Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, May 24, 2022.

Being a right adjoint, the functor

$$\mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}} \xrightarrow{\mathrm{Spec}} \mathsf{Schemes}$$

preserves all limits that exist in its domain. The purpose of this problem set is to show that it also preserves finite coproducts, or equivalently, that it preserves initial objects and binary coproducts.

Problem 1. (1) Show that $\mathrm{Spec}\colon \mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}} \to \mathsf{Schemes}$ preserves initial objects.

(2) Show that the category of schemes admits binary coproducts, that is, show that for every pair of schemes (Y_1, Y_2), there exists a diagram of schemes

$$Y_1 \xrightarrow{i_1} Y_1 \sqcup Y_2 \xleftarrow{i_2} Y_2$$

such that for every diagram of schemes of the form

$$Y_1 \xrightarrow{f_1} X \xleftarrow{f_2} Y_2,$$

there exists a unique map of schemes $f\colon Y_1 \sqcup Y_2 \to X$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[r, "i_1"] \arrow[dr, bend right, "f_1"'] & Y_1 \sqcup Y_2 \arrow[d, "f"] & Y_2 \arrow[l, "i_2"'] \arrow[dl, bend left, "f_2"] \\ & X & \end{tikzcd}$$

commutes.

(3) Show that the category of rings admits binary products, that is, show that for every pair of rings (R_1, R_2), there exists a diagram of rings

$$R_1 \xleftarrow{p_1} R_1 \times R_2 \xrightarrow{p_2} R_2$$

such that for every diagram of rings of the form

$$R_1 \xleftarrow{\varphi_1} S \xrightarrow{\varphi_2} R_2,$$

there exists a unique map of rings $\varphi\colon S \to R_1 \times R_2$ such that the diagram

$$\begin{tikzcd} R_1 & R_1 \times R_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] & R_2 \\ & S \arrow[ul, bend left, "\varphi_1"] \arrow[u, "\varphi"'] \arrow[ur, bend right, "\varphi_2"'] & \end{tikzcd}$$

commutes.

Clearly, a category $\mathcal{C}$ admits binary products if and only if the opposite category $\mathcal{C}^{\mathrm{op}}$ admits binary coproducts.

(4) Show that $\mathrm{Spec}\colon \mathrm{CAlg}(\mathsf{Ab})^{\mathrm{op}} \to \mathsf{Schemes}$ preserves binary coproducts in the sense that for every pair of rings (R_1, R_2), the canonical map of schemes

$$\mathrm{Spec}(R_1) \sqcup \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R_1 \times R_2)$$

is an isomorphism.