E_8(\mathbb{C})$ from above. In particular we may consider the subgroups of $3E_6(\mathbb{C})$ as subgroups of $E_8(\mathbb{C})$. Since the Weyl group of $E^3_{3E_6}$ in $3E_6(\mathbb{C})$ has order divisible by 13 by Theorem (1.3.5), we see as above that $E^3_{3E_6}$ is nontoral in $E_8(\mathbb{C})$. Hence $E^4_{3E_6}$ is also nontoral in $E_8(\mathbb{C})$. Using the informations about the inclusions of conjugacy classes given above and the ones given in (1.4.2) we see that $E^4_{3E_6}$ has class distribution $\mathbf{3B^{80}}$ in $E_8(\mathbb{C})$. Thus Theorem (1.5.4) shows that $E^4_{3E_6}$ is conjugate to $E^{4c}_{E_8}$ in $E_8(\mathbb{C})$.

Now since $E^4_{3E_6}$ contains the element $g_1 = z$ we get

$$\begin{aligned} C_{E_8(\mathbb{C})}(E^4_{3E_6}) &= C_{C_{E_8(\mathbb{C})}(g_1)}(E^4_{3E_6}) = C_{\mathrm{SL}_3(\mathbb{C})\circ_{C_3} 3E_6(\mathbb{C})}(E^4_{3E_6}) \\ &= \mathrm{SL}_3(\mathbb{C}) \circ_{C_3} C_{3E_6(\mathbb{C})}(E^4_{3E_6}) = \mathrm{SL}_3(\mathbb{C}) \circ_{C_3} E^4_{3E_6}, \end{aligned}$$

using the computation above of $C_{3E_6(\mathbb{C})}(E^4_{3E_6})$. From this we get

$$Z(C_{E_8(\mathbb{C})}(E^4_{3E_6})) = C_3 \circ_{C_3} E^4_{3E_6} = E^4_{3E_6}.$$

Since $E^{4c}_{E_8}$ is conjugate to $E^4_{3E_6}$ in $E_8(\mathbb{C})$ the results on $E^{4c}_{E_8}$ now follow immediately. $\square$

(1.6.10) Corollary. *The Weyl groups of the elementary abelian 3-subgroups* $E^3_{3E_6}, E^4_{3E_6} \subseteq 3E_6(\mathbb{C})$, $E^{3b}_{E_6}, E^{4a}_{E_6} \subseteq E_6(\mathbb{C})$, $E^3_{2E_7}, E^4_{2E_7} \subseteq 2E_7(\mathbb{C})$ *and* $E^{3b}_{E_8}, E^{4c}_{E_8} \subseteq E_8(\mathbb{C})$ *equal the lower bounds given earlier, i.e. we have*

$$W(E^3_{3E_6}) = W(E^{3b}_{E_6}) = W(E^3_{2E_7}) = W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbb{F}_3),$$

$$W(E^4_{3E_6}) = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right], W(E^{4a}_{E_6}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & det & * \\ 0 & 0 & 0 & det \end{array}\right],$$

$$W(E^4_{2E_7}) = W(E^{4c}_{E_8}) = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

(**1.6.11**) **Proof**: By Theorem (1.5.4) we know that $W(E^{3b}_{E_8})$ contains $\mathrm{SL}_3(\mathbb{F}_3)$ and by Proposition (1.6.8) and Lemma (1.6.6) we have $W(E^{3b}_{E_8}) \neq \mathrm{GL}_3(\mathbb{F}_3)$. Thus we conclude that $W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbb{F}_3)$. Note that by Theorem (1.5.4) $W(E^{4c}_{E_8})$ contains the group

$$W' = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right],$$

whose only minimal overgroup in $\mathrm{GL}_4(\mathbb{F}_3)$ equals

$$\left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{GL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

However, since $E^{3b}_{E_8} = \langle \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle$ and $E^{4c}_{E_8} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle$, $W(E^{4c}_{E_8})$ cannot contain this group as this would imply that $W(E^{3b}_{E_8}) = \mathrm{GL}_3(\mathbb{F}_3)$. We conclude that $W(E^{4c}_{E_8}) = W'$ as claimed.

Using the inclusion $2E_7(\mathbb{C}) \subseteq E_8(\mathbb{C})$ and Proposition (1.6.8) we get

$$W_{2E_7(\mathbb{C})}(E^3_{2E_7}) \subseteq W_{E_8(\mathbb{C})}(E^{3b}_{E_8}), \qquad W_{2E_7(\mathbb{C})}(E^4_{2E_7}) \subseteq W_{E_8(\mathbb{C})}(E^{4c}_{E_8}).$$

Combining this and Theorem (1.4.3) with the results just proved for $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ we conclude that $W(E^3_{2E_7}) = \mathrm{SL}_3(\mathbb{F}_3)$ and $W(E^4_{2E_7}) = W'$.

In the proof of Proposition (1.6.8) we saw that $E^4_{3E_6}$ is conjugate to $E^{4c}_{E_8}$ in $E_8(\mathbb{C})$ under the inclusion $3E_6(\mathbb{C}) \subseteq E_8(\mathbb{C})$ from above. Similarly we obtain that $E^3_{3E_6}$ is conjugate to $E^{3b}_{E_8}$ in $E_8(\mathbb{C})$. In particular we have $W(E^3_{3E_6}) = \mathrm{SL}_3(\mathbb{F}_3)$ by Theorem (1.3.5), and $W(E^4_{3E_6}) \subseteq W'$. Since z is central in $3E_6(\mathbb{C})$ we also have

$$W(E^4_{3E_6}) \subseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{GL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right],$$

and combining this with $W(E^4_{3E_6}) \subseteq W'$ and the lower bound from Theorem (1.3.5), it follows that

$$W(E^4_{3E_6}) = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbb{F}_3) & \\ 0 & & & \end{array}\right].$$

Thus we get $W(E^{3b}_{E_6}) = \mathrm{SL}_3(\mathbb{F}_3)$ since $E^4_{3E_6} = \langle z, a, x_2, y_2 \rangle$ and $E^{3b}_{E_6} = \langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$.

Finally consider the group $E_{E_6}^{4a} = \langle \overline{a}, \overline{y_2}, \overline{y_1}, \overline{x_2} \rangle$ which contains the groups $E_{E_6}^{2a} = \langle \overline{y_1}, \overline{x_2} \rangle$ and $E_{E_6}^{3b} = \langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$. From Theorem (1.3.11) we know that $W(E_{E_6}^{4a})$ contains the group

$$W'(E_{E_6}^{4a}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array}\right].$$

Using our computer program we obtain that it has the following 3 minimal overgroups in $\mathrm{GL}_4(\mathbb{F}_3)$:

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \varepsilon \end{array}\right], \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \varepsilon & * \\ 0 & 0 & 0 & \det \end{array}\right], \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \varepsilon & * \\ 0 & 0 & 0 & \varepsilon \end{array}\right].$$

Thus if $W(E_{E_6}^{4a}) \neq W'(E_{E_6}^{4a})$ then $W(E_{E_6}^{4a})$ would have to contain one of the diagonal matrices $\mathrm{diag}(1,1,1,-1)$, $\mathrm{diag}(1,1,-1,1)$ or $\mathrm{diag}(1,1,-1,-1)$. However the first two of these fix the subgroup $E_{E_6}^{2a}$ and induce the automorphisms $\mathrm{diag}(1,-1)$ and $\mathrm{diag}(-1,1)$ on $E_{E_6}^{2a}$. Since $W(E_{E_6}^{2a}) = \mathrm{SL}_2(\mathbb{F}_3)$ we conclude that $\mathrm{diag}(1,1,1,-1)$ and $\mathrm{diag}(1,1,-1,1)$ does not belong to $W(E_{E_6}^{4a})$. The element $\mathrm{diag}(1,1,-1,-1)$ fixes the subgroup $E_{E_6}^{3b}$ and induces the automorphism $\mathrm{diag}(1,-1,1)$ on $E_{E_6}^{3b}$. Since $W(E_{E_6}^{3b}) = \mathrm{SL}_3(\mathbb{F}_3)$ we conclude that $\mathrm{diag}(1,1,-1,-1) \notin W(E_{E_6}^{4a})$ and thus $W(E_{E_6}^{4a}) = W'(E_{E_6}^{4a})$ as was to be proved. □

1.7 Corrections and misprints

In this section we have gathered some corrections to the papers [7], [8] and [13].

(**1.7.1**) **Remark**. There are some minor misprints in [7]. On p. 374, the term $30t^7$ should be replaced by $39t^7$ in the power series expansion of the generating function for the number of elements of order a divisor of i.

In table 4, p. 375–376, the following corrections should be made: For the class $6J$ the power maps should be $[B, C]$ and for the class $6L$ it should be $[A, C]$. The representative of the class $7G$ should be replaced by $(2, 4, 0, 0, 0, 0, 2, 0)$.

In table 6, p. 384–385, the following corrections should be made: The centralizers types of the classes $3A$, $3E$, $5E$ and $6Q$ should be A_6T_1, D_6T_1, $A_3A_2A_1T_1$ and $A_3A_1A_1A_1T_1$ respectively. The power maps for the class $6V$ should be $[A, D]$.

(**1.7.2**) **Remark**. There are some minor misprints in [8]. On p. 108 the formula defining the multiplication on $\mathbf{K}$ (the second formula) should read

$$\langle x \times y, z\rangle = 6 \langle x, y, z\rangle .$$

On the bottom of p. 109, the formula for the action of n_1 should read

$$h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)^{n_1} = h(\alpha, \beta, \gamma, (\gamma\delta)^{-1}, \varepsilon, \zeta).$$

On p. 110, line 5, the element n_1n_7 should be replaced by $n_1n_7h(-1, -1, 1, -1, 1, 1)$, since n_1n_7 is not an element of $F = F_4(\mathbb{C})$. In the third displayed formula on p. 110 giving the generating function for the number of conjugacy classes of elements of finite order in F, the first term 1 should be removed. In the next formula the extra factor $(1 - xt)$ should be inserted in denominator.

In table 2, p. 111–112, the following corrections should be made: The representative of the class $4H$ should be $(1, 1, 1, 0, 2, 3)$. The $\mathbf{g}$-multiplicity for the class $5D$ should be $20, 13, 16, 16, 13$. The representative of the class $6C$ should be $(1, 0, 1, 0, 4, 3)$. The centralizer type of the class $6R$ should be $A_2A_2T_2$. The representative of the class $7A$ should be $(1, 5, 1, 0, 1, 6)$. The centralizer type of the class $7E$ should be $A_3A_1T_2$. The representatives of the classes $7Q$ and $7R$ should be $(3, 1, 6, 1, 3, 4)$ and $(1, 1, 2, 1, 4, 5)$.

Finally, on p. 149, the reference for [5] should be J. Algebra 131 (1990) 23–39.

(**1.7.3**) **Remark**. There are some errors and misprints in [13]. We consider as above the case of the ground field $\mathbb{C}$.

