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Non-Symmetric Translation Invariant Dirichlet Forms

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Introduction

In order to treat certain "non-symmetric" potential theories, Itô [4] and Bliedtner [1] have generalized Beurling and Deny's theory of Dirichlet spaces by replacing the inner product in the Dirichlet space with a bilinear form defined on a real, regular functional space. This bilinear form, called a Dirichlet form, is supposed to be continuous and coercive, and furthermore to satisfy a "contraction"-condition.

When the underlying space is a locally compact abelian group, there is a complete characterization of translation invariant Dirichlet spaces in terms of real, negative definite functions on the dual group. The main purpose of the paper is to obtain an analogous characterization (Theorem 3.7) of translation invariant Dirichlet forms, in terms of complex, negative definite functions on the dual group.

The contents of the paper may be summarized in the following way: In § 1 we study positive closed sesquilinear forms on an abstract complex Hilbert space, and using the Lax-Milgram theorem, it is shown, that with every such form is associated a resolvent. The theory is specialized in § 2 to sesquilinear forms on the Hilbert space $L^2(X, \xi)$, where some relations with normal contractions are discussed. In particular an example of a positive closed form β is given, such that the unit contraction operates with respect to β, but not with respect to the form β^*, which is adjoint to β. However, in the case of a translation invariant, positive closed form β on $L^2(G)$, where G is a locally compact abelian group, it is shown in § 3, that the unit contraction operates with respect to β if and only if it operates with respect to β^*, and we finish by establishing the above mentioned characterization of translation invariant Dirichlet forms.

§ 1. Positive Closed Sesquilinear Forms

Let E be a complex Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Furthermore let V be a dense subspace and β: $V\times V\to\mathbb{C}$ a sesquilinear form. The adjoint form β^* is defined by $\beta^*(x,y)=\overline{\beta(y,x)}$, and the hermitian part of β by $\alpha=\frac{1}{2}(\beta+\beta^*)$.

1.1. *Definition.* A sesquilinear form β, or more precisely (β, V), is called a *positive closed form* on E, if the following three conditions are fulfilled:

i) $\operatorname{Re}\beta(x,x)\geqq 0$ for all $x\in V$.

ii) There is a constant $C>0$ such that

$$|\beta(x,y)|\leqq C\,\|x\|_\alpha\|y\|_\alpha \quad \text{for all } x,y\in V,$$

where $\|x\|_\alpha=(\alpha(x,x))^{\frac{1}{2}}$ is the Hilbert seminorm associated with the positive hermitian form α.

iii) The subspace V is complete under the norm $\|\cdot\|_V$, given by $\|x\|_V^2=\|x\|^2+\|x\|_\alpha^2=\|x\|^2+\operatorname{Re}\beta(x,x)$, and with associated scalar product

$$(x,y)_V=(x,y)+\alpha(x,y).$$

1.2. *Remark.* If E is a real Hilbert space, V a dense subspace of E and $\beta\colon V\times V\to\mathbb{R}$ a real bilinear form satisfying the three conditions of 1.1, we will also say that β is a (real) positive closed form on E.

Let $\mathscr{H}$ be a real, regular functional space on (X,ξ) and β a continuous, coercive bilinear form on $\mathscr{H}$ (cf. Bliedtner [1]). The restriction of β to $(\mathscr{H}\cap L^2(X,\xi))\times(\mathscr{H}\cap L^2(X,\xi))$ is then a positive closed form on the real Hilbert space $L^2(X,\xi)$.

1.3. Let (β, V) be a positive closed form on E. Then there exists a continuous linear mapping P from the Hilbert space E to the Hilbert space $(V,\|\cdot\|_V)$ such that

$$(x,y)=(Px,y)_V \quad \text{for all } x\in E,\ y\in V.$$

This follows from the estimate

$$|(x,y)|\leqq\|x\|\cdot\|y\|\leqq\|x\|\cdot\|y\|_V,$$

which also shows that $\|Px\|_V\leqq\|x\|$.

For later use we note that $P(E)$ is dense in $(V,\|\cdot\|_V)$.

