Algebra III/Introduction to Algebra III: Scheme Theory

Due: Please upload solutions to NUCT by Tuesday, June 14, 2022.

Problem 1. Let k be a field, and $\mathbb{P}^1_k$ the projective line over k. Thus, $\mathbb{P}^1_k$ has an open subscheme given by $\operatorname{Spec}(k[x])$ and another open subscheme given by $\operatorname{Spec}(k[y])$, and they intersect in

$$\operatorname{Spec}(k[x,y]/(xy-1)) = \operatorname{Spec}(k[x^{\pm 1}]) = \operatorname{Spec}(k[y^{\pm 1}]).$$

Prove the following statements.

(1) Every section $s\colon \operatorname{Spec}(k) \to \mathbb{P}^1_k$ of the structure map $f\colon \mathbb{P}^1_k \to \operatorname{Spec}(k)$ is a closed immersion.
(2) The closed subscheme defined by such a section $s\colon \operatorname{Spec}(k) \to \mathbb{P}^1_k$ is an effective Cartier divisor $D_s \subset \mathbb{P}^1_k$.
(3) The associated line bundle $\mathcal{O}(D_s)$ is isomorphic to $\mathcal{O}(1)$, independently of the choice of section $s\colon \operatorname{Spec}(k) \to \mathbb{P}^1_k$.

(The statements (1)–(3) are true for every ring k.)

Problem 2. Let k be a field. Show that every line bundle on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}(n)$ for some integer n.

[Hint: Start by showing that every line bundle on $\mathbb{A}^1_k$ is trivial.]