The group $2E_7(\mathbb{C})$ for $p = 3$ is considered in Table II, p. 259 and in Theorem (11.16), p. 296. Here the centralizer of the rank 3 subgroup should have the shape $3^3 \times \mathrm{SL}_2(\mathbb{C})$. Since the Weyl group of the rank 3 group is $\mathrm{SL}_3(\mathbb{F}_3)$ the shape of its normalizer is wrong as well. The claims about the normalizers of both the rank 3 and the rank 4 groups mapping onto $\mathrm{AGL}_3(\mathbb{F}_3)$ also seem inconsistent with our Weyl group computations, although we do not know the exact structure of the normalizers. Finally, in the table it is stated that the factor 2 in the normalizers splits, but in the theorem and its proof it is claimed not to split.

The group $E_8(\mathbb{C})$ for $p = 3$ is considered in Table II, p. 259 and in Lemma (11.5), p. 289–290. The information for the maximal elementary abelian 3-subgroup of type 1 should be as follows. The centralizer should be $3^3 \times \mathbb{T}_2$ (this is given wrongly in the table) and the Weyl group should have the shape $3^3 : GL(3,3) \times \Sigma_3$. Thus the stated structure of the normalizer is wrong. Furthermore for the type 1 group there is a unique subgroup of order 3 which is normal in $N(E)$, namely $E_{00} = \langle \overline{x_2} \rangle$. However we have $E_0 \cap \mathbb{T}_2 = \left\langle \overline{x_2}, \overline{x_3 a^{-1}} \right\rangle$, so $E_{00} \neq E_0 \cap \mathbb{T}_2$ contrary to the claim on p. 290. For the group of type 2 group, the statement about the centralizer types of the rank 2 subgroups of $E_0 = \langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2} \rangle$ is wrong since not every such subgroup has a centralizer of type $T_4 D_4$. For example the centralizer type of the group $\langle \overline{x_1}, \overline{x_2} \rangle$ is seen to be ${A_2}^4$ (this also follows by the information given about the rank 2 subgroups in the type 1 case).

In Proposition (2.13)(vi), p. 265, the condition "H is semisimple" should be corrected to "H is simply connected". The numbering in the proof is not correct, there is never a reference to (iv), but there are two references to (vii). Moreover the references [BoLAG] and [St2] does not occur in the bibliography.

On the bottom of p. 265 and in Table VI, p. 266, the centralizer type of an element in the class $3E[2E_7(\mathbb{C})]$ should be $D_6 T_1$.

In Table VI, p. 266, the eigenvalue multiplicities for the classes in $2E_7(\mathbb{K})$ should be as follows. For **3C** it should be $43, 45, 45$, for **3E** it should be $67, 33, 33$, for **4A** it should be $63, 0, 70, 0$, for **4B** it should be $39, 32, 30, 32$ and for **4I** it should be $39, 32, 30, 32$.

On p. 268, line 10, $3E(\mathbb{K})$ should be corrected to $3E_6(\mathbb{K})$, and on line 12 from the bottom the word "subgroups" should be replaced by "2-subgroups".

The statement of Theorem (11.14), p. 294, is somewhat misleading since the groups occurring in part (ii) are not maximal.

On p. 295, line 17, Σ_3 should be replaced by the dihedral group Dih_{12}. On line 16 from the bottom, the minimum weight of the tetracode should be 3, and on line 5 from the bottom the equation should read $\mathbf{3C} \cap E = Z(E)v \cup Z(E)v^{-1}$.

In remark (13.2), p. 298 the Weyl group of F_3 should be $\mathrm{SL}_3(\mathbb{F}_3)$ and not $\mathrm{GL}_3(\mathbb{F}_3)$.

Finally, in remark (A2.6), p. 303, the reference [Gr7] does not occur in the bibliography.

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Part II

Fusion, Solvability and Swan's theorem

Chapter 2

Fusion, Solvability and Swan's theorem

2.1 Introduction

The purpose of this note is to discuss the relations between various notions of solvability of finite groups and control of fusion. More precisely, we prove the following result.

(**2.1.1**) **Theorem**. *Let G be a finite group, p a prime number and P a Sylow p-subgroup of G.*

(1) If G is p-supersolvable then $N_G(P)$ controls p-fusion strongly in G.

(2) If G is p-solvable and P controls p-fusion weakly in G then G is p-supersolvable.

(3) If $N_G(P)$ controls p-fusion weakly in G then $N_G(P)$ controls p-fusion strongly in G.

We also give a counterexample to the original version of Swan's theorem [16; 17] on the cohomology of p-normal groups. The results of this note represent work in progress, and we hope to return later to some of the questions that we leave open. It should also be mentioned that work of Stammbach [14] gives cohomological characterizations of solvable and supersolvable groups and it would be interesting to find relations between this and the present work.

I would like to thank Antonio Viruel for showing me another way to prove part (1). I am also happy to thank Jesper Grodal for pointing out the reference [5], which

led me to prove part (3), and for many stimulating conversations. In particular, Theorem (2.2.22) (which is the key to proving part (2)) is really joint work.[1]

Finally it should be added that the theorem is inspired by numerous calculations using the computer algebra system MAGMA [2].

[1] I would also like to thank Radu Stancu, who after the approval of my thesis showed that it is possible to remove a superfluous condition in Theorem (2.2.22).

2.2 Solvability and fusion

In the following all groups are finite and p denotes a prime. We start by recalling the definitions of p-nilpotence, p-solvability, p-supersolvability and control of p-fusion and p-transfer.

(**2.2.1**) **Definition**. A *series* of G is a series of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_s = G$. The quotients G_i/G_{i-1}, $1 \leq i \leq s$, are called the *factors* of the series. The series is called *proper* if $G_{i-1} \triangleleft G_i$ for $1 \leq i \leq s$, i.e. if the factors all are nontrivial. A factor G_i/G_{i-1}, $1 \leq i \leq s$, is called *central* if $G_{i-1} \trianglelefteq G$ and $G_i/G_{i-1} \leq Z(G/G_{i-1})$. A proper series without proper refinements is called a *composition series*. Equivalently the factors should be simple groups. The factors are called the *composition factors*. A series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_s = G$ satisfying $G_i \trianglelefteq G$, $0 \leq i \leq s$, is called a *normal series* of G. A proper normal series having no normal series as a proper refinement, is called a *chief series*. The factors are called the *chief factors*.

(**2.2.2**) **Remark**. The chief factors are characteristically simple groups (see [12, 7.38]). By [7, Theorem 2.1.4], [8, Satz I.9.12] or [12, Theorem 8.10] a group is characteristically simple if and only if it is a direct product H^r of finitely many copies of a single simple group H.

(**2.2.3**) **Definition**. A group G is called *p-nilpotent* if G has a normal p-complement, i.e. a normal subgroup K such that $KP = G$ and $K \cap P = 1$ for a Sylow p-subgroup P of G. Equivalently every chief factor of G of order divisible by p should be central ([8, Satz IV.4.4] or [12, Exercise 563]). A group is called *p-supersolvable* if its chief factors are either cyclic of order p or p'-groups. Finally a group G is called *p-solvable* if G has a series where all factors are either p-groups or p'-groups. Equivalently the composition factors of G are either p-groups or p'-groups. This is again equivalent to demanding that the chief factors of G are either p-groups or p'-groups.

(**2.2.4**) **Remark**. A group is nilpotent if and only if it is p-nilpotent for all primes p. Similarly G is solvable if and only if G is p-solvable for all primes p, and G is supersolvable if and only if G is p-supersolvable for all primes p. We also have the implications

$$G \ p\text{-nilpotent} \Longrightarrow G \ p\text{-supersolvable} \Longrightarrow G \ p\text{-solvable}.$$

These implications can not be reversed. The group Σ_3 is supersolvable but not 3-nilpotent, and the group A_4 is solvable but not 2-supersolvable.

We next turn to the definitions of control of p-fusion in a group.

(**2.2.5**) **Definition** (Fusion). A subgroup $H \leq G$ is said to control p-fusion weakly in G if H contains a Sylow p-subgroup of G and whenever $Q, Q^g \leq H$ for some p-subgroup Q and some $g \in G$ we may write $g = nh$ with $n \in N_G(Q)$ and

$h \in H$. The subgroup $H \leq G$ is said to control p-fusion strongly in G if H contains a Sylow p-subgroup of G and whenever $Q, Q^g \leq H$ for some p-subgroup Q and some $g \in G$ we may write $g = ch$ with $n \in C_G(Q)$ and $h \in H$.

(**2.2.6**) **Remark**. Obviously, if H controls p-fusion strongly in G, then H controls p-fusion weakly in G. Furthermore if $H \leq K \leq G$ and H controls p-fusion weakly or strongly respectively in G, then K also controls p-fusion weakly or strongly respectively in G.

Our interest in control of fusion stems from the following result which relates it to group cohomology. Here, for an abelian group M, we denote by $M_{(p)}$ its p-primary submodule, i.e. the submodule consisting of the elements of p-power order.

(**2.2.7**) **Theorem**. *Let G be a finite group, H a subgroup of G and p a prime number. Then the following conditions are equivalent.*

(1) H controls p-fusion strongly in G.

(2) The restriction map $H^*(G, A)_{(p)} \longrightarrow H^*(H, A)_{(p)}$ is an isomorphism for any trivial $\mathbb{Z}[G]$-module A.

(3) The restriction map $H^*(G, \mathbb{F}_p) \longrightarrow H^*(H, \mathbb{F}_p)$ is an isomorphism.

(**2.2.8**) **Proof**: The implication (1) $\Longrightarrow$ (2) is well known ([10, Proposition 2.1], [9, Theorem 12.8]) and goes back to [16]. The implication (2) $\Longrightarrow$ (3) is obvious and (3) $\Longrightarrow$ (1) is due to Mislin [11]. $\square$

Denote by $O^p(G)$ the smallest normal subgroup K of G, such that G/K is a p-group. The following result is due to Tate.

(**2.2.9**) **Theorem** (Transfer). *Let H be a subgroup of G containing a Sylow p-subgroup P of G. Then the following conditions are equivalent:*

(1) $G/O^p(G) \cong H/O^p(H)$

(2) $|G/O^p(G)| = |H/O^p(H)|$

(3) $H \cap O^p(G) = O^p(H)$

(4) $G/(O^p(G)G') \cong H/(O^p(H)H')$

(5) $|G/(O^p(G)G')| = |H/(O^p(H)H')|$

(6) $H \cap O^p(G)G' = O^p(H)H'$

(7) $P \cap G' = P \cap H'$

(8) $PG'/G' \cong PH'/H'$

(**2.2.10**) **Proof**: See [9, Lemma X.6.15 and Theorem X.6.18]. □

(**2.2.11**) **Remark**. In case the conditions are satisfied we say that H controls p-transfer in G. By condition (7) we see that if $H \leq K \leq G$ and H controls p-transfer in G then K also controls p-transfer in G. Note also that by the Focal Subgroup Theorem ([9, X.6.2] and [12, 10.34]) the subgroup $P \cap G'$ occurring in (7) is the focal subgroup of P in G.

(**2.2.12**) **Proposition**. Let H be a subgroup of G containing a Sylow p-subgroup P of G. If H controls p-fusion strongly in G then H controls p-transfer in G.

