IML COURSE CREDIT HOMEWORK ASSIGNMENT

The exercises below must be handed in to the T.A. on May 29, 2012, in section. There are two parts, which will be graded individually. You must receive a passing grade on both part 1 and 2 in order to take the final exam.

1. Preliminary discussion

The purpose of these homework exercises is to introduce and develop the notions of *ultraproduct* and *ultrapower*. This relies on the set-theoretic notions of filter and ultrafilter, which you may have seen before in other courses. In case you haven't, we start by giving the definitions and a bit of discussion.

Let I be a non-empty set, and let $\mathcal{P}(I)$ denote the power set of I. A *filter* on I is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ which satisfy

(1) $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$;
(2) if $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$;
(3) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

An *ultrafilter* is a filter which additionally satisfies

(4) For all $A \subseteq I$, either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$.

The basic idea is that the sets in a filter are declared to somehow be "large". In general, a filter may fail to make a decision about whether a given $A \subseteq I$ is large, or its complement $I \setminus A$ is large. An ultrafilter is a filter that does not have this shortcoming: It always decides whether a set or its complement is considered large. (Note that (3) and (1) imply that it never can be the case that both A and $I \setminus A$ are in the filter.)

An example of a filter is $\text{cofin}(I) = \{A \subseteq I : I \setminus A \text{ is finite}\}$ (when I is not itself finite.) This is *not* an ultrafilter (why?). On the other hand, if we fix $i_0 \in I$ and define $\mathcal{F}_{i_0} = \{A \in \mathcal{P}(I) : i_0 \in A\}$, then this *is* an ultrafilter (check this). An ultrafilter like $\mathcal{F}_{i_0}$ is called a *principal* ultrafilter, and they are not very interesting. Using the Axiom of Choice, one can show that there are non-principal ultrafilters. In fact, one has:

Theorem 1. *Let I be a non-empty set. Then any filter on I can be extended to an ultrafilter on I. More precisely this means: For any filter $\mathcal{F}$ on I there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$, $\mathcal{U} \subseteq \mathcal{P}(I)$.*

If you are familiar with Zorn's lemma, you can probably prove this easily. If not, take it on faith. Note that if we apply this theorem to $\text{cofin}(I)$, where I is infinite, then it follows that: *On any infinite set there is a non-principal ultrafilter.* This, and the definition of an ultrafilter, is all the information you need to solve the problems below (that and the material we've covered in the course.) Before moving on to the real homework exercises, you may want to do the following warm-up, though you should *not* hand in your solution to the warm-up exercise.

Warm-up exercise. Show that if $\mathcal{A} \subseteq \mathcal{P}(I)$ is a collection of sets not containing $\emptyset$ and which is closed under finite intersections, then there is a filter $\mathcal{F} \supseteq \mathcal{A}$, $\mathcal{F} \subseteq \mathcal{P}(I)$.

Part 1

Let $\mathcal{L}$ be a language[1]. Fix a non-empty set I and let $\mathfrak{A}_i$, $i \in I$, be models of $\mathcal{L}$ with universes A_i. Fix also an ultrafilter $\mathcal{U}$ on I. Consider the cartesian product

$$\prod_{i\in I} A_i = \{f : I \to \mathbb{V} : (\forall i \in I) f(i) \in A_i\}.$$

Define for $f, g \in \prod_{i\in I} A_i$,

$$f \sim_{\mathcal{U}} g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

Exercise 1. Prove that $\sim_{\mathcal{U}}$ is an equivalence relation.

We let $[\![f]\!]_{\mathcal{U}} = \{g : g \sim_{\mathcal{U}} f\}$ denote the equivalence class of f, and let

$$A = (\prod_{i\in I} A_i)/\sim_{\mathcal{U}} = \{[\![f]\!]_{\mathcal{U}} : f \in \prod_{i\in I} A_i\}$$

be the set of all equivalence classes. We define a model $\mathfrak{A}$ with universe A by

