# IML HOMEWORK AND COMMENTS FOR WEEK 1

**Homework**: Exercises 2.1, 3, 5 from the book, plus the exercises below. The last exercise below is optional.

**Set theoretic notation.** I remind you that in lecture we introduced the following set-theoretic[1] notation: $0 = \emptyset$, and in general the natural number $n$ is defined to be

$$n = \{0, \dots, n-1\}.$$

We also defined

$$\omega = \{0, 1, 2, \dots, n, \dots\},$$

that is, $\omega$ is the set of non-negative integers.

We let $\mathbb{V}$ denote the (sometimes non-existent) totality of all sets, that is, $x \in \mathbb{V}$ if and only if $x$ is a set. We emphasize that this simply is practical short-hand notation, since $\mathbb{V}$ is not usually a set itself. (This follows from the well-known *Russel's paradox.*)

(Warning: The book defines $V$ to be the set of all our variable symbols. It should not be confused with $\mathbb{V}$. Most set theory books use $V$ to denote what we have called $\mathbb{V}$.)

To avoid confusion with the rounded parenthesis that show up in our language (and in terms and formulas), we will use angled brackets, $\langle\rangle$ to enclose finite ordered sequences in set theory. As always, we allow a slight *abus de langage* and identify $a$ and $\langle a \rangle$.

In this course, a *relation* is a subset $R$ of some (finite) cartesian product of sets. To be precise, an *$n$-ary* relation is a subset $R \subseteq A_1 \times \cdots \times A_n$ of some $n$-fold cartesian product. One often writes $R(x_1, \dots, x_n)$ rather than $\langle x_1, \dots, x_n \rangle \in R$; if this is the case then we say that $x_1, \dots, x_n$ are *related by $R$*.

Let $A$ be a set. A *sequence of length* $n \in \omega$ in $A$ is a function $S : n \to A$. We naturally identify $A^n$ with the set $\{s : n \to A : s \text{ is a function}\}$. We let

$$A^{<\omega} = \bigcup_{n \in \omega} A^n;$$

that is, $A^{<\omega}$ is the set the set of all finite sequences in $A$. For $s \in A^{\omega}$, we let $\ell\mathrm{h}(s) = \mathrm{dom}(s)$, the *length* of $s$.

For any function $f : B \to A$, where $B$ and $A$ are some sets, and a subset $B_0 \subseteq B$, we let $f \upharpoonright B_0$ denote the *restriction* of $f$ to $B_0$, that is, $f \upharpoonright B_0$ is the function with domain $B_0$ defined by $(f \upharpoonright B_0)(x) = f(x)$ for all $x \in B_0$.

**Exercise 1.** Some people[2] like to be really formal (and some have good reasons to). One such person defines a language as follows: A language is a pair $\langle f, p \rangle$ where

(1) $f : \omega \to \mathbb{V}$;
(2) $p : \omega \setminus \{0\} \to \mathbb{V}$;

[1]Until further notice, we will use sets naïvely in this course, that is, sets are treated just like they are in other math course you've taken.

[2]We will call such a person a *Really Formal Person*, or RFP.

(3) $(\forall m \in \omega)(\forall n \in \omega)(f(m) \cap p(n) = \emptyset \wedge (m \neq n \implies (f(m) \cap f(n) = \emptyset = p(m) \cap p(m))))$;
(4) for all $n \in \omega$, $f(n)$ and $p(n)$ are disjoint from $\{2n : n \in \omega\} \cup \{1, 3, 5, 7, 9, 11\}$;
(5) no function whose domain is in $\omega \setminus \{\emptyset\}$ belongs to any $f(n)$ or $p(n)$.

Explain. (Hint: The same person also defines that the variables $v_0, v_1, v_2, \ldots$ officially are the sets $0, 2, 4, \ldots$, that ( is 1, that ) is 3, that = is 5, that $\neg$ is 7, $\rightarrow$ is 9, and $\forall$ is 11.) □

**Terms.** Let $\mathcal{L}$ be language[3]. In lecture we defined

$$\mathrm{Term}_0^{\mathcal{L}} = \{a : a \text{ is a variable or constant symbol}\},$$

and then, recursively, that $t \in \mathrm{Term}_{n+1}^{\mathcal{L}}$ if and only if there is some $m$ and an $m$-place function symbol $F$ in $\mathcal{L}$, and terms $t_1, \ldots, t_m \in \bigcup_{0 \leq i \leq n} \mathrm{Term}_i^{\mathcal{L}}$ such that $t$ is $F(t_1 \ldots t_m)$. (We decided to use a start and end parenthesis in lecture that the book skips. It makes no real difference. Commas, however, are not part of our language, and so only informally may we write $F(t_1, \ldots, t_n)$ rather than $F(t_1 \ldots t_n)$.) We finally defined $\mathrm{Term}(\mathcal{L}) = \bigcup_{n \in \omega} \mathrm{Term}_n^{\mathcal{L}}$, and convinced ourselves that $t$ is a term (in $\mathcal{L}$) if and only if $t \in \mathrm{Term}(\mathcal{L})$. The *complexity* of a term $t$ is the least $n$ such that $t \in \mathrm{Term}_n^{\mathcal{L}}$.