1.4. *Definition.* The *generator* (A, D_A) of a positive closed form β is the operator in E with domain

$$D_A=\{x\in V\,|\,\exists\, x^*\in E\ \forall\, y\in V\colon \beta(x,y)=-(x^*,y)\}$$

and defined by

$$Ax=x^* \quad \text{for } x\in D_A.$$

We shall now see that (A, D_A) is the infinitesimal generator of a strongly continuous contraction semigroup $(P_t)_{t>0}$ of operators in E. By the Hille-Yosida theorem (cf. Meyer [5] and Yosida [6]) this is contained in the following theorem.

1.5. Theorem. *Let β be a positive closed form on E. For each $\lambda>0$ there exists a uniquely determined bounded operator R_λ in E, which maps E into V such that*

$$\beta(R_\lambda x, y)=(x-\lambda R_\lambda x, y) \quad \textit{for all } x\in E,\ y\in V. \tag{*}$$

The family $(R_\lambda)_{\lambda>0}$ is a strongly continuous contraction resolvent satisfying $(\lambda I-A)^{-1}=R_\lambda$ for all $\lambda>0$.

Proof. For $\lambda>0$ we define a sesquilinear form β_λ on V by

$$\beta_\lambda(x, y)=\beta(x, y)+\lambda(x, y),$$

and the inequalities

$$\begin{aligned}|\beta_\lambda(x, y)| &\leqq C\,\|x\|_\alpha\,\|y\|_\alpha+\lambda\,\|x\|\,\|y\| \\ &\leqq \max(C, \lambda)\,\|x\|_V\,\|y\|_V\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\beta_\lambda(x, x) &= \operatorname{Re}\beta(x, x)+\lambda\,\|x\|^2 \\ &\geqq \min(1, \lambda)\,\|x\|_V^2\end{aligned}$$

show that β_λ is a continuous coercive sesquilinear form on the Hilbert space $(V, \|\cdot\|_V)$.

By the Lax-Milgram theorem (cf. Yosida [6], p. 92) there is an isomorphism S_λ of $(V, \|\cdot\|_V)$ such that

$$(x, y)=(Px, y)_V=\beta_\lambda(S_\lambda Px, y) \quad \text{for all } x\in E,\ y\in V.$$

This shows that the operator $R_\lambda=S_\lambda P$ satisfies

$$\beta_\lambda(R_\lambda x, y)=(x, y) \qquad \text{for all } x\in E,\ y\in V \tag{**}$$

and hence

$$\beta(R_\lambda x, y)=(x-\lambda R_\lambda x, y) \quad \text{for all } x\in E,\ y\in V.$$

Obviously R_λ is uniquely determined by (**).

The following calculation shows that λR_λ is a contraction in E.

$$\begin{aligned}\lambda\,\|R_\lambda x\|^2 &= -(x-\lambda R_\lambda x, R_\lambda x)+(x, R_\lambda x) \\ &= -\beta(R_\lambda x, R_\lambda x)+(x, R_\lambda x) \\ &= -\operatorname{Re}\beta(R_\lambda x, R_\lambda x)+\operatorname{Re}(x, R_\lambda x) \\ &\leqq \operatorname{Re}(x, R_\lambda x) \\ &\leqq \|x\|\,\|R_\lambda x\|.\end{aligned}$$

From (*) it follows by the definition of (A, D_A) that $R_\lambda(E)\subseteq D_A$ and that $(\lambda I-A)R_\lambda x=x$ for all $x\in E$.

Furthermore, by (**) and the definition of (A, D_A) we get

$$\beta_\lambda(x, y) = ((\lambda I - A)x, y) = \beta_\lambda(R_\lambda(\lambda I - A)x, y) \quad \text{for all } x \in D_A, \; y \in V,$$

and on account of the coerciveness of β_λ on $(V, \|\cdot\|_V)$ it follows that

$$x = R_\lambda(\lambda I - A)x \quad \text{for all } x \in D_A.$$

We have now shown that $R_\lambda = (\lambda I - A)^{-1}$, $\lambda > 0$ is a contraction resolvent, so in order to prove the strong continuity of $(R_\lambda)_{\lambda>0}$ on E, it suffices to verify that D_A is dense in E (cf. Meyer [5], p. 260), and this follows from Lemma 1.6 below, since V is dense in E. □

1.6. **Lemma.** *The subspace D_A is dense in $(V, \|\cdot\|_V)$ and $\lim_{\lambda\to\infty} \lambda R_\lambda = I$ strongly in $(V, \|\cdot\|_V)$.*