(1) for a constant symbol c in $\mathcal{L}$, define $c^{\mathfrak{A}} = [\![f]\!]_{\mathcal{U}}$, where $f(i) = c^{\mathfrak{A}_i}$ for all $i \in I$;
(2) for a k-place function symbol F in $\mathcal{L}$, define $F^{\mathfrak{A}}([\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_k]\!]_{\mathcal{U}}) = [\![f]\!]_{\mathcal{U}}$ where $f(i) = F^{\mathfrak{A}_i}(f_1(i), \dots, f_k(i))$;
(3) for a k-place relation symbol R in $\mathcal{L}$, define

$$R^{\mathfrak{A}}([\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_k]\!]_{\mathcal{U}}) \iff \{i \in I : R^{\mathfrak{A}_i}(f_1(i), \dots, f_k(i))\} \in \mathcal{U}.$$

Exercise 2. Show that $\mathfrak{A}$ is well-defined. That is, the definitions of $F^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ above do not depend of the choice of representatives for the classes $[\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_k]\!]_{\mathcal{U}}$.

The model $\mathfrak{A}$ is called the *ultraproduct* of the family $(\mathfrak{A}_i)_{i\in I}$ with respect to $\mathcal{U}$, and it is denoted $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U}$.

Exercise 3. Show that if we take $\mathcal{U}$ to be the principal ultrafilter $\mathcal{F}_{i_0}$, for some $i_0 \in I$, then the ultraproduct $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_{i_0}$.

Exercise 4. Prove *Łos' Theorem*: For any formula $\varphi(x_1, \dots, x_n)$ in $\mathcal{L}$ with free variables shown, and any $f_1, \dots, f_n \in \prod_{i\in I} A_i$, it holds that

$$\mathfrak{A} \models \varphi[[\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_n]\!]_{\mathcal{U}}] \iff \{i \in I : \mathfrak{A}_i \models \varphi[f_1(i), \dots, f_n(i)]\} \in \mathcal{U}.$$

(*Hint*: Use induction on φ.)

(The name Łos is Polish, and is pronounced something like "Wosh".)

Exercise 5. Use ultraproducts to prove the *Compactness Theorem*: If Σ is a set of sentences in $\mathcal{L}$ such that every finite subset $\Gamma \subseteq \Sigma$ is satisfiable, then Σ is satisfiable. (*Hint*: Let I be the set of finite subsets of Σ; Use Theorem 1 above on an appropriately chosen filter.)

When $\mathfrak{A}_i = \mathfrak{B}$ for all $i \in I$ for some fixed model $\mathfrak{B}$, then we call the ultraproduct $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U}$ the *ultrapower* of $\mathfrak{B}$ (with respect to I and $\mathcal{U}$), denoted $\mathfrak{B}^I/\mathcal{U}$.

[1]As always, we will assume that equality = is in our language.

Exercise 6. For any ultrapower $\mathfrak{B}^I/\mathcal{U}$, find a natural (!) elementary embedding[2] $j : \mathfrak{B} \to \mathfrak{B}^I/\mathcal{U}$. Prove that your embedding is elementary.

Part 2

Consider now the *language of ordered rings*, $\mathcal{L} = \{c_0, c_1, a, m, L\}$, where c_0 and c_1 are constant symbols, a and m are binary function symbols, and L is a binary relation symbol (we include equality as a logical symbol.) A natural model of $\mathcal{L}$ is the ordered field of real numbers, $\mathfrak{R} = \langle \mathbb{R}; 0, 1, +, \cdot, < \rangle$, with the natural interpretations. Recall that $\mathfrak{R}$ satisfies the "axioms of ordered fields", that is, the universal closure of the following formulas:

(A.1) $a(a(v_0v_1)v_2) = a(v_0a(v_1v_2))$ (associativity of addition);
(A.2) $a(v_0v_1) = a(v_1v_0)$ (commutativity of addition);
(A.3) $a(v_0c_0) = v_0$ (additive identity);
(A.4) $(\exists v_1)a(v_0v_1) = c_0$ (existence of additive inverses);

(M.1) $m(m(v_0v_1)v_2) = m(v_0m(v_1v_2))$ (associativity of multiplication);
(M.2) $m(v_0v_1) = m(v_1v_0)$ (commutativity of multiplication);
(M.3) $m(v_0c_1) = v_0$ (multiplicative identity);
(M.4) $v_0 = c_0 \vee (\exists v_1)m(v_0v_1) = c_1$ (existence of multiplicative inverses);