(**2.2.13**) **Proof**: See [9, Lemma X.9.2]. A cohomological proof runs as follows. Since H controls p-fusion strongly in G we have $H^2(G,\mathbb{Z})_{(p)} \cong H^2(H,\mathbb{Z})_{(p)}$. However for any group K we have $H^2(K,\mathbb{Z})_{(p)} = H_1(K,\mathbb{Z})_{(p)} \cong K/(O^p(K)K')$ and we conclude that condition (4) is satisfied. □

(**2.2.14**) **Definition**. Let G be a finite group, p a prime number and P a Sylow p-subgroup of G. Then a p-subgroup Q of P is said to be *weakly closed in P with respect to G* if $Q^g \leq P$ implies $Q^g = Q$, i.e. if Q is the only conjugate of Q contained in P. The group G is called *p-normal* if $Z(P)$ is weakly closed in P with respect to G.

(**2.2.15**) **Theorem**. *Let G be a finite group, p a prime number and P a Sylow p-subgroup of G. Then $N_G(P)$ controls p-fusion strongly in G if and only if P has a central series consisting of subgroups which are weakly closed in P with respect to G.*

(**2.2.16**) **Proof**: This is a special case of the main result in [5] translated to our notation (the definition of strong control of fusion in [5] differs slightly from ours). □

(**2.2.17**) **Remark**. This theorem generalizes work of Thevénaz [20], whose main theorem states that if P has a strongly characteristic central series $1 = P_0 \trianglelefteq P_1 \trianglelefteq \ldots \trianglelefteq P_s = P$, then $N_G(P)$ controls p-fusion strongly in G. Actually the original inductive proof in [20] fails, but if one in addition assumes that P_i/P_j is strongly characteristic in P/P_j for $1 \leq j \leq i \leq s$, then the inductive argument does go through [19]. However, as already mentioned, due to Theorem (2.2.15) one does not need this stronger hypothesis.

(**2.2.18**) **Lemma**. *Let G be a finite group, p a prime number and P a Sylow p-subgroup of G. If G is p-nilpotent then P controls p-fusion strongly in G.*

(**2.2.19**) **Proof**: We first give a group theoretical proof, similar to [8, proof of Satz IV.4.9]. Let $K \trianglelefteq G$ be a normal p-complement of P, and suppose $Q, Q^g \leq P$. We can then write $g = kp$ where $k \in K$ and $p \in P$. For $x \in Q$ we have $x^k = (x^g)^{p^{-1}} \leq (Q^g)^{p^{-1}} \leq P^{p^{-1}} = P$ so $x^{-1}x^k \in P$. On the other hand we also have $x^{-1}x^k = (x^{-1}k^{-1}x)k \in K$ since $K \trianglelefteq G$. So $x^{-1}x^k \in P \cap K = 1$ for any $x \in Q$, which proves $k \in C_G(Q)$. Since $g = kp$ we get that P controls p-fusion strongly in G. A cohomological proof runs as follows. Let $\iota : P \to G$ be the inclusion and let $\pi : G \to G/K \cong P$ be the projection. Consider the diagram

$$H^*(P, \mathbb{F}_p) \xrightarrow{\pi^*} H^*(G, \mathbb{F}_p) \xrightarrow{\iota^*} H^*(P, \mathbb{F}_p) \xrightarrow{\mathrm{cor}_P^G} H^*(G, \mathbb{F}_p)$$

where cor_P^G is the corestriction. Since the composition of the first two is the identity and the composition of the last two is multiplication by $[G : P]$ which is prime to p, we see that ι^* is an isomorphism. Thus P controls p-fusion strongly in G by Theorem (2.2.7). □

(**2.2.20**) **Lemma**. *Let G be a finite group, p a prime number and P a Sylow p-subgroup of G. If G has a normal p-nilpotent subgroup of index prime to p, then $N_G(P)$ controls p-fusion strongly in G.*

(**2.2.21**) **Proof**: Let $K \trianglelefteq G$ be a p-nilpotent subgroup of index prime to p. Then $P \leq K$ and by the Frattini argument [8, Satz I.7.8] we have $G = KN_G(P)$. Assume now that $Q, Q^g \leq P$, and write $g = kn$ with $k \in K$ and $n \in N_G(P)$. Then $Q^k = Q^{gn^{-1}} \leq P^{n^{-1}} = P$. By Lemma (2.2.18) P controls p-fusion strongly in K, so we may write $k = cp$ with $c \in C_K(Q)$ and $p \in P$. Thus $g = cpn$ with $c \in C_K(Q)$ and $pn \in N_G(P)$ proving that $N_G(P)$ controls p-fusion strongly in G. A cohomological proof runs as follows. As before we have $P \leq K$ and $G = KN_G(P)$ by the Frattini argument. From the Lyndon-Hochschild-Serre spectral sequence [4] we get that the restriction $H^*(G, \mathbb{F}_p) \longrightarrow H^*(K, \mathbb{F}_p)^{N_G(P)}$ is an isomorphism since K has index prime to p. By Lemma (2.2.18) and Theorem (2.2.7) the restriction $H^*(K, \mathbb{F}_p) \longrightarrow H^*(P, \mathbb{F}_p)$ is also an isomorphism. Finally $H^*(N_G(P), \mathbb{F}_p) \longrightarrow H^*(P, \mathbb{F}_p)^{N_G(P)}$ is an isomorphism, so the commutative diagram

$$\begin{array}{ccc} H^*(G, \mathbb{F}_p) & \xrightarrow{\cong} & H^*(K, \mathbb{F}_p)^{N_G(P)} \\ \downarrow & & \downarrow{\cong} \\ H^*(N_G(P), \mathbb{F}_p) & \xrightarrow{\cong} & H^*(P, \mathbb{F}_p)^{N_G(P)} \end{array}$$

shows that $H^*(G, \mathbb{F}_p) \longrightarrow H^*(N_G(P), \mathbb{F}_p)$ is an isomorphism, and we conclude that $N_G(P)$ controls p-fusion strongly in G by Theorem (2.2.7). □

(**2.2.22**) **Theorem**. *Let G be a finite group, p a prime number and $K \trianglelefteq G$ a normal subgroup. Assume that $H \leq G$ controls p-fusion weakly or strongly respectively in G. If K is p-solvable, then HK/K controls p-fusion weakly or strongly respectively in G/K.* [2]

(**2.2.23**) **Proof**: We start by considering the case where K is either a p-group or a p'-group. Note that we may assume $K \leq H$. Let P be a Sylow p-subgroup of G

[2] Actually, the condition that K is p-solvable may be removed, as has been shown to us by Stancu [15]. The proof will appear elsewhere. This renders Remark (2.2.24) below obsolete, however I have chosen to retain it in order to keep this version of my thesis as close to the original as possible.

contained in H. Then PK/K is a Sylow p-subgroup of G/K contained in H/K. Let $R, R^{\overline{g}} \leq PK/K$ be conjugate in G/K. We may then find $Q_1, Q_2 \leq P$ such that $R = Q_1K/K$ and $R^{\overline{g}} = Q_2K/K$, i.e. we have $Q_1^g K = Q_2 K$.

If K is a p'-group, then both Q_1^g and Q_2 are Sylow p-subgroups of $Q_1^g K = Q_2 K$. We can thus find $k \in K$, such that $(Q_1^g)^k = Q_2$. In the case where K is a p-group we have $K \leq P$ since K is normal. We thus conclude that $Q_1^g \leq Q_1^g K = Q_2 K \leq P$.

Thus setting $k = 1 \in K$ in the last case, we see that we have $Q_1, Q_1^{gk} \leq P$ in both cases. We may then write $gk = xh$ where $h \in H$ and $x \in N_G(Q_1)$ if H controls p-fusion weakly in G or $x \in C_G(Q_1)$ if H controls p-fusion strongly in G. As $R = Q_1K/K$ we see that $\overline{x} \in N_{G/K}(R)$ or $\overline{x} \in C_{G/K}(R)$ respectively. Since $\overline{g} = \overline{x}\overline{h}$ we conclude that H/K controls p-fusion weakly or strongly respectively in G/K.

The special case where K is a p'-group and we want to prove that H/K controls p-fusion strongly in G/K may also be handled directly by cohomological means. Consider the diagram

$$\begin{tikzcd}
H^*(G, \mathbb{F}_p) \arrow[r, "\cong"] & H^*(H, \mathbb{F}_p) \\
H^*(G/K, \mathbb{F}_p) \arrow[r] \arrow[u, "\cong"] & H^*(H/K, \mathbb{F}_p) \arrow[u, "\cong"']
\end{tikzcd}$$

where the vertical maps are isomorphisms by the Lyndon-Hochschild-Serre spectral sequence since K is a p'-group. Thus the restriction $H^*(G/K, \mathbb{F}_p) \longrightarrow H^*(H/K, \mathbb{F}_p)$ is an isomorphism and by Theorem (2.2.7) we conclude that H/K controls p-fusion strongly in G/K.

We now consider the general case, i.e. the case where K is p-solvable. Then the upper p-series of K,

$$1 = K_0 \trianglelefteq K_1 = O_p(K) \trianglelefteq K_2 = O_{p,p'}(K) \trianglelefteq \ldots \trianglelefteq K_n = K$$

terminates by [7, Theorem 6.3.1]. By construction the subgroups K_i are characteristic in K, and since K is normal in G we have $K_i \trianglelefteq G$. Since the factors K_i/K_{i-1} are either p-groups or p'-groups, the general case follows by induction from the special case considered above. □

(**2.2.24**) **Remark**. We do not know if the assumption that K is p-solvable can be removed. Assume that G is a minimal counterexample, i.e. that the theorem does not hold for the group G and that the order of G is chosen minimal with respect to this property. Choose $K \trianglelefteq G$ and $H \leq G$ such that H controls p-fusion weakly or strongly respectively in G, but HK/K does not control p-fusion weakly or strongly respectively in G/K. Now the Fitting subgroup $F(K)$ is a nilpotent characteristic subgroup of K [8, §III.4]. In particular $F(K)$ is p-solvable, so since G is a minimal counterexample we conclude that $F(K) = 1$ by the theorem.

Now consider the generalized Fitting subgroup $F^*(K)$, cf. [9, §X.13]. Then $F^*(K)$ is quasinilpotent by [9, Theorem 13.10] and thus $Z_\infty(K) = F(K) = 1$ by [9, Corollary 13.7]. By [9, Theorem 13.6] we then conclude that $F^*(K)$ is semisimple, i.e. the direct product of nonabelian simple groups. Writing $F^*(K) = S_1^{e_1} \times \ldots \times S_n^{e_n}$ where the groups S_i are isomorphic nonabelian simple groups, we see that the subgroup $S_1^{e_1}$ is characteristic in $F^*(K)$. Since $F^*(K)$ is characteristic in K and $K \trianglelefteq G$ we conclude that $S_1^{e_1} \trianglelefteq G$. Thus to prove the theorem without the assumption of p-solvability, it suffices to consider the case where K is a direct product of isomorphic nonabelian simple groups.