Proof. We first show that the operators $(\lambda R_\lambda)_{\lambda>0}$ are uniformly bounded in $(V, \|\cdot\|_V)$. By (*) we have

$$\|x - \lambda R_\lambda x\|^2 = \beta(R_\lambda x, x - \lambda R_\lambda x) \geqq 0 \quad \text{for all } x \in V,$$

and this implies

$$\begin{aligned} \lambda \|R_\lambda x\|_\alpha^2 &= \lambda \operatorname{Re} \beta(R_\lambda x, R_\lambda x) \\ &\leqq \operatorname{Re} \beta(R_\lambda x, x) \\ &\leqq C \|R_\lambda x\|_\alpha \|x\|_\alpha, \end{aligned}$$

hence

$$\|\lambda R_\lambda x\|_\alpha \leqq C \|x\|_\alpha.$$

Since also $\|\lambda R_\lambda x\| \leqq \|x\|$ we find

$$\|\lambda R_\lambda x\|_V \leqq \max(C^2, 1)^{\frac{1}{2}} \|x\|_V \quad \text{for all } x \in V.$$

By Lax-Milgram's theorem S_λ is an isomorphism of the Hilbert space V onto itself, and this implies that $D_A = R_\lambda(E) = S_\lambda(P(E))$ is dense in $(V, \|\cdot\|_V)$ because $P(E)$ is so.

On account of the uniform boundedness of λR_λ on $(V, \|\cdot\|_V)$ the second part of the lemma is proved when we have seen that $\lim_{\lambda\to\infty} \lambda R_\lambda x = x$ in $(V, \|\cdot\|_V)$ for all x in the dense subspace D_A. For $x \in D_A$ we find

$$\begin{aligned} \|x - \lambda R_\lambda x\|_V^2 &= \|x - \lambda R_\lambda x\|^2 + \operatorname{Re} \beta(x - \lambda R_\lambda x, x - \lambda R_\lambda x) \\ &= \|x - \lambda R_\lambda x\|^2 + \operatorname{Re} \beta(x, x - \lambda R_\lambda x) - \lambda \|x - \lambda R_\lambda x\|^2 \\ &\leqq \|x - \lambda R_\lambda x\|^2 - \operatorname{Re}(Ax, x - \lambda R_\lambda x) \\ &\leqq \|x - \lambda R_\lambda x\|^2 + \|Ax\| \, \|x - \lambda R_\lambda x\|, \end{aligned}$$

which tends to zero as $\lambda \to \infty$, because $(R_\lambda)_{\lambda>0}$ is strongly continuous on E. □

1.7. *Remark.* To every positive closed form β we have associated a strongly continuous contraction semigroup $(P_t)_{t>0}$ of operators in E, with resolvent $(R_\lambda)_{\lambda>0}$ and infinitesimal generator (A, D_A).

The corresponding objects for the adjoint form β^* are the adjoint operators $(P_t^*)_{t>0}$, $(R_\lambda^*)_{\lambda>0}$, (A^*, D_{A^*}). To see this, let $\hat{R}_\lambda$ denote the resolvent for the form β^*. We then have

$$\beta^*(\hat{R}_\lambda y, x) = (y - \lambda \hat{R}_\lambda y, x) \quad \text{for all } y \in E,\ x \in V,$$

and in particular

$$\beta^*(\hat{R}_\lambda y, R_\lambda x) = (y - \lambda \hat{R}_\lambda y, R_\lambda x) \quad \text{for all } x, y \in E.$$

Likewise we get from $(*)$ that

$$\beta(R_\lambda x, \hat{R}_\lambda y) = (x - \lambda R_\lambda x, \hat{R}_\lambda y) \quad \text{for all } x, y \in E,$$

and it follows that

$$(x - \lambda R_\lambda x, \hat{R}_\lambda y) = (R_\lambda x, y - \lambda \hat{R}_\lambda y) \quad \text{for all } x, y \in E,$$

and then

$$(R_\lambda x, y) = (x, \hat{R}_\lambda y) \quad \text{for all } x, y \in E,$$

and the proof is easily accomplished.

§ 2. Dirichlet Forms

Let X be a locally compact space endowed with a positive Radon measure ξ. We shall consider positive closed forms (β, V) on the complex Hilbert space $L^2(X, \xi)$.