(D.1) $m(v_0a(v_1v_2)) = a(m(v_0v_1)m(v_0v_2))$ (distributive law);

(O.1) $\neg L(v_0v_0)$;
(O.2) $v_0 = v_1 \vee L(v_0v_1) \vee L(v_1v_0)$ (trichotomy);
(O.3) $L(v_0v_1) \wedge L(v_1v_2) \to L(v_0v_2)$ (transitivity);
(O.4) $L(c_0v_0) \wedge L(c_0v_1) \to L(c_0a(v_0v_1))$;
(O.5) $L(c_0v_0) \wedge L(c_0v_1) \to L(c_0m(v_0v_1))$;
(O.6) $L(v_0v_1) \to L(a(v_0v_2)a(v_1v_2))$.

Any model of $\mathcal{L}$ which satisfies (the universal closures of) the above axioms is called an *ordered field.*

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.

Exercise 7. Is $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$ an ordered field? Prove your answer.

Consider the natural elementary embedding $j : \mathfrak{R} \to \mathfrak{R}^{\mathbb{N}}/\mathcal{U}$.

Exercise 8. Show that there is an element $[\![f]\!]_{\mathcal{U}} \in \mathfrak{R}^{\mathbb{N}}/\mathcal{U} \setminus \{c_0^{\mathfrak{R}^{\mathbb{N}}/\mathcal{U}}\}$ such that $L^{\mathfrak{R}^{\mathbb{N}}/\mathcal{U}}(c_0^{\mathfrak{R}^{\mathbb{N}}/\mathcal{U}}, [\![f]\!]_{\mathcal{U}})$ and

$$L^{\mathfrak{R}^{\mathbb{N}}/\mathcal{U}}([\![f]\!]_{\mathcal{U}}, j(x))$$

for all $x \in \mathfrak{R}$ with $x > 0$.

An element as in the previous exercise is an example of an *infinitesimal* element in the ultrapower $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$.

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[2]Recall that an elementary embedding $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding (i.e., an injective homorphism, where homomorphism is understood in the strong sense of our textbook) with the property that for any formula $\phi(x_1, \ldots, x_n)$ with free variables shown we have: For all $a_1, \ldots, a_n \in A$, $\mathfrak{A} \models \phi[a_1, \ldots, a_n] \iff \mathfrak{B} \models \phi[f(a_1), \ldots, f(a_n)]$.

Exercise 9. Show that the set $j((0,+\infty))$ is bounded below, but has no greatest lower bound. Conclude that $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$ is not complete in the usual sense of real analysis. Is $j((0,+\infty))$ bounded above? Prove your answer.

Remark 1: The theory of or $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$ was developed extensively by Abraham Robinson in the 1960s under the moniker "non-standard analysis". The elements in $(\mathfrak{R}^{\mathbb{N}}/\mathcal{U}) \setminus j(\mathbb{R})$ are called *non-standard reals*. One of the main points of studying $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$ is that in $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$ infinitesimals exist, while in classical real analysis infinitesimals formally don't exist, but appear only informally as "convenient notation" in the form of symbols like dx, dy, etc. By setting up the laws of reasoning with infinitesimals in $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$, one can then formalize many of the hitherto *informal* proofs in analysis that use infinitesimals and prove actual theorems of (non-standard) analysis in $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$. Łos theorem then provides a way of porting back to classical analysis (i.e., analysis in $\mathbb{R}$) these non-standard theorems. In some sense, Robinson's non-standard analysis also explains why so many informal proofs with infinitesimals actually give correct results: While they are formally incorrect in $\mathbb{R}$, they are perfectly legitimate proofs in $\mathfrak{R}^{\mathbb{N}}/\mathcal{U}$.

In Ch. 2.8 in the textbook there is a discussion of non-standard analysis, albeit without using ultrapowers.

Remark 2: There is also something called the *metric* ultraproduct, which appears in many places in abstract analysis (e.g., in Banach space theory and in operator algebras.) It is a slightly different, but very similar, construction, with many of the same basic properties as the ultraproducts we have introduced above.

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