Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, April 26, 2022.

Problem 1. Let X be a topological space and let A be a set. Let $\varphi\colon \mathcal{F} \to \mathcal{F}'$ be the canonical map from the presheaf of constant functions with values in A to the sheaf of locally constant functions with values in A.

(1) By checking on stalks, show that the map $\varphi\colon \mathcal{F} \to \mathcal{F}'$ is a sheafification.

(2) Show that $\varphi\colon \mathcal{F} \to \mathcal{F}'$ is an isomorphism, if the topological space X is irreducible.

Problem 2. Show that the canonical map $\varphi\colon \mathcal{F} \to \mathcal{F}'$ from the presheaf of bounded continuous real functions on $\mathbb{R}^d$ to the sheaf of continuous real functions on $\mathbb{R}^d$ is a sheafification.

Problem 3. Here we give an example of a map of sheaves, which is surjective as a map of sheaves, but not as a map of presheaves. Let $X = \mathbb{C}$, let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, and consider the map of presheaves

$$\mathcal{O}_X \xrightarrow{d/dz} \mathcal{O}_X$$

that maps f to its derivative.

(1) Show that d/dz has the following property: for every open subset $U \subset X$ and section $g \in \mathcal{O}_X(U)$, there is an open cover $(U_i)_{i\in I}$ of U such that $g\,|_{U_i}$ is in the image of $d/dz\colon \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_i)$ for all $i \in I$.

(2) Show that there exists an open subset $U \subset X$ such that $d/dz\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is not surjective.