2.1. *Definition.* We will say that a normal contraction T of the complex plane *operates on V with respect to β*, if the following holds:

For every $f \in V$ we have $Tf \in V$ and

$$\operatorname{Re} \beta(f + Tf, f - Tf) \geqq 0.$$

Here Tf is defined as the composite function $T \circ f$ (cf. Deny [2] and Bliedtner [1]).

The *modulus contraction* $T(z) = |z|$ operates on V with respect to β if and only if it operates on V with respect to β^*, as is easily seen. An analogous result is however false for the *unit contraction* T_I (T_I is the projection of $\mathbb{C}$ onto the unit interval $I = [0, 1]$) as the following example shows. This gives an answer to Itô [4].

2.2. *Example.* Let X be a two point set $X = \{1, 2\}$ and $\xi = \varepsilon_1 + \varepsilon_2$. The space $L^2(X, \xi)$ can thus be identified with $\mathbb{C}^2$. The sesquilinear forms in question are given as 2×2-matrices: To the matrix $(\beta_{ij})_{i,j=1,2}$ we associate

$$\beta(x,y)=\sum_{i,j=1}^{2}\beta_{ij}x_i\bar{y}_j.$$

We consider the form β defined by $(\beta_{ij})=\begin{pmatrix}2 & 0\\ -1 & \frac{1}{2}\end{pmatrix}$. It is easily seen that $\operatorname{Re}\beta(x,x)\geqq\frac{1}{4}\|x\|^2$ and it follows that β is a positive closed form. It can be seen directly, that T_I operates with respect to β, by going through the nine possibilities for $f=(f_1,f_2)$

$$f_i<0,\quad 0\leqq f_i\leqq 1,\quad 1<f_i,\qquad i=1,2,$$

in which cases $T_I f$ can be given explicitly. (It is enough to look at real f's.)

On the other hand T_I does not operate with respect to β^*, for if $f=(1,\frac{3}{2})$ we have $T_I f=(1,1)$, and

$$\beta^*(f+T_I f, f-T_I f)=-\tfrac{3}{8}.$$

2.3. *Definition.* A positive closed form (β, V) on $L^2(X,\xi)$ is called a *Dirichlet form*, if the unit contraction T_I operates on V with respect to β as well as β^*, i.e.

For all $f\in V$ we have $T_I f\in V$ and

$$\operatorname{Re}\beta(f+T_I f, f-T_I f)\geqq 0$$
$$\operatorname{Re}\beta(f-T_I f, f+T_I f)\geqq 0.$$

The hermitian part α of a Dirichlet form β is easily seen to be a Dirichlet form in the sense of Deny [2], i.e.

For all $f\in V$ we have $T_I f\in V$ and

$$\alpha(T_I f, T_I f)\leqq\alpha(f,f).$$

§ 3. Translation Invariant Dirichlet Forms

Let G be a locally compact abelian group with Haar measure dx.

3.1. A *vaguely continuous convolution semigroup* on G is a semigroup of positive measures $(\mu_t)_{t>0}$ with total mass $\int d\mu_t\leqq 1$, and such that $\lim_{t\to 0}\mu_t=\varepsilon_0$ vaguely. Vaguely continuous convolution semigroups $(\mu_t)_{t>0}$ are in one-to-one correspondence with continuous, negative definite functions ψ defined on the dual group Γ, by the Fourier transformation (cf. Deny [2]),

$$\hat{\mu}_t(\gamma)=e^{-t\psi(\gamma)}\qquad\text{for all } t>0,\ \gamma\in\Gamma.$$

3.2. A positive measure μ on G with total mass $\int d\mu\leqq 1$ determines a bounded operator M on $L^2(G)$ by convolution, $Mf=\mu * f$ for $f\in L^2$.

The operator M commutes with the translations on G and is sub-markovian, and conversely any bounded operator on $L^2(G)$ with these properties is given as convolution with a positive measure μ on G, of total mass $\int d\mu \leqq 1$.

3.3. If $(\mu_t)_{t>0}$ is a vaguely continuous convolution semigroup on G, we get a strongly continuous contraction semigroup $(P_t)_{t>0}$ of operators on $L^2(G)$, defined by $P_t f = \mu_t * f$ for $f \in L^2(G)$.

