VALUATIONS

Fix a language $\mathcal{L}$. To avoid an extensive rear guard campaign into Chapter 1 of the book, and the tedium of truth-tables, we decided to treat tautologies and tautological implication as follows:

Recall that a formula is called *prime* if it is either atomic or of the form $(\forall x)\varphi$, for some variable x and formula φ. Let $\mathrm{PFormula}_0^{\mathcal{L}}$ denote the set of prime formulas in the language $\mathcal{L}$. Then we define recursively that $\mathrm{PFormula}_{n+1}^{\mathcal{L}}$ consists of all the formulas either of the form $\neg\psi$ or of the form $(\psi \to \varphi)$, where $\psi, \varphi \in \mathrm{PFormula}_i^{\mathcal{L}}$ for some $i \leq n$.

Exercise 1. Show that $\bigcup_{n\geq 0} \mathrm{PFormula}_n^{\mathcal{L}} = \mathrm{Formula}(\mathcal{L})$.

Recall that a *valuation* is a function $v : \mathrm{PFormula}_0^{\mathcal{L}} \to \{0, 1\}$.

Exercise 2. Show that given a valuation v, there is a unique function $v^* : \mathrm{Formula}(\mathcal{L}) \to \{0, 1\}$ such that $v^* \restriction \mathrm{PFormula}_0^{\mathcal{L}} = v$, $v^*(\neg\varphi) = 1 - v^*(\varphi)$ and $v^*((\psi \to \varphi)) = \max\{1 - v^*(\psi), v^*(\varphi)\}$.

Let us recall the two important notions we defined in class using valuations. We emphasize that our definitions of these notions agree with the identically named notions introduced in the book. However, the virtue of our definitions is that they do not refer to the sentential logic of Chapter 1.

Definition 0.1.

(1) A *tautology* is a formula φ such that $v^*(\varphi) = 1$ for any valuation v.

(2) Let Γ be a set of formulas and φ a formula. We say that Γ *tautologically implies* φ, or *truth-functionally implies*[1] φ, if whenever v is a valuation such that $v^*(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v^*(\varphi) = 1$. If this is the case, we write $\Gamma \models_t \varphi$.

[1]Probably the term *truth-functionally* arose because *truth-table-y* is too silly.