Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, May 10, 2022.

There is one more tool from category theory that we need to have in our toolbox, namely, Kan extensions. Given a (small) category, let us write

$$\mathcal{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathsf{Set})$$

for the category, whose objects are functors $\mathcal{F}\colon K^{\mathrm{op}} \to \mathsf{Set}$, and whose morphisms are natural transformations between such functors. So if X is a topological space, then $\mathcal{P}(X) = \mathcal{P}(X_{\mathrm{Zar}})$.

If $u\colon K \to L$ is a functor between (small) categories, then the functor

$$\mathcal{P}(L) \xrightarrow{u^*} \mathcal{P}(K)$$

defined by $u^*(\mathcal{G}) = \mathcal{G} \circ u^{\mathrm{op}}$ is called the restriction along u. It has both a left adjoint functor $u_!$ called the left Kan extension along u and a right adjoint functor u_* called the right Kan extension along u. So the functor u gives rise to three adjoint functors

$$\begin{tikzcd} \mathcal{P}(L) \arrow[r, "u^*"] & \mathcal{P}(K). \arrow[l, bend right, "u_!"'] \arrow[l, bend left, "u_*"] \end{tikzcd}$$

The values of $u_!(\mathcal{F})$ and $u_*(\mathcal{F})$ on an object $V \in L$ are given by

$$u_!(\mathcal{F})(V) \simeq \mathrm{colim}_{(U, V \to u(U))} \mathcal{F}(U)$$
$$u_*(\mathcal{F})(V) \simeq \lim_{(U, u(U) \to V)} \mathcal{F}(U).$$

Warning: In general, the index categories can be very complicated and need not be filtered nor cofiltered.

Problem 1. Let $f\colon Y \to X$ be a continuous map between topological space. It defines the functor $u = f^{-1}\colon X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$, so we obtain the three adjoint functors

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "u^*"] & \mathcal{P}(X). \arrow[l, bend right, "u_!"'] \arrow[l, bend left, "u_*"] \end{tikzcd}$$

We rename $f^p = u_!$ and $f_p = u^*$, and we discard the functor u_*.

(1) Show that the functor $f_p = u^*$ preserves sheaves in the sense that there is a unique functor f_* such that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f_p"] & \mathcal{P}(X) \\ \mathrm{Sh}(Y) \arrow[u, "\iota_Y"'] \arrow[r, "f_*"] & \mathrm{Sh}(X) \arrow[u, "\iota_X"'] \end{tikzcd}$$

(2) The functor $f^p = u_!$ does not preserve sheaves, except in trivial cases. However, show that the composite functor $f^* = \mathrm{ass}_Y \circ f^p \circ \iota_X$ is left adjoint to f_*. Conclude that the "inverse image" functor f^* preserves all (small) colimits and that the "direct image" functor f_* preserves all (small) limits.

(3) Show that the index category of the colimit that defines $f^p(\mathcal{F})(V)$ is filtered, and conclude that f^* preserves finite limits.

Problem 2. Let X be a topological space, and let $j\colon U \to X$ is the inclusion of an open subspace. In this case, we have a functor $v\colon U_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$ that to $V \subset U$ assigns $V \subset X$. This again gives us three adjoint functors

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "v^*"] & \mathcal{P}(U). \arrow[l, bend right, "v_!"'] \arrow[l, bend left, "v_*"] \end{tikzcd}$$

(1) Show that the functor v^* preserves sheaves in the sense that there is a unique functor $j^!$ such that the diagram

$$\begin{tikzcd} \mathcal{P}(U) & \mathcal{P}(X) \arrow[l, "v^*"'] \\ \mathrm{Sh}(U) \arrow[u, "\iota_Y"] & \mathrm{Sh}(X) \arrow[l, "j^!"'] \arrow[u, "\iota_X"'] \end{tikzcd}$$

commutes.

(2) The functor $v_!$ does typically not preserve sheaves. However, show that the composite functor $j_! = \mathrm{ass}_X \circ v_! \circ \iota_Y$ is left adjoint to $j^!$.

(3) Show that there is a natural isomorphism $j^p \to v^*$, and conclude that, since the common functor preserves sheaves, this defines a natural isomorphism

$$j^! \longrightarrow j^*.$$

[Hint: Show that the index category of the colimit that defines $j^p(\mathcal{F})(V)$ has a final object and use the general fact that if $p\colon K \to \mathcal{C}$ is a diagram and $1 \in K$ is a final object, then the canonical map $\mathrm{colim}_K\, p \to p(1)$ is an isomorphism.]

Remark. As a left adjoint, the functor $j_!$ preserves all (small) colimits, and as a right adjoint, the functor $j^!$ preserves all (small) limits. However, the functor $j_!$ does generally *not* preserve finite limits, since $v_!$ fails to do so. Indeed,

$$v_!(\mathcal{G})(V) \simeq \mathrm{colim}_{(W, V \subset j(W))}\, \mathcal{G}(W) \simeq \begin{cases} \mathcal{G}(V) & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U, \end{cases}$$

so the image by $v_!$ of a final object of $\mathcal{P}(X)$, which is the limit of the empty diagram, is not a final object in $\mathcal{P}(U)$, unless $U = X$. In particular, the adjoint pair of functors $(j_!, j^!)$ do not directly induce an adjunction between the associated categories of abelian group objects. Nevertheless, there is an adjunction

$$\begin{tikzcd} \mathsf{Ab}(\mathrm{Sh}(X)) \arrow[r, shift right, "j^!"'] & \mathsf{Ab}(\mathrm{Sh}(U)) \arrow[l, shift right, "j_!"'] \end{tikzcd}$$

but the left adjoint must be constructed separately using the additive functor

$$\mathsf{Ab}(\mathcal{P}(U) \xrightarrow{v_!^+} \mathsf{Ab}(\mathcal{P}(X),$$

given by

$$v_!^+(\mathcal{G})(V) \simeq \begin{cases} \mathcal{G}(V) & \text{if } V \subset U \\ \{0\} & \text{if } V \not\subset U. \end{cases}$$

Note that the abelian group $\{0\}$ has one element, where as the empty set $\emptyset$ has zero elements. So $\{0\} \neq \emptyset$, and hence, $v_!^+ \neq v_!$.