We shall now prove a useful characterization of the infinitesimal generator (A, D_A) of the semigroup $(P_t)_{t>0}$, by using the Fourier-Plancherel transformation of $L^2(G)$ onto $L^2(\Gamma)$ (here $L^2(\Gamma)$ means $L^2(\Gamma, d\gamma)$, where $d\gamma$ is a Haar measure on Γ, normalized in such a way that the Plancherel theorem holds with respect to dx and $d\gamma$).

Proposition. *The domain D_A of A is given by*

$$D_A = \{f \in L^2(G) \mid \hat{f}\psi \in L^2(\Gamma)\},$$

and

$$\widehat{Af} = -\hat{f}\psi \qquad \text{for } f \in D_A.$$

Proof. If $f \in D_A$ we have

$$\lim_{t\to 0} \frac{1}{t}(P_t f - f) = Af \qquad \text{in } L^2(G)$$

and then

$$\lim_{t\to 0} \frac{1}{t}(e^{-t\psi} - 1)\hat{f} = \widehat{Af} \quad \text{in } L^2(\Gamma).$$

Since $\lim_{t\to 0} \frac{1}{t}(e^{-t\psi(\gamma)} - 1) = -\psi(\gamma)$ for all $\gamma \in \Gamma$, it follows that

$$\widehat{Af} = -\hat{f}\psi \qquad \text{in } L^2(\Gamma).$$

Next, suppose that $f \in L^2(G)$ and $\hat{f}\psi \in L^2(\Gamma)$. By the mean-value theorem we get

$$\left|\frac{1}{t}(e^{-t\psi} - 1)\right| \leqq |\psi| \qquad \text{for all } t > 0,$$

because $\operatorname{Re}\psi \geqq 0$, and then we have

$$\lim_{t\to 0} \frac{1}{t}(e^{-t\psi} - 1)\hat{f} = -\hat{f}\psi \qquad \text{in } L^2(\Gamma),$$

which shows that $f \in D_A$. □

3.4. *Definition.* A positive closed form (β, V) on $L^2(G)$ is called *translation invariant* if the following holds.

For all $f\in V$, $x\in G$ we have $f*\varepsilon_x\in V$ and

$$\beta(f*\varepsilon_x, g*\varepsilon_x)=\beta(f,g) \quad \text{for all } f,g\in V.$$

It is easy to see that (β, V) is translation invariant if and only if the associated resolvent operators (R_λ), $\lambda>0$ commute with the translations:

$$R_\lambda(f*\varepsilon_x)=(R_\lambda f)*\varepsilon_x \quad \text{for all } \lambda>0, f\in L^2(G),\ x\in G.$$

3.5. **Theorem.** *Let (β, V) be a translation invariant positive closed form on $L^2(G)$ and suppose that $T_I f\in V$ for all $f\in V$ (where T_I is the unit contraction, cf. 2.1). The following properties are equivalent.*

(i) *T_I operates with respect to β.*

(ii) *λR_λ is sub-markovian for all $\lambda>0$.*

(iii) *λR_λ^* is sub-markovian for all $\lambda>0$.*

(iv) *T_I operates with respect to β^*.*

Proof. (i) $\Rightarrow$ (ii). We prove that $\lambda R_{\lambda+\mu}$ is sub-markovian for all $\lambda, \mu>0$, and this implies (ii).

Let $f\in L^2(G)$, $0\leqq f\leqq 1$ loc.-p.p., and let $\lambda, \mu>0$. We put $g=\lambda R_{\lambda+\mu}f$. By hypothesis we have

$$\operatorname{Re}\beta(g+T_I g, g-T_I g)\geqq 0,$$

and it is evident that

$$\int_G |T_I g|^2\,dx\leqq \int_G |g|^2\,dx.$$

This implies that

$$\operatorname{Re}\beta_\mu(g, g-T_I g)\geqq -\operatorname{Re}\beta_\mu(T_I g, g-T_I g).$$

We have the following inequalities

$$\begin{aligned}\mu\,\|g-T_I g\|^2 &\leqq \operatorname{Re}\ \beta_\mu(g-T_I g, g-T_I g)\\ &\leqq 2\operatorname{Re}\beta_\mu(g, g-T_I g)\\ &=2\lambda\operatorname{Re}(f-g, g-T_I g)\\ &=-2\lambda\,\|f-g\|^2+2\lambda\operatorname{Re}(f-g, f-T_I g)\\ &\leqq -2\lambda\,\|f-g\|^2+2\lambda\,\|f-g\|\,\|f-T_I g\|.\end{aligned}$$

We now use that $f=T_I f$ so that

$$\|f-T_I g\|=\|T_I f-T_I g\|\leqq\|f-g\|,$$

which shows, that the last expression in the above inequalities is $\leqq 0$. It follows that $\mu\,\|g-T_I g\|^2=0$, and this implies that $0\leqq g\leqq 1$ loc.-p.p.