Our main result Theorem (2.1.1) is contained in the following.

(**2.2.25**) **Theorem**. *Let G be a finite group, p a prime and P a Sylow p-subgroup of G. Consider the following properties:*

(1) G is p-nilpotent.

(2) P controls p-fusion strongly in G.

(3) P controls p-fusion weakly in G.

(4) $N_G(P)$ controls p-fusion strongly in G.

(5) $N_G(P)$ controls p-fusion weakly in G.

(6) G is p-normal.

(7) $N_G(Z(P))$ controls p-fusion strongly in G.

(8) $N_G(Z(P))$ controls p-fusion weakly in G.

(9) G is p-supersolvable.

(10) G is p-solvable.

(11) P controls p-transfer in G.

(12) $N_G(P)$ controls p-transfer in G.

(13) $N_G(Z(P))$ controls p-transfer in G.

Then we have the following implications:

$$\begin{tikzcd}
(1) \Leftrightarrow (2) \Leftrightarrow (11) \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & (3) \arrow[d, Rightarrow] & \\
(9) \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & (4) \Leftrightarrow (5) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (12) \arrow[d, Rightarrow] \\
(10) & & (6) \Leftrightarrow (7) \Leftrightarrow (8) \arrow[r, Rightarrow] & (13)
\end{tikzcd}$$

$$(3) + (10) \Longrightarrow (9)$$

(**2.2.26**) **Proof**: Most of the implications are trivial. By Remark (2.2.6) we get (2) $\Longrightarrow$ (3), (4) $\Longrightarrow$ (5) and (7) $\Longrightarrow$ (8). Since $Z(P)$ is a characteristic subgroup of P, we get $P \leq N_G(P) \leq N_G(Z(P))$. Thus by Remark (2.2.6) we also have (3) $\Longrightarrow$ (5) $\Longrightarrow$ (8) and similarly (12) $\Longrightarrow$ (13) by Remark (2.2.11).

Proposition (2.2.12) shows (2) $\Longrightarrow$ (11), (4) $\Longrightarrow$ (12) and (7) $\Longrightarrow$ (13). The implications (1) $\Longrightarrow$ (9) $\Longrightarrow$ (10) are obvious from the definitions, cf. Remark (2.2.4).

The implication (6) $\Longrightarrow$ (7) is proved in [9, proof of corollary X.12.9] and the implication (8) $\Longrightarrow$ (6) follows directly from the definitions. Namely, assume that (8) is satisfied, and let $g \in G$ with $Z(P)^g \leq P$. We may then write $g = nh$ with $n, h \in N_G(Z(P))$ so we get $g \in N_G(Z(P))$ and thus $Z(P)^g = Z(P)$ as desired.

Assume that P controls p-transfer in G. By Theorem (2.2.9) we have $|G/O^p(G)| = |P|$. In particular P has a normal p-complement $O^p(G)$, so G is p-nilpotent. This argument, which is due to Tate [18], proves (11) $\Longrightarrow$ (1). The implication (1) $\Longrightarrow$ (2) follows from Lemma (2.2.18).

We now prove have (9) $\Longrightarrow$ (4). Assume that G is p-supersolvable and let G_i/G_{i-1} be a chief factor of G of order divisible by p. By definition, G_i/G_{i-1} is then cyclic of order p, and $\mathrm{Aut}(G_i/G_{i-1})$ is cyclic of order $p-1$. Since it is abelian of order prime to p, both G' and P acts trivially on G_i/G_{i-1}. Since this holds for all chief factors, we see that PG' is p-nilpotent by [8, Satz VI.5.4(b)]. Thus PG' is a normal p-nilpotent subgroup of G of index prime to p, and $N_G(P)$ controls p-fusion strongly in G by Lemma (2.2.20).

To prove (5) $\Longrightarrow$ (4), we note that by Theorem (2.2.15) P has a central series consisting of subgroups which are weakly closed in P with respect to $N_G(P)$, since $N_G(P)$ obviously controls p-fusion strongly in $N_G(P)$. Now assume that $Q \leq P$ is any subgroup which is weakly closed in P with respect to $N_G(P)$, and that $Q^g \leq P$ for some $g \in G$. By assumption we may write $g = nh$, with $n \in N_G(Q)$ and $h \in N_G(P)$. Thus $Q^h = Q^g \leq P$. Since Q is weakly closed in P with respect to $N_G(P)$ we then get $Q^h = Q$. Thus $Q^g = Q^h = Q$ and we conclude that Q is weakly closed in P with respect to G. Thus the central series of P from above actually consists of subgroups which are weakly closed in P with respect to G, so $N_G(P)$ controls p-fusion strongly in G by Theorem (2.2.15).

Finally, we prove that (3) + (10) $\Longrightarrow$ (9). Let H/K be a chief factor of G of order divisible by p. Then H/K is by assumption a p-group and we have to prove that $|H/K| = p$. Assume to the contrary that $|H/K| > p$. Note that $H/K \trianglelefteq PK/K$, and since PK/K is a p-group we have $H/K \cap Z(PK/K) \neq 1$ by [8, Satz I.6.9]. Thus we may find R such that $R \trianglelefteq PK/K$ and $1 < R < H/K$. Since P controls p-fusion weakly in G it follows from Theorem (2.2.22) that PK/K controls p-fusion weakly in G/K. Now let $\overline{g} \in G/K$. Then we have $R^{\overline{g}} < (H/K)^{\overline{g}} = H/K \leq PK/K$. Thus we may write $\overline{g} = \bar{n}\bar{p}$ with $\overline{n} \in N_{G/K}(R)$ and $\overline{p} \in PK/K$. Thus since $R \trianglelefteq PK/K$ we get $R^{\overline{g}} = R^{\bar{n}\bar{p}} = R^{\overline{p}} = R$. Thus $R \trianglelefteq G/K$, but since $1 < R < H/K$ this contradicts the fact that H/K is a chief factor of G. We conclude that $|H/K| = p$ and we are done. $\square$

(2.2.27) Remark. Let P be a p-group and let G be a finite group such that P is a Sylow p-subgroup of G. If the restriction $H^*(G, \mathbb{F}_p) \longrightarrow H^*(P, \mathbb{F}_p)^{N_G(P)}$ is an isomorphism for any such group G, then P is called a *Swan group*. By Theorem (2.2.7) this condition is equivalent to $N_G(P)$ controlling p-fusion strongly in G for any such group G. It is well known that if P is abelian then P is a Swan group [16, Lemma 1]. This follows also from Theorem (2.2.25) since in this case G is obviously p-normal. Since $Z(P) = P$ we then obtain condition (4).

If P is cyclic we obtain directly from the definition that P controls p-fusion weakly in G, since P then contains at most one subgroup of a given order.

Finally, if P is a T.I. set (trivial intersection set), i.e. if $P^g = P$ or $P^g \cap P = 1$ for all $g \in G$, then we also get directly that $N_G(P)$ controls p-fusion strongly in G. In fact, if P is a T.I. set, then we have an even stronger cohomological result, namely that the restriction map $H^*(G, A)_{(p)} \longrightarrow H^*(H, A)_{(p)}$ is an isomorphism for any $\mathbb{Z}[G]$-module A [1, Corollary 3.6.19].

The following table gives all possible combinations of truth values allowed by Theorem (2.2.25)). For each case we also list a group with given properties. In the examples $\widetilde{\Sigma_5}$ denotes either of the two nonisomorphic double covers of Σ_5 ([13]) and G_{576} denotes the solvable group of order 576 with the presentation:

$$\begin{aligned} G_{576} = \big\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 | & x_1^2 = x_4^2 = x_5^2 = x_6^2 = x_7^2 = x_8, x_2^3 = x_3^3 = 1, \\ & x_8^2 = 1, x_3^{x_1} = x_3^2, x_4^{x_1} = x_6x_8, x_7^{x_2} = x_7^{x_3} = x_6, x_4^{x_2} = x_5^{x_3} = x_4x_5x_8, x_4^{x_3} = x_5, \\ & x_5^{x_2} = x_4, x_5^{x_4} = x_7^{x_1} = x_5x_8, x_5^{x_1} = x_7^{x_6} = x_7x_8, x_6^{x_1} = x_4x_8, x_6^{x_2} = x_6^{x_3} = x_6x_7, \\ & [x_1, x_2] = [x_2, x_3] = [x_4, x_6] = [x_5, x_6] = [x_4, x_7] = [x_5, x_7] = [x_1, x_8] = \\ & [x_2, x_8] = [x_3, x_8] = [x_4, x_8] = [x_5, x_8] = [x_6, x_8] = [x_7, x_8] = 1 \rangle . \end{aligned}$$

This group is an extension of $C_3 \wr C_2$ by the extraspecial 2-group 2_+^{1+4} generated by x_4, x_5, x_6, x_7 and x_8. The center of G_{576} is the cyclic group of order 2 generated by x_8, and we let $G_{288} = G_{576}/Z(G_{576})$.

The group ASL denotes the affine special linear group, which also occur is [6] where it is called the *quadratic group Qd*.

(1), (2), (11)	(3)	(4), (5)	(6), (7), (8)	(9)	(10)	(12)	(13)	Groups
+	+	+	+	+	+	+	+	G abelian
−	+	+	+	+	+	+	+	Σ_3, $p = 3$
−	+	+	+	−	−	+	+	A_5, $p = 3, 5$
−	−	+	+	+	+	+	+	$C_3 \wr C_2$, $p = 3$
−	−	+	+	−	+	+	+	A_4, $p = 2$
−	−	+	+	−	−	+	+	A_5, $p = 2$
−	−	−	+	−	+	+	+	G_{576}, $p = 2$
−	−	−	+	−	+	−	+	$\mathrm{GL}_2(\mathbb{F}_3)$, $p = 2$
−	−	−	+	−	−	+	+	$A_5 \times G_{576}$, $p = 2$
−	−	−	+	−	−	−	+	$\widetilde{\Sigma}_5$, $p = 2$
−	−	−	−	−	+	+	+	G_{288}, $p = 2$ $\mathrm{ASL}_2(\mathbb{F}_3)$, $p = 3$
−	−	−	−	−	+	−	+	$\mathrm{GL}_2(\mathbb{F}_3) \times G_{288}$, $p = 2$
−	−	−	−	−	+	−	−	Σ_4, $p = 2$
−	−	−	−	−	−	+	+	$A_5 \times \mathrm{ASL}_2(\mathbb{F}_3)$, $p = 3$
−	−	−	−	−	−	−	+	$\widetilde{\Sigma}_5 \times G_{288}$, $p = 2$
−	−	−	−	−	−	−	−	Σ_5, $p = 2$

(**2.2.28**) **Remark**. A finite group G is called a *monomial* group (or simply an M-group) if every irreducible complex character of G is induced from a linear character of a subgroup of G. Then every supersolvable group is an M-group and every M-group is solvable [8, Satz V.18.5(a) and Satz V.18.6(b)]. However, there seems to be no good notion of a p-monomial group. The obvious definition would require that every irreducible complex character of G of degree divisible by p is induced from a linear character of a subgroup of G. With this definition G is an M-group if and only if G is p-monomial for all p, but it is not true that any p-monomial group is p-solvable. To see this consider the group A_5, whose irreducible characters have degrees 1, 3, 3, 4 and 5 [3, §14D]. Since the character of degree 5 is induced from any of the two nontrivial linear characters of the subgroup A_4, we see that A_5 is 5-monomial. However it is not 5-solvable since A_5 is simple.