**Exercise 2.** Show that $\mathrm{Term}_n^{\mathcal{L}} \subseteq \mathrm{Term}_{n+1}^{\mathcal{L}}$ for $n > 0$. What about $n = 0$? Is it possible for $\mathrm{Term}_n^{\mathcal{L}} = \emptyset$ when $n > 0$? What about $n = 0$? □

**Exercise 3.** Prove by induction on complexity that terms have an equal number of start parenthesis '(' and stop parenthesis ')'. (If you use the definition of terms from the book this exercise is *really* easy.) □

Given $g : m \to \mathbb{V}$ and $h : n \to \mathbb{V}$, where $m, n \in \omega$, we let $g^\frown h : m + n \to \mathbb{V}$ be given by

$$g^\frown h(k) = \begin{cases} g(k) & \text{if } k < m, \\ h(j) & \text{if } k = m + j \text{ and } j < n. \end{cases}$$

If $h$ is a finite sequence of finite sequences (that is, informally, $h \in (\mathbb{V}^{<\omega})^{<\omega}$), we define $\mathrm{concat}(h)$, the *concatenation* of $h$, recursively as:

$$\mathrm{concat}(h) = \begin{cases} \emptyset & \text{if } \ell\mathrm{h}(h) = 0, \\ (\mathrm{concat}(h \restriction n))^\frown h(n) & \text{if } \ell\mathrm{h}(h) = n + 1. \end{cases}$$

**Exercise 4.** Let $\mathcal{L} = (f, p)$ be a language in the really formal sense. A RFP defines terms as follows:

$$\mathrm{Term}_0^{\mathcal{L}} = \{\langle a \rangle : a \text{ is a variable or constant symbol}\},$$

and recursively, $\mathrm{Term}_{n+1}^{\mathcal{L}}$ is defined to be the set of all $\mathrm{concat}(h)$ such that for some $k \in \omega \setminus \{0\}$,

(a) $h : k + 3 \to \mathbb{V}$;
(b) $h(0) \in \{\langle a \rangle : a \in f(k)\}$;
(c) $h(1) = \langle 1 \rangle$;
(d) $h(k + 2) = \langle 3 \rangle$
(e) $(\forall j < k) h(2 + j) \in \bigcup_{m \leq n} \mathrm{Term}_m^{\mathcal{L}}$.

[3]Recall that in lecture we decided to include equality, =, as a logical symbol, whereas the book treats it as an optional part of the language. This will not make a big difference, but it slightly alters the structure of some definitions.

Then define a term of $\mathcal{L}$ to be any element of $\mathrm{Term}(\mathcal{L}) = \bigcup_{n\in\omega} \mathrm{Term}_n^{\mathcal{L}}$. Explain how this really formal definition relates to our usual definition of a term. □

In lecture, we defined $\mathrm{Formula}_0^{\mathcal{L}}$, the set of *atomic formulas*, to be the set of $\varphi$ such that $\varphi$ is either $t_1 = t_2$, where $t_1, t_2 \in \mathrm{Term}(\mathcal{L})$, or $\varphi$ is $R(t_1 \dots t_k)$ where $R$ is a $k$-place relation symbol, and $t_1, \dots, t_k \in \mathrm{Term}(\mathcal{L})$. We then defined, by recursion on $n \in \omega$, that $\mathrm{Formula}_{n+1}^{\mathcal{L}}$ consists of the $\varphi$ such that one of the following three holds:

(A) $\varphi$ is $\neg\psi$ for some $\psi \in \bigcup_{m\leq n} \mathrm{Formula}_m^{\mathcal{L}}$;
(B) $\varphi$ is $(\psi \to \theta)$, where $\psi, \theta \in \bigcup_{m\leq n} \mathrm{Formula}_m^{\mathcal{L}}$;
(C) $\varphi$ is $(\forall v_i)\psi$ for some $v_i \in V$ and $\psi \in \bigcup_{m\leq n} \mathrm{Formula}_m^{\mathcal{L}}$.

We let $\mathrm{Formula}(\mathcal{L}) = \bigcup_{n\in\omega} \mathrm{Formula}_n^{\mathcal{L}}$. In lecture we convinced ourselves that $\varphi$ is a formula in $\mathcal{L}$ if and only if $\varphi \in \mathrm{Formula}(\mathcal{L})$. (Note that the use of parenthesis in the lectures differ slightly from that of the book. This is a purely cosmetic difference of no mathematical consequence.) The *complexity* of a formula $\varphi$ is the least $n$ such that $\varphi \in \mathrm{Formula}_n^{\mathcal{L}}$.

**Exercise 5.** Show that $\mathrm{Formula}_n^{\mathcal{L}} \subseteq \mathrm{Formula}_{n+1}^{\mathcal{L}}$ for $n > 0$. What happens with $n = 0$? □

**Exercise 6.** Prove the *induction principle for formulas*: If $\Phi \subseteq \mathrm{Formula}(\mathcal{L})$ is a set of formulas such that $\mathrm{Formula}_0^{\mathcal{L}} \subseteq \Phi$ and it holds that

(A) if $\varphi \in \Phi$ then $\neg\varphi \in \Phi$;
(B) if $\varphi, \psi \in \Phi$ then $(\varphi \to \psi) \in \Phi$;
(C) if $\varphi \in \Phi$ then for any $v_i \in V$, $(\forall v_i)\varphi \in \Phi$;

then $\Phi = \mathrm{Formula}(\mathcal{L})$. (Hint: Use induction on complexity.) □

**Exercise 7.** Use the induction principle for formulas to show that in any formula $\varphi$ there is an equal number of start parenthesis and end parenthesis. □

**Exercise 8*.** Give a definition of formulas and of $\mathrm{Formula}_n^{\mathcal{L}}$ worthy of a RFP. □