(ii) ⇒ (iii). Under the assumption (ii) there exists a positive measure ρ_λ with $\int \lambda\, d\rho_\lambda \leqq 1$ such that $R_\lambda f = \rho_\lambda * f$ (cf. 3.2). It follows that the adjoint operator R_λ^* is given by $R_\lambda^* f = \check{\rho}_\lambda * f$, where $\check{\rho}_\lambda$ is the reflection of ρ_λ in the neutral element in G. This shows that λR_λ^* is sub-markovian.

(ii) ⇒ (i). There exists a positive measure ρ_λ with $\int \lambda\, d\rho_\lambda \leqq 1$, such that $R_\lambda f = \rho_\lambda * f$ for $f \in L^2(G)$, $\lambda > 0$.

Let $f \in V$. It suffices to prove that

$$\operatorname{Re} \beta(T_I f, f - T_I f) \geqq 0,$$

because we then have

$$\operatorname{Re} \beta(f + T_I f, f - T_I f) = \operatorname{Re} \beta(f - T_I f, f - T_I f) + 2 \operatorname{Re} \beta(T_I f, f - T_I f) \geqq 0.$$

By Lemma 1.6 it follows that

$$\lim_{\lambda \to \infty} \beta(\lambda R_\lambda f, g) = \beta(f, g) \quad \text{for all } f, g \in V$$

or equivalently

$$\lim_{\lambda \to \infty} \lambda(f - \lambda R_\lambda f, g) = \beta(f, g) \quad \text{for all } f, g \in V.$$

It is therefore enough to prove that

$$\operatorname{Re}\big(T_I f - \lambda R_\lambda(T_I f), f - T_I f\big) \geqq 0.$$

The left hand side of this inequality is equal to $I_1 + I_2$ where

$$I_1 = \big(1 - \int \lambda\, d\rho_\lambda\big) \int_G T_I f(x)\big(\operatorname{Re} f(x) - T_I f(x)\big)\, dx$$

and

$$I_2 = \lambda \int_G \Big(\int_G (T_I f(x) - T_I f(x-y))(\operatorname{Re} f(x) - T_I f(x))\, dx\Big)\, d\rho_\lambda(y).$$

Since

$$T_I f(x)\big(\operatorname{Re} f(x) - T_I f(x)\big) = \begin{cases} 0 & \text{if } \operatorname{Re} f(x) < 0 \\ 0 & \text{if } 0 \leqq \operatorname{Re} f(x) \leqq 1 \\ \operatorname{Re} f(x) - 1 & \text{if } 1 < \operatorname{Re} f(x) \end{cases}$$

we find that $I_1 \geqq 0$.

If we put $A = \{x \in G \mid \operatorname{Re} f(x) < 0\}$, $B = \{x \in G \mid \operatorname{Re} f(x) > 1\}$, the inner integral of I_2 can be written

$$-\int_A T_I f(x-y) \operatorname{Re} f(x)\, dx + \int_B \big(1 - T_I f(x-y)\big)\big(\operatorname{Re} f(x) - 1\big)\, dx$$

which is non-negative for all $y \in G$. This implies that $I_2 \geqq 0$. □

3.6. **Corollary.** *A translation invariant positive closed form* (β, V) *on* $L^2(G)$ *is a Dirichlet form if and only if the unit contraction operates on* V *with respect to* β.

We now come to our main result.

3.7. **Theorem.** *Let* (β, V) *be a translation invariant Dirichlet form on* $L^2(G)$. *Then there exists a continuous, negative definite function* $\psi: \Gamma \to \mathbb{C}$ *satisfying an inequality of the form*

$$|\operatorname{Im} \psi| \leqq C \cdot \operatorname{Re} \psi \tag{1}$$

for a suitable constant $C > 0$, *such that*

$$V = \{f \in L^2(G) \mid \int |\hat{f}|^2 \operatorname{Re} \psi \, d\gamma < \infty\} \tag{2}$$

and such that

$$\beta(f, g) = \int \hat{f} \bar{\hat{g}} \, \psi \, d\gamma \qquad \text{for } f, g \in V. \tag{3}$$

Conversely, let $\psi: \Gamma \to \mathbb{C}$ *be a continuous, negative definite function satisfying* (1) *for a* $C > 0$. *Then the pair* (β, V) *with* V *as in* (2) *and* β *defined by* (3), *constitutes a translation invariant Dirichlet form on* $L^2(G)$.