We end this section by mentioning a question inspired by Theorem (2.2.25) and the table above.

(**2.2.29**) **Question**. Let G be a finite group, p a prime number, P a Sylow p-subgroup and H a subgroup containing $N_G(P)$. Is it then true that H controls p-fusion strongly in G if H controls p-fusion weakly in G ? By Theorem (2.2.25) this holds for $H = N_G(P)$ and $H = N_G(Z(P))$. We have checked that this is true for all groups of order less than 384.

2.3 Swan's theorem

In this section we give a counterexample to the original version of Swan's theorem on the cohomology of p-normal group [16]. The theorem was later corrected [17] due to a counterexample of Evens, however the example was never published. The corrected version of the theorem is as follows:

(**2.3.1**) **Theorem**. *Let G be p-normal and let P be a Sylow p-subgroup of G. If A is a trivial $\mathbb{Z}[G]$-module, then the restriction map*

$$H^*(G, A)_{(p)} \to H^*(N_G(Z(P)), A)_{(p)}$$

is an isomorphism.

(**2.3.2**) **Proof**: Since G is p-normal, we know by Theorem (2.2.25) that $N_G(Z(P))$ controls p-fusion strongly in G. Thus the theorem follows by Theorem (2.2.7). □

(**2.3.3**) **Example**. We now give an example showing that the original version of Swan's theorem which left out the condition of trivial coefficients, does not hold. Let $p = 2$ and

$$G = D_{12} = \left\langle \sigma, \tau | \sigma^6 = \tau^2 = 1, \sigma^\tau = \sigma^{-1} \right\rangle$$

be the dihedral group of order 12. Then $P = \langle \sigma^3, \tau \rangle$ is a Sylow 2-subgroup, and since P is abelian, G is 2-normal. Moreover $N_G(Z(P)) = P$. Let $C_2 =< \sigma^3 >$, $C_3 =< \sigma^2 >$ and $C_6 =< \sigma >$ be the cyclic subgroups of the subgroup of rotations in G. Let M be an arbitrary $\mathbb{Z}[G]$-module. By [21, 7.3, p. 158] we then have

$$H^*(G, M)_{(2)} = H^*(P, M)_{(2)} + \frac{1}{2}\left[H^*(C_6, M)_{(2)} - H^*(C_2, M)_{(2)}\right].$$

The first term is the one predicted by Swan's theorem. Therefore the original version of Swan's theorem does not hold if we can construct M with $H^*(C_2, M)_{(2)} \ncong H^*(C_6, M)_{(2)}$.

Now C_2 is normal in G and $G/C_2 \cong \Sigma_3 \cong GL(2, \mathbb{F}_2)$ acts naturally on $M = \mathbb{F}_2^2$. Using this, M gets the structure of a $\mathbb{Z}[G]$-module. The action of σ and τ is up to conjugation given by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \tau \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Using the Lyndon-Hochschild-Serre spectral sequence we get the isomorphism

$$H^*(C_6, M) \cong H^*(C_6/C_3, M^{C_3}).$$

A direct computation shows that $M^{C_3} = 0$, so $H^*(C_6, M) = 0$. However the action of C_2 on M is trivial, so $H^n(C_2, M) = \mathbb{F}_2^2$ for all n.

(**2.3.4**) **Remark**. As remarked above, by [1, Corollary 3.6.19] the restriction map

$$H^*(G, A)_{(p)} \to H^*(N_G(P), A)_{(p)}$$

is an isomorphism for any $\mathbb{Z}[G]$-module A if the Sylow p-subgroup P is a T.I. set. In particular this happens if $|P| = p$, and thus we see that the order of a counterexample has to be at least $2^2 \cdot 3 = 12$.

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Part III

On the Poincaré series of cohomology rings of finite groups

Chapter 3

On the Poincaré series of cohomology rings of finite groups

3.1 Introduction

Let G be a finite group and let p be a prime. By the Venkov-Evens theorem [13; 29], the cohomology ring $H^*(G, \mathbb{F}_p)$ is a finitely generated $\mathbb{F}_p$-algebra. If the degrees of the generators are $n_1, \ldots, n_s$ then by the Hilbert-Serre theorem ([2, Theorem 2.1.1] or [26, Theorem 4.1.1]), the Poincaré series $P_G(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{F}_p} H^n(G, \mathbb{F}_p) t^n$ is a rational function of the form

$$P_G(t) = \frac{f(t)}{\prod_{i=1}^{s}(1 - t^{n_i})} \tag{i}$$

where $f(t)$ is a polynomial in t with integer coefficients. The purpose of this paper is to discuss some of the properties of $P_G(t)$ and their relations to properties of the cohomology ring $H^*(G, \mathbb{F}_p)$. Along the way we correct a formula for the Poincaré series of wreath products given by Webb [31].

3.2 Zeroes and poles of the Poincaré series

The series $P_G(t)$ reflects some of the properties of the cohomology ring $H^*(G, \mathbb{F}_p)$. For example the order of the pole of $P_G(t)$ at $t = 1$ equals the Krull dimension of $H^*(G, \mathbb{F}_p)$ by [2, Theorem 2.2.7] or [26, Theorem 5.3.3]. By Quillen's theorem [23] this also equals the p-rank of G (i.e. the rank of the largest elementary abelian p-subgroup of G).

Concerning the other poles of $P_G(t)$ we have less information. Of course by (i) they must be roots of unity as well. In [24, p. 367] it is claimed that if G is a 2-group and $p = 2$ then all the poles are $|G|$'th roots of unity, but no proof is given. For odd primes p, this result is not true. Consider the semidirect product of cyclic groups of order p^α and p^β:

$$G = \left\langle x, y | x^{p^\alpha} = y^{p^\beta} = 1, x^{-1}yx = y^{1+u} \right\rangle, \qquad \text{(ii)}$$

where $1 + u$ satisfies $(1 + u)^{p^\alpha} - 1 = m \cdot p^\beta$ for some $m \in \mathbb{Z}$. Then if $\alpha, \beta \geq 1$ and $p \nmid m$ we have

$$P_G(t) = \frac{1 + t}{(1 - t)(1 - t^{2p})} \qquad \text{(iii)}$$

by [17]. Note that the condition is satisfied if $\alpha = 1$, $\beta = 2$ and $u = p$, in which case $G = p_-^{1+2}$, the extraspecial group of order p^3 and exponent p^2. Unfortunately no details of the computation are given in [17], however their result may be verified by using [30] [1] which gives the additive structure of the integral cohomology ring for any split metacyclic group. We will return to this example again later.

Lacking evidence to the contrary, we conjecture that for $p = 2$ the poles of $P_G(t)$ are all $|G|$'th roots of unity and that for p odd they are $2|G|$'th roots of unity. By the Hilbert-Serre theorem such a result pertains to the degrees of the generators of the cohomology ring $H^*(G, \mathbb{F}_p)$ and thus potentially interesting for concrete computations.

The location of the *zeros* of $P_G(t)$ is more mysterious. By the Hilbert-Serre theorem they are algebraic numbers. In [24, p. 367] Rusin gives an example showing that they need not be roots of unity, and asks whether they are always algebraic integers. In Example (3.2.8) below we show that this is not the case. In general we have the following result linking ring theoretic properties of $H^*(G, \mathbb{F}_p)$ to properties of $P_G(t)$. The commutative algebra we will need may be found in [7] and [19].

(**3.2.1**) **Theorem**. *Let G be a finite group, p a prime number, $P_G(t)$ the Poincaré series of $H^*(G, \mathbb{F}_p)$ and d the Krull dimension of $H^*(G, \mathbb{F}_p)$. Consider the following conditions:*

(1) *$H^*(G, \mathbb{F}_p)$ is a complete intersection.*

[1] The summations in the formulas for $f_{2n}d_1$ and k_m in [30, p. 254] should start at $j = 0$ instead of $j = 1$.

(2) *$H^*(G,\mathbb{F}_p)$ is Gorenstein.*

(2′) *$H^*(G,\mathbb{F}_p)$ is Cohen-Macaulay.*

(3) *The roots of $P_G(t)$ are roots of unity.*

(4) *$P_G(t)$ satisfies $P_G(1/t) = (-t)^d P_G(t)$.*

(5) *$P_G(t)$ satisfies $P_G(1/t) = (-1)^d t^\rho P_G(t)$ for some $\rho \in \mathbb{Z}$.*

(6) *The roots of $P_G(t)$ are algebraic integers.*

We then have the implications:

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (2) \arrow[r, Leftrightarrow] & (2') \arrow[r, Rightarrow] & (4) \arrow[d, Rightarrow] & \\
& (3) \arrow[rr, Rightarrow] & & (5) \arrow[r, Rightarrow] & (6)
\end{tikzcd}$$

(**3.2.2**) **Remark**. In [15, p. 2079] the properties (4) and (5) are called *strongly quasi-Gorenstein* and *quasi-Gorenstein* respectively.

(**3.2.3**) **Proof**: The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (2′) are well known and are proved in [7, Proposition 3.1.20] for strictly commutative rings, but the proof carries over verbatim to graded commutative rings. The implication (4) $\Longrightarrow$ (5) is obvious, and (2′) $\Longrightarrow$ (4) and (2′) $\Longrightarrow$ (2) are [3, Theorem 1.1]. From [27, Corollary 3.3] we see that if (1) is satisfied we may write $P_G(t)$ in the form

$$P_G(t) = \frac{\prod_{j=1}^{r}(1-t^{f_j})}{\prod_{i=1}^{s}(1-t^{e_i})}, \tag{iv}$$

for some positive integers $f_1,\dots,f_r,e_1,\dots e_s$ (note that even though [27] is written for strictly commutative rings, the proof immediately carries over to graded commutative rings as well). This immediately implies (3). If the roots of $P_G(t)$ are roots of unity we see by comparison with (i) that all roots of $f(t)$ are roots of unity as well. Note that $f(0) = 1$ since $H^0(G,\mathbb{F}_p) = \mathbb{F}_p$. Since $f(t)$ has integer coefficients it then follows that $f(t)$ is a product of cyclotomic polynomials. Since these may be written in the form (iv) it then follows that $P_G(t)$ may also be written in the form (iv). Then by direct computation we get that

$$\frac{P_G(1/t)}{P_G(t)} = (-1)^{r-s} t^\rho,$$

where $\rho = \sum_{i=1}^{s} e_i - \sum_{j=1}^{r} f_j$. Since the Krull dimension is the order of the pole of $P_G(t)$ at $t = 1$ we get $d = r - s$. This proves (3) $\Longrightarrow$ (5). To show that (5) $\Longrightarrow$ (6) we write $P_G(t)$ in the form (i). We then have

$$\frac{P_G(1/t)}{P_G(t)} = \frac{f(1/t)}{f(t)}(-1)^s t^{\sum_{i=1}^{s} n_i},$$

Thus if (5) is satisfied, we see that $\frac{f(1/t)}{f(t)}$ must have the form $\pm t^r$ for some $r \in \mathbb{Z}$. Now write $f(t) = a_0 + a_1 t + \ldots + a_m t^m$, where $a_m \neq 0$. Note that $a_0 = 1$ since $f(0) = 1$. We then get

$$\pm t^r = \frac{f(1/t)}{f(t)} = \frac{a_m + a_{m-1}t + \ldots + a_0 t^m}{a_0 + a_1 t + \ldots + a_m t^m} t^{-m}.$$

Since $a_0 = 1$ and $a_m \neq 0$ we conclude that $a_m = \pm a_0 = \pm 1$. In particular all roots of $f(t)$ must be algebraic integers, which proves (6). □

(**3.2.4**) **Remark**. The Cohen-Macaulay property is satisfied in many interesting cases. Let P be a Sylow p-subgroup of G. Then if $H^*(P, \mathbb{F}_p)$ is Cohen-Macaulay then so is $H^*(G, \mathbb{F}_p)$ by [3, Proposition 6.8]. This condition is satisfied if P is abelian or more generally if all elements of order p in P are central [12]. If P is an extraspecial 2-group then $H^*(P, \mathbb{F}_2)$ is Cohen-Macaulay by [22]. However if p is odd and P is an extraspecial p-group, then $H^*(P, \mathbb{F}_p)$ is not Cohen-Macaulay unless $p = 3$ and $P = 3^{1+2}_+$ is the extraspecial 3-group of order 27 and exponent 3 [20].

