Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, May 17, 2022.

Problem 1. Let k be an algebraically closed field, let V be a finite-dimensional k-vector space, and let $T\colon V \to V$ be a k-linear map. Consider the (commutative) subring R of the non-commutative ring $\mathrm{End}_k(V)$ generated by T, and let $|X|$ be the Zariski space of R.

(1) Show that $|X|$ is canonically homeomorphic to the spectrum of T, that is, the set of eigenvalues of T with the discrete topology.

We next consider the sheaf $\widetilde{V}$ on $|X|$ associated with V, considered as an R-module by restriction of scalars along the inclusion $R \to \mathrm{End}_k(V)$.

(2) Let λ be an eigenvalue of $T\colon V \to V$, and let $x \in |X|$ be the point corresponding to λ under the homeomorphism in (1). Show that the restriction map

$$V = \widetilde{V}(|X|) \longrightarrow \widetilde{V}(\{x\})$$

induces an isomorphism from the generalized λ-eigenspace

$$V_\lambda = \{\boldsymbol{v} \in V \mid (T - \lambda\,\mathrm{id})^m(\boldsymbol{v}) = 0 \text{ for some } m > 0\} \subset V$$

onto $\widetilde{V}(\{x\})$.