Proof. The hermitian part (α, V) of (β, V) is a translation invariant Dirichlet form, in the sense of Deny [2], on $L^2(G)$, and it follows (cf. [2], p. 190) that there exists a continuous, negative definite function $\psi^\alpha: \Gamma \to \mathbb{R}$ such that

$$V = \{f \in L^2(G) \mid \int |\hat{f}|^2 \, \psi^\alpha \, d\gamma < \infty\}$$

and such that

$$\alpha(f, g) = \int \hat{f} \bar{\hat{g}} \, \psi^\alpha \, d\gamma \qquad \text{for } f, g \in V.$$

The operators $(\lambda R_\lambda)_{\lambda > 0}$, where $(R_\lambda)_{\lambda > 0}$ is the resolvent associated with β, are sub-markovian and commute with the translations of G, and this implies that the semigroup $(P_t)_{t > 0}$ associated with β consists of sub-markovian operators which commute with the translations of G. It is then easy to see that the associated family of positive measures $(\mu_t)_{t > 0}$ on G with total mass $\int d\mu_t \leqq 1$ (cf. 3.2) constitutes a vaguely continuous convolution semigroup on G. Let $\psi: \Gamma \to \mathbb{C}$ be the corresponding continuous, negative definite function (cf. 3.1).

For all $f \in D_A$ and $g \in V$ we find (cf. 1.4 and 3.3)

$$\beta(f, g) = (-A f, g) = \int \hat{f} \bar{\hat{g}} \, \psi \, d\gamma,$$

and in particular

$$\alpha(f, f) = \operatorname{Re} \beta(f, f) = \int |\hat{f}|^2 \operatorname{Re} \psi \, d\gamma$$

for all $f \in D_A$. By Proposition 3.3 we have

$$\bar{\mathscr{F}}\varphi \in D_A \subseteq V \quad \text{for all } \varphi \in \mathscr{K}(\Gamma),$$

where $\mathscr{K}(\Gamma)$ is the set of continuous, complex functions on Γ with compact support, and this implies that

$$\int |\varphi|^2 \operatorname{Re}\psi \, d\gamma = \int |\varphi|^2 \psi^\alpha \, d\gamma \quad \text{for all } \varphi \in \mathscr{K}(\Gamma)$$

hence that

$$\operatorname{Re}\psi = \psi^\alpha.$$

Since β is continuous with respect to $\|\cdot\|_\alpha$, there is a constant $C>0$ such that for all $\varphi \in \mathscr{K}(\Gamma)$

$$|\beta(\bar{\mathscr{F}}\varphi, \bar{\mathscr{F}}\varphi)| \leqq C \cdot \alpha(\bar{\mathscr{F}}\varphi, \bar{\mathscr{F}}\varphi)$$

or

$$\left|\int |\varphi|^2 \psi \, d\gamma\right| \leqq C \int |\varphi|^2 \psi^\alpha \, d\gamma$$
$$= C \int |\varphi|^2 \operatorname{Re}\psi \, d\gamma,$$

hence that

$$\left|\int \varphi \psi \, d\gamma\right| \leqq C \int \varphi \operatorname{Re}\psi \, d\gamma \quad \text{for all } \varphi \in \mathscr{K}^+(\Gamma),$$

and it is easy to see that this implies

$$|\operatorname{Im}\psi| \leqq C \cdot \operatorname{Re}\psi \quad \text{on } \Gamma.$$

Let $f, g \in V$. Since D_A is dense in $(V, \|\cdot\|_V)$ (cf. Lemma 1.6) there exists a sequence (f_n) with $f_n \in D_A$ and such that $\|f_n - f\|_\alpha \to 0$. By the inequality (1) the integral

$$\int \hat{f} \bar{\hat{g}} \psi \, d\gamma$$

is well-defined, and we find

$$\left|\beta(f, g) - \int \hat{f} \bar{\hat{g}} \psi \, d\gamma\right| \leqq \left|\beta(f - f_n, g) - \int (\hat{f} - \hat{f}_n) \bar{\hat{g}} \psi \, d\gamma\right|$$
$$\leqq C \|f - f_n\|_\alpha \|g\|_\alpha + \int |\hat{f} - \hat{f}_n| |\hat{g}| |\psi| \, d\gamma$$
$$\leqq (2C+1) \|f - f_n\|_\alpha \|g\|_\alpha$$

which shows that (3) is fulfilled.