We now proceed to give examples showing that the theorem is the best possible in the sense that no further implications hold. For this we need the following result on the cohomology of wreath products.

(**3.2.5**) **Theorem**. *Let G be a finite group and p a prime number. The cohomology ring $H^*(G \wr C_p, \mathbb{F}_p)$ of the wreath product $G \wr C_p$ has Poincaré series*

$$P_{G\wr C_p}(t) = \frac{1}{p}\left(P_G(t)^p - P_G(t^p)\right) + \frac{P_G(t^p)}{1-t}.$$

The Krull dimension and depth is given by

$$\begin{aligned} \dim H^*(G \wr C_p, \mathbb{F}_p) &= \max\{1, p \cdot \dim H^*(G, \mathbb{F}_p)\}, \\ \operatorname{depth} H^*(G \wr C_p, \mathbb{F}_p) &= \operatorname{depth} H^*(G, \mathbb{F}_p) + 1. \end{aligned}$$

(**3.2.6**) **Proof**: The result on the Poincaré series follows from [5] (an translation may be found in [6]). We will return to Webb's generalization [31] in section 3.3. The result on the depth is [9, Theorem 2.1]. Finally, the Krull dimension may be computed as follows. Let $x \in G \wr C_p$ be an element which permutes the factors of $G^p \subseteq G \wr C_p$ cyclically. Let d denote the p-rank of G, and let E be an elementary abelian p-subgroup of $G \wr C_p$. If $E \subseteq G^p$ then $E \subseteq \pi_1(E) \times \ldots \times \pi_p(E)$ and thus $\operatorname{rank} E \leq pd$, where $\pi_i : G^p \to G$ is the projection on the i'th factor. If $E \not\subseteq G^p$ then E contains an element of the form $y = (g_1, \ldots, g_p)x^i$, $1 \leq i \leq p-1$. We may assume that $i = 1$ after replacing y by a suitable power of itself. Since y has order p we get $g_1 \cdot \ldots \cdot g_p = 1$. Conjugating E by the element $h = (h_1, \ldots, h_p)$ where $h_i = g_i \cdot \ldots \cdot g_p$ we get that E^h contains x. Thus $E^h \subseteq C_{G\wr C_p}(x) = \Delta G \times \langle x \rangle$, where $\Delta G = \{(g, \ldots, g) | g \in G\} \cong G$ is the diagonal subgroup. From this we get $\operatorname{rank} E = \operatorname{rank} E^h \leq d + 1$. Conversely, let E be an elementary abelian p-subgroup of G of rank d. Then the subgroups E^p and $\Delta E \times \langle x \rangle$ are elementary abelian p-subgroups of $G \wr C_p$ of rank pd and $d+1$ respectively. Thus the p-rank of $G \wr C_p$ is exactly $\max\{pd, d+1\} = \max\{1, pd\}$. By Quillen's theorem the p-rank equals the Krull dimension of the cohomology ring and we are done. Alternatively the formula for the Krull dimension may be proved by combining the formula for the Poincaré series of the wreath product with the fact that the Krull dimension is the order of the pole of the Poincaré series at $t = 1$. □

We can now give the promised examples, which will mainly be wreath products of 2-groups. For our main examples we will not need to know very much about the cohomology of these groups except for Theorem (3.2.5). For completeness we will also give some other examples which have lower order, however their verification requires a more detailed knowledge of the cohomology rings involved. For 2-groups of order at most 32 these results may be found in [24]. The cohomology of the 2-groups of order at most 64 (except for a few) may be found on Carlson's web page [8]. These groups were first classified by Hall and Senior ([14]) and we will use their names for the groups. The following table (which gives all possible combination of truth values allowed by Theorem (3.2.1)), summarizes our examples. Incidentally we do not know if any of the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) or (3) $\Longrightarrow$ (5) can be reversed in odd characteristic. However we do not believe that this is the case, and that the lack of examples is probably only due to the fact that very few computations have been carried out in odd characteristic compared to characteristic 2.

(1)	(2), (2′)	(3)	(4)	(5)	(6)	Groups
+	+	+	+	+	+	G abelian
−	+	+	+	+	+	Example (3.2.10)
−	+	−	+	+	+	Example (3.2.12)
−	−	+	+	+	+	Example (3.2.7)
−	−	+	−	+	+	Example (3.2.9)
−	−	−	+	+	+	Example (3.2.7)
−	−	−	−	+	+	Example (3.2.13)
−	−	−	−	−	+	Example (3.2.11)
−	−	−	−	−	−	Example (3.2.8)

(3.2.7) Example. Consider the group $G = {C_2}^n \wr C_2$, $n \geq 1$ and let $p = 2$. The cohomology ring $H^*({C_2}^n, \mathbb{F}_2)$ is a polynomial algebra on n generators in degree 1 so it is Cohen-Macaulay of Krull dimension n and its Poincaré series is $1/(1-t)^n$. From Theorem (3.2.5) we then see that G satisfies (2′) if and only if $n = 1$. From Theorem (3.2.5) we also get

$$\begin{aligned} P_G(t) &= \frac{1}{2}\left(\frac{1}{(1-t)^{2n}} - \frac{1}{(1-t^2)^n}\right) + \frac{1}{(1-t)(1-t^2)^n} \\ &= \frac{1}{(1-t)^{2n}(1+t)^{n-1}} \cdot \frac{1}{2}\left[(1+t)^{n-1} + (1-t)^{n-1}\right] \\ &= \frac{1}{(1-t)^{2n}(1+t)^{n-1}} \cdot \sum_{j=0}^{[(n-1)/2]} \binom{n-1}{2j} t^{2j}. \end{aligned}$$

The leading term in the sum is t^{n-1} for n odd and $(n-1)t^{n-2}$ for n even. From this we see that (6) is satisfied if and only if n is odd or $n = 2$. Since the Krull dimension of $H^*(G, \mathbb{F}_2)$ is $2n$ we see from this by direct computation that (4) is satisfied exactly when n is odd and (5) holds exactly when $n = 2$ or n is odd. We also see that (3) holds if and only if all roots of the polynomial $(1+t)^{n-1} + (1-t)^{n-1}$ are roots of unity. However it is easily seen that this is the case only for $n = 1$, 2 and 3.

We conclude that for n odd, $n \geq 5$, the group ${C_2}^n \wr C_2$ satisfies (4), (5) and (6), but not (1), (2), (2′) or (3). For $n = 3$ the group satisfies (3), (4), (5) and (6), but not (1), (2) or (2′). Another example with the same properties is the group $32\Gamma_7 a_2$. In particular the Poincaré series of these groups satisfy the functional equation (4), but their mod-2 cohomology rings are not Cohen-Macaulay. The example $32\Gamma_7 a_2$ is also given in [4], which generalizes the implication (2′) $\Longrightarrow$ (4) of Theorem (3.2.1).

(**3.2.8**) **Example**. From the above example we also conclude that for n even, $n \geq 4$, the group ${C_2}^n \wr C_2$ satisfies none of the properties in Theorem (3.2.1). In particular the roots of the Poincaré series are not algebraic integers. As mentioned above, this answers a question of Rusin [24, p. 267]. From [8] we also find some examples of order 64, namely the groups $64\Gamma_{15}c_3$, $64\Gamma_{15}c_4$, $64\Gamma_{15}e_3$, $64\Gamma_{15}f_1$, $64\Gamma_{16}c_1$ and $64\Gamma_{17}b_2$.

We can also construct some examples in odd characteristic. Consider the group $G = C_p \wr C_p$ which is the Sylow p-subgroup of the symmetric group Σ_{p^2} (see for example [1, p. 184–185]). From Theorem (3.2.5) we get that for odd p the Poincaré series of $H^*(G, \mathbb{F}_p)$ is given by:

$$\begin{aligned} P_G(t) &= \frac{1}{p}\left(\frac{1}{(1-t)^p} - \frac{1}{1-t^p}\right) + \frac{1}{1-t} \cdot \frac{1}{1-t^p} \\ &= \frac{1}{(1-t)^p(1-t^p)}\left[1 + \sum_{i=1}^{p-2} \frac{p-i-1}{p} \begin{pmatrix} p \\ i \end{pmatrix} (-t)^i\right]. \end{aligned}$$

The leading term is $\frac{p-1}{2}(-t)^{p-2}$, so for $p \geq 5$ we see that the roots of $P_G(t)$ are not algebraic integers. Thus none of the properties in Theorem (3.2.1) are satisfied.

(**3.2.9**) **Example**. From Example (3.2.7) we see that the group ${C_2}^2 \wr C_2 = 32\Gamma_4 a_1$ satisfies (3), (5) and (6), but not (1), (2), (2′) or (4). From [24] we also get that this holds for the groups $16\Gamma_2 c_1$, $16\Gamma_2 d$ and $16\Gamma_3 a_2$ of order 16. The group $16\Gamma_3 a_2$ is the semidihedral group of order 16 and occurs as the Sylow 2-subgroup of the Mathieu group M_{11} and of $\mathrm{GL}_2(\mathbb{F}_q)$ for $q \equiv 3 \pmod 8$. It also occurs as the Sylow 2-subgroup of the simple groups $\mathrm{PSL}_3(q)$ for $q \equiv 3 \pmod 8$ and $\mathrm{PSU}_3(q)$ for $q \equiv 5 \pmod 8$. Both $16\Gamma_2 d$ and $16\Gamma_3 a_2$ are split metacyclic groups, so the Poincaré series of their cohomology rings may also be found using [30].

