Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, July 19, 2022.

This problem set applies the fundamental theorems of coherent cohomology to the group of line bundles on schemes proper over a finite field. In general, the set of isomorphism classes of line bundles on a scheme X form a group $\mathrm{Pic}(X)$ under tensor product, and we will show later that

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

Problem 1. Let k be a finite field, and let $f\colon X \to \mathrm{Spec}(k)$ be a proper map. Let $i\colon X' \to X$ be a closed immersion defined by a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ and suppose that $\mathcal{I}^N = 0$ for some $N \geq 0$. (In this situation, we say that $i\colon X' \to X$ is a nilpotent thickening.) Show that the map

$$\mathrm{Pic}(X) \xrightarrow{\;i^*\;} \mathrm{Pic}(X')$$

has finite kernel and finite cokernel.

[Hint: First use induction to reduce to the case $N = 2$. Then try to establish a short exact sequence relating $\mathcal{O}_X^\times$ and $\mathcal{O}_{X'}^\times$.]