In odd characteristic we have the example (ii) above. From its Poincaré series (we assume that $\alpha, \beta \geq 1$ and $p \nmid m$) we immediately see that this group satisfies (3) and hence (5) and (6), but not (4) and hence neither (1), (2) nor (2′). In particular this applies to the extraspecial groups p_-^{1+2} of order p^3 and exponent p^2.

Another example with the same properties is $C_3 \wr C_3$ for $p = 3$, which by Example (3.2.8) has the Poincaré series

$$P_{C_3 \wr C_3}(t) = \frac{1}{(1-t)^2(1-t^3)}.$$

(**3.2.10**) **Example**. Consider the group $G = C_4 \wr C_2 = 32\Gamma_3 e$ and let $p = 2$. This group occurs as the Sylow 2-subgroup of the groups $\mathrm{GL}_2(\mathbb{F}_q)$ for $q \equiv 5 \pmod 8$, $\mathrm{PSL}_3(\mathbb{F}_q)$ for $q \equiv 5 \pmod 8$ and $\mathrm{PSU}_3(\mathbb{F}_q)$ for $q \equiv 3 \pmod 8$. Since the Poincaré of C_4 is $\frac{1}{1-t}$ we see from Theorem (3.2.5) that the cohomology ring $H^*(G, \mathbb{F}_2)$ is Cohen-Macaulay of dimension 2 and has the Poincaré series $\frac{1}{(1-t)^2}$. Thus the conditions (2), (2′), (3), (4), (5) and (6) of Theorem (3.2.1) are all satisfied. However the cohomology ring is not a complete intersection, which may be seen from Rusin's calculation [24]. Of course the cohomology ring may also be directly computed from Nakaoka's theorem [21, Theorem 3.3]. From [24] we find that the groups $32\Gamma_2 h$, $32\Gamma_3 c_2$ and $32\Gamma_4 c_3$ have the same properties.

(**3.2.11**) **Example**. Consider the group $G = C_2 \wr (C_2 \times C_2) = 64\Gamma_{25} a_1$, where $C_2 \times C_2$ acts regularly on a set of size 4 and let $p = 2$. This group is the Sylow 2-subgroup of the simple groups $A_8 \cong \mathrm{GL}_4(\mathbb{F}_2) \cong O_6^+(\mathbb{F}_2)$, $PSU_4(\mathbb{F}_{2^2}) \cong O_5(\mathbb{F}_3) \cong O_6^-(\mathbb{F}_2)$ and $PSp_4(\mathbb{F}_5) \cong O_5(\mathbb{F}_5)$ and it is also the Sylow 2-subgroup of the affine general linear group $\mathrm{AGL}_3(\mathbb{F}_2)$. By [31, Example 3.4] the Poincaré series of the cohomology ring is

$$P_G(t) = \frac{1 + t^2 - t^3}{(1-t)^3(1-t^4)}.$$

Thus G satisfies (6), but none of the properties (1), (2), (2′), (3), (4) or (5). Other groups with the same properties are $32\Gamma_4 {c_1}^2$, $32\Gamma_4 d$, $32\Gamma_6 a_2$, $32\Gamma_7 a_3$, $16\Gamma_2 d \wr C_2$ and $16\Gamma_3 a_2 \wr C_2$ (the groups $16\Gamma_2 d$ and $16\Gamma_3 a_2$ also occurred in Example (3.2.9) above).

For odd primes we have more examples. Consider the extraspecial group $G = p_+^{1+2}$ of order p^3 and exponent p. The cohomology ring of this group is given in [18] and its Poincaré series is found to be

$$P_G(t) = \frac{1 + t^2 - t^4 - t^{2p}}{(1-t)^2(1-t^{2p})}$$

(cf. [25, Theorem 7]). From this we see that if $p \geq 5$, only condition (6) in Theorem (3.2.1) is satisfied.

We also have the following example. Consider again the group

$$P = \left\langle x, y | x^{p^\alpha} = y^{p^\beta} = 1, x^{-1}yx = y^{1+u} \right\rangle$$

[2] The Poincaré series for this group equals $\frac{1+t+t^3}{(1-t)^2(1-t^4)}$, but is given in [24, p. 367] as $\frac{1+t^2+t^3}{(1-t)^2(1-t^4)}$.

from (ii) above, where $1+u$ satisfies $(1+u)^{p^\alpha}-1 = m\cdot p^\beta$ for some $m\in\mathbb{Z}$. Assume furthermore that p is an odd prime, that $\alpha,\beta\geq 1$ and that $p\nmid m$. Then by [10, Proposition 3.2(1)] we get $\mathrm{Out}(P) = O_p \rtimes C_{p-1}$, where O_p is a Sylow p-subgroup of $\mathrm{Out}(P)$. Choose any divisor d of $p-1$ and consider the semidirect product $G = P\rtimes C_d$, where $C_d \leq C_{p-1}\leq \mathrm{Out}(P)$. Then by [11, Theorem 1.1(2)] we have

$$H^*(G,\mathbb{F}_p)\cong \mathbb{F}_p[v,z]\otimes E[b,\alpha_{2i-1}, i=1,\ldots,p]/R$$

where the relations R are given by

$$\alpha_{2i-1}\alpha_{2j-1} = 0 \text{ for } 1\leq i,j\leq p$$
$$\alpha_{2i-1}v = 0 \text{ for } 1\leq i\leq p-1,$$

with $\deg b = 1$, $\deg v = 2$, $\deg z = 2p$ and $\deg\alpha_{2i-1} = 2i-1+2pd(i)$, where $0\leq d(i) < d$ is the residue of $-i$ mod d. From this we can easily compute the Poincaré series:

$$\begin{aligned} P_G(t) &= \frac{1+t}{1-t^{2p}}\left[\frac{1}{1-t^2}+\frac{t^{\deg\alpha_{2p-1}}}{1-t^2}+\sum_{i=1}^{p-1} t^{\deg\alpha_{2i-1}}\right] \\ &= \frac{1+t^{2d-1}-t^{2d}-t^{2d+1}+t^{2pd+1}-t^{2pd+2d-1}}{(1-t)(1-t^{2d})(1-t^{2p})}. \end{aligned}$$

If $d>1$ we see from this that condition (5) is not satisfied and that condition (6) is satisfied. Thus the group satisfies none of the conditions in Theorem (3.2.1) except for (6). Note that for $d=1$ we have $G=P$ and we recover the Poincaré series (iii) given earlier. For $d>1$ we also see that the a-invariant ([7, Definition 4.3.6]) of $H^*(G,\mathbb{F}_p)$, which by definition is the degree of the Poincaré series as a rational function, equals $2pd-2p-2$. Thus it can be arbitrarily large compared with the Krull dimension which is 2. Hence the analogue of [16, Conjecture 22] does not hold for cohomology rings of finite groups.

(**3.2.12**) **Example**. Let $p=2$ and consider the group $Q_8\wr C_2$, where Q_8 is the quaternion group of order 8. By [1, Theorem IV.2.9] we have $P_{Q_8}(t) = \frac{1+2t+2t^2+t^3}{1-t^4}$. By Theorem (3.2.5) we then see that $H^*(Q_8\wr C_2,\mathbb{F}_2)$ is Cohen-Macaulay of dimension 2 and has the Poincaré series

$$P_{Q_8\wr C_2}(t) = \frac{(1+t+t^2)(1+2t^2-t^3+2t^4+t^6)}{(1-t)^2(1+t^2)^2(1+t^4)}.$$

Thus we see that $Q_8\wr C_2$ satisfies (2), (2′), (4), (5) and (6) but not (1) and (3). Other groups with this behavior are $64\Gamma_4k_2$, $64\Gamma_9b_4$, $64\Gamma_9d_2$ and $64\Gamma_9e$ which is the Sylow 2-subgroup of the simple group Sz(8).

(**3.2.13**) **Example**. As a final example note that by combining Example (3.2.9) and Example (3.2.12) we obtain that the group $(Q_8\wr C_2)\times\left((C_2)^2\wr C_2\right)$ satisfies only conditions (5) and (6) of Theorem (3.2.1). Other examples are given by the groups $32\Gamma_2 d \wr C_2$, $32\Gamma_2 f \wr C_2$, $32\Gamma_3 a_2 \wr C_2$, $32\Gamma_4 b_2 \wr C_2$, $32\Gamma_4 c_2 \wr C_2$ and $32\Gamma_6 a_1 \wr C_2$[3]. Here $32\Gamma_2 d = C_2 \times 16\Gamma_2 d$ and $32\Gamma_3 a_2 = C_2 \times 16\Gamma_3 a_2$, where the groups $16\Gamma_2 d$ and $16\Gamma_3 a_2$ occurs in Example (3.2.9) above. Moreover the group $32\Gamma_3 a_2$ is the Sylow 2-subgroup of $\mathrm{GL}_3(\mathbb{F}_q)$ for $p \equiv q \pmod 8$. Note also that $32\Gamma_6 a_1 = C_8 \rtimes \mathrm{Aut}(C_8)$.

[3]The Poincaré series of the groups $32\Gamma_2 d$, $32\Gamma_2 f$, $32\Gamma_3 a_2$, $32\Gamma_4 b_2$, $32\Gamma_4 c_2$ and $32\Gamma_6 a_1$ is $\frac{1}{(1-t)^3(1+t^2)}$ and thus $\dim H^n(G, \mathbb{F}_p) = 1, 3, 7, 10, 14, \ldots$ for $n = 0, \ldots$. These dimensions are stated wrongly in [24, p. 368].

3.3 Poincaré series of wreath products

In this section we obtain a formula for the Poincaré series of the cohomology ring of a wreath product of finite groups. Such a formula was first obtained by Webb [31], but unfortunately his formula is not correct in general. Here we show how the correct formula may be obtained by modifying Webb's formula. To save space and time we will assume complete familiarity with the definitions and results in [31].

Now let p be a prime number, F a finite group, G a subgroup of Σ_m and consider the wreath product $F \wr G = F^m \rtimes G$. We are then interested in computing the Poincaré series of $H^*(F \wr G, \mathbb{F}_p)$. As in [31, p. 450] we start by using Nakaoka's theorem [21, Theorem 3.3], which gives an isomorphism $H^*(F \wr G, \mathbb{F}_p) \cong H^*(G, H^*(F, \mathbb{F}_p)^{\otimes m})$ as graded rings. Here G acts on the factors of the m-fold tensor product $H^*(F, \mathbb{F}_p)^{\otimes m}$ not simply by permuting the factors (as claimed in [31]), but also by including the appropriate sign. For example if $\sigma = (1, 2) \in \Sigma_2$ then

$$\sigma \cdot (\xi_1 \otimes \xi_2) = (-1)^{\deg \xi_1 \deg \xi_2} (\xi_2 \otimes \xi_1).$$

Since there are no problems with the sign for $p = 2$ we will from now on assume that p is odd unless stated otherwise. Now let Ξ be a graded basis of $H^*(F, \mathbb{F}_p)$, i.e. a graded set such that $\Xi(n)$ is a basis of $H^n(F, \mathbb{F}_p)$, and let B be the set of elements $\xi_1 \otimes \ldots \otimes \xi_m$, where $\xi_i \in \Xi$. Then B forms a basis of $H^*(F, \mathbb{F}_p)^{\otimes m}$, but unfortunately G does not act on B since we also have to include the appropriate sign. To overcome this problem we introduce the 'double'

$$\widetilde{B} = B \amalg -B = \{\pm b | b \in B\} \subseteq H^*(F, \mathbb{F}_p)^{\otimes m},$$

which is a G-set, and we let $\widetilde{M} = \mathbb{F}_p \left[\widetilde{B}\right]$ be the associated permutation module. Then $B_+ = \{b + (-b) | b \in B\} \subseteq \widetilde{M}$ is a G-set and the G-action is just given by permuting the coordinates (without the sign). Let $M_+ = \mathbb{F}_p [B_+]$ be the associated permutation module. Define also $B_- = \{b - (-b) | b \in B\} \subseteq \widetilde{M}$ and let M_- be the submodule of $\widetilde{M}$ spanned by B_-. Since p is odd we find that $\widetilde{M} = M_+ \oplus M_-$ and $M_- \cong H^*(F, \mathbb{F}_p)^{\otimes m}$ as $\mathbb{F}_p [G]$-modules. For a subgroup $J \leq G$ we now define the power series $\widetilde{f}_J$ and f_J by

$$\widetilde{f}_J(t) = \sum_{i=0}^{\infty} \left|\widetilde{B}(i)^J\right| t^i, \; f_J(t) = \sum_{i=0}^{\infty} \left|B_+(i)^J\right| t^i.$$

Now using [31, Proposition 1.2] and replacing G-sets by permutation modules we obtain

$$P_{\widetilde{M}}(t) = \sum_{K \leq J} \frac{\mathbb{F}_p [G/K] \, \mu(K, J) \widetilde{f}_J(t)}{[G : K]}, \; P_{M_+}(t) = \sum_{K \leq J} \frac{\mathbb{F}_p [G/K] \, \mu(K, J) f_J(t)}{[G : K]},$$

where the sums are over all pairs of subgroups $K \leq J$ of G and μ is the Möbius function of the poset of subgroups of G [28, 3.7]. As in [31] both sides of these equations are to be interpreted as elements in the ring of power series over the Green ring $A(G)$. Combining this with the above we see that the series of the graded $\mathbb{F}_pG$-module $H^*(F,\mathbb{F}_p)^{\otimes m}$ is

$$\sum_{K\leq J}\frac{\mathbb{F}_p\,[G/K]\,\mu(K,J)\left(\widetilde{f}_J(t)-f_J(t)\right)}{[G:K]}.$$

Thus using [31, Lemma 3.2] we obtain that the Poincaré series of the cohomology ring $H^*(F\wr G,\mathbb{F}_p)$ equals

$$\sum_{K\leq J}\frac{g_K(t)\mu(K,J)\left(\widetilde{f}_J(t)-f_J(t)\right)}{[G:K]},$$

where $g_K(t)$ is the Poincaré series of $H^*(K,\mathbb{F}_p)$.

To compute the series $\widetilde{f}_J$ and f_J we may proceed as follows. Let $\{1,\ldots,m\}=\Omega_1\cup\ldots\cup\Omega_n$ be the decomposition of $\{1,\ldots,m\}$ into J-orbits, $f(t)$ be the Poincaré series of $H^*(F,\mathbb{F}_p)$ and let $b=\xi_1\otimes\ldots\otimes\xi_m\in B$, $\xi_i\in\Xi$. Since G acts on B_+ simply by permuting the coordinates we see that $b+(-b)\in B_+$ is invariant under J if and only if $\xi_i=\xi_j$ whenever $i,j\in\{1,\ldots,m\}$ are in the same J-orbit. Thus we get

$$f_J(t)=f(t^{|\Omega_1|})\cdot\ldots\cdot f(t^{|\Omega_n|})$$

as in [31, p. 450 and Proposition 1.4(1)]. Computing the series $\widetilde{f}_J$ is more difficult since we have to take care of the signs. If $b\in\widetilde{B}$ is invariant under the action of J then $\xi_i=\xi_j$ whenever $i,j\in\{1,\ldots,m\}$ are in the same J-orbit; however this is enough to guarantee that b is J-invariant. Now assume that b satisfies this condition. From the decomposition $\{1,\ldots,m\}=\Omega_1\cup\ldots\cup\Omega_n$ we get the homomorphism

$$\varphi: J\longrightarrow\Sigma_{\Omega_1}\times\ldots\times\Sigma_{\Omega_n}\overset{\sigma_1\times\ldots\times\sigma_n}{\longrightarrow}\mathbb{F}_2\times\ldots\times\mathbb{F}_2=V,$$

where $\sigma_i:\Sigma_{\Omega_i}\to\mathbb{F}_2$ is the sign homomorphism. Let $i_1\in\Omega_1,\ldots,i_n\in\Omega_n$ be orbit representatives and let $\bullet$ be the inner product on V, determined by the formula $\mathbf{u}\bullet\mathbf{v}=u_1v_1+\ldots+u_nv_n$. With a slight abuse of notation we then have

$$g\cdot b=(-1)^{\varphi(g)\bullet(\deg\xi_{i_1},\ldots,\deg\xi_{i_n})}\cdot b$$

for $g\in J$. Thus b is J-invariant if and only if $(\deg\xi_{i_1},\ldots,\deg\xi_{i_n})\bullet\varphi(g)=0$ for all $g\in J$, i.e. when $(\deg\xi_{i_1},\ldots,\deg\xi_{i_n})\in\varphi(J)^{\perp}$. Now set

$$f_0(t)=\sum_{i=0,\ i\text{ even}}^{\infty}\dim H^i(F,\mathbb{F}_p)\,t^i=\frac{1}{2}\left(f(t)+f(-t)\right),$$

$$f_1(t)=\sum_{i=0,\ i\text{ odd}}^{\infty}\dim H^i(F,\mathbb{F}_p)\,t^i=\frac{1}{2}\left(f(t)-f(-t)\right)$$

and let

$$f_{\mathbf{v}}(t) = f_{v_1}(t^{|\Omega_1|}) \cdot \ldots \cdot f_{v_n}(t^{|\Omega_n|})$$

for $\mathbf{v} = (v_1, \ldots, v_n) \in V$. Thus we get the formula

$$\widetilde{f_J}(t) = 2 \cdot \sum_{\mathbf{v} \in \varphi(J)^{\perp}} f_{\mathbf{v}}(t),$$

where the factor 2 comes from the fact that $-b$ is J-invariant if b is. We summarize the results of the above discussion in the following theorem which corrects [31, Theorem 3.1] (as noted above the original formula is correct for $p = 2$).

(**3.3.1**) **Theorem**. *Let p be a prime number, F a finite group and G a subgroup of Σ_m. For a subgroup $J \leq G$ define the series $\widetilde{f_J}(t)$,$f_J(t)$ and $g_J(t)$ as above. Then the Poincaré series of the cohomology ring $H^*(F \wr G, \mathbb{F}_p)$ is given by*

$$P_{F \wr G}(t) = \sum_{K \leq J} \frac{g_K(t)\mu(K,J) f_J(t)}{[G:K]},$$

for $p = 2$ and by

$$P_{F \wr G}(t) = \sum_{K \leq J} \frac{g_K(t)\mu(K,J)\left(\widetilde{f_J}(t) - f_J(t)\right)}{[G:K]},$$

for p odd. □

(**3.3.2**) **Example**. Let p be any prime number and let $G = C_p$ act regularly on a set of size p. Let F be any finite group with Poincaré series $f(t)$. We then have

$$g_1(t) = 1, \qquad f_1(t) = f(t)^p$$
$$g_G(t) = \frac{1}{1-t}, \qquad f_G(t) = f(t^p).$$

Now assume that p is odd. Noting that $(1, \ldots, p) \in G$ is an even permutation we get

$$\widetilde{f_1}(t) = 2f(t)^p, \qquad \widetilde{f_G}(t) = 2f(t^p).$$

Thus the Poincaré series of $H^*(F \wr C_p, \mathbb{F}_p)$ is given as

$$\begin{aligned} P_{F \wr C_p}(t) &= \frac{g_G(t) \cdot 1 \cdot \left(\widetilde{f_G}(t) - f_G(t)\right)}{[G:G]} + \frac{g_1(t) \cdot (-1) \cdot \left(\widetilde{f_G}(t) - f_G(t)\right)}{[G:1]} \\ &+ \frac{g_1(t) \cdot 1 \cdot \left(\widetilde{f_1(t)} - f_1(t)\right)}{[G:1]} = \frac{1}{p}\left(f(t)^p - f(t^p)\right) + \frac{f(t^p)}{1-t}. \end{aligned}$$

For $p = 2$ it is easily seen that the same formula holds, since $\widetilde{f_J}(t) - f_J(t) = f_J(t)$. Thus we recover the formula of Bogačenko ([5]) from Theorem (3.2.5) above. Incidentally this is also the result obtained from Webb's original formula.

(**3.3.3**) **Example**. Let p be an odd prime and let $G = C_2$ act regularly on a set of size 2. Let F be any finite group with Poincaré series $f(t)$. We then have

$$\begin{aligned} g_1(t) &= 1, \qquad & f_1(t) &= f(t)^2 \\ g_G(t) &= 1, \qquad & f_G(t) &= f(t^2). \end{aligned}$$

Since $\sigma = (1,2) \in G$ is an odd permutation we get

$$\widetilde{f_1}(t) = 2f(t)^2, \qquad \widetilde{f_G}(t) = 2f_0(t^2)$$

and hence the Poincaré series of $H^*(F \wr C_2, \mathbb{F}_p)$ equals

$$\begin{aligned} P_{F\wr C_2}(t) = {} & \frac{g_G(t)\cdot 1 \cdot \left(\widetilde{f_G}(t) - f_G(t)\right)}{[G:G]} + \frac{g_1(t)\cdot(-1)\cdot\left(\widetilde{f_G}(t) - f_G(t)\right)}{[G:1]} \\ & + \frac{g_1(t)\cdot 1\cdot\left(\widetilde{f_1}(t) - f_1(t)\right)}{[G:1]} = \frac{1}{2}\left(f(-t^2) + f(t)^2\right). \end{aligned}$$

In this case the result does not agree with Webb's original formula which gives $\frac{1}{2}\left(f(t^2) + f(t)^2\right)$. In the case $F = C_p$ we have $f(t) = \frac{1}{1-t}$ and we get

$$P_{C_p\wr C_2}(t) = \frac{1+t^3}{(1-t)(1-t^4)}.$$

This may be checked easily since we have the isomorphism

$$C_p \wr C_2 = (C_p \times C_p) \rtimes C_2 = C_p \times (C_p \rtimes C_2) = C_p \times D_{2p}$$

because the matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are conjugate in $\mathrm{GL}_2(\mathbb{F}_p)$. Since $H^*(D_{2p}, \mathbb{F}_p)$ has Poincaré series $\frac{1+t^3}{1-t^4}$ by [30] this verifies our calculation.

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