Conversely, suppose that $\psi: \Gamma \to \mathbb{C}$ is a continuous, negative definite function satisfying an inequality of the form (1) with a constant $C>0$. The function $\operatorname{Re}\psi$ is then a real, continuous, negative definite function on Γ, and it defines a translation invariant Dirichlet form (α, V) on $L^2(G)$, in the sense of Deny [2]. Here the domain V is given by

$$V = \left\{ f \in L^2(G) \,\middle|\, \int |\hat{f}|^2 \operatorname{Re}\psi \, d\gamma < \infty \right\}$$

and for $f, g \in V$ we have

$$\alpha(f, g) = \int \hat{f}\bar{\hat{g}} \cdot \operatorname{Re}\psi \, d\gamma.$$

For $f, g \in V$ we now put

$$\beta(f, g) = \int \hat{f}\bar{\hat{g}} \, \psi \, d\gamma,$$

which is well-defined in view of (1). It is clear that $\beta: V \times V \to \mathbb{C}$ is a sesquilinear form, and for $f, g \in V$ we find

$$\begin{aligned} |\beta(f, g)| &= \left|\int \hat{f}\bar{\hat{g}} \cdot i \operatorname{Im}\psi \, d\gamma + \int \hat{f}\bar{\hat{g}} \cdot \operatorname{Re}\psi \, d\gamma\right| \\ &\leqq \int |\hat{f}| \cdot |\hat{g}| \, C \operatorname{Re}\psi \, d\gamma + \int |\hat{f}| \cdot |\hat{g}| \operatorname{Re}\psi \, d\gamma \\ &= (1 + C) \int |\hat{f}| (\operatorname{Re}\psi)^{\frac{1}{2}} \, |\hat{g}| \, (\operatorname{Re}\psi)^{\frac{1}{2}} \, d\gamma \end{aligned}$$

hence

$$\begin{aligned} |\beta(f, g)|^2 &\leqq (1 + C)^2 \int |\hat{f}|^2 \operatorname{Re}\psi \, d\gamma \cdot \int |\hat{g}|^2 \operatorname{Re}\psi \, d\gamma \\ &= (1 + C)^2 \, \alpha(f, f) \, \alpha(g, g), \end{aligned}$$

and it follows that (β, V) is a positive closed form on $L^2(G)$.

It is clear that the form β is translation invariant, and the proof will thus be finished when we have seen that the unit contraction operates on V with respect to β, or equivalently (cf. 3.5) that the family $(\lambda R_\lambda)_{\lambda > 0}$, where $(R_\lambda)_{\lambda > 0}$ is the resolvent associated with β, consists of sub-markovian operators. This is indeed the case, because (R_λ) is for $\lambda > 0$ given as convolution with the measure ρ_λ defined by

$$\hat{\rho}_\lambda = (\lambda + \psi)^{-1}$$

(the function $(\lambda + \psi)^{-1}$ is continuous and positive definite). To see this, we remark that $f \in L^2(G)$ implies $\rho_\lambda * f \in V$ because

$$|\widehat{\rho_\lambda * f}|^2 \operatorname{Re}\psi = |\hat{f}|^2 \operatorname{Re}\psi \, |\lambda + \psi|^{-2} \leqq \lambda^{-1} \, |\hat{f}|^2.$$

Furthermore we have for all $f \in L^2(G)$ and $g \in V$ that

$$\begin{aligned} \beta(\rho_\lambda * f, g) &= \int \hat{f} \cdot (\lambda + \psi)^{-1} \, \bar{\hat{g}} \cdot \psi \, d\gamma \\ &= (f, g) - \lambda(\rho_\lambda * f, g), \end{aligned}$$

and this shows that

$$R_\lambda f = \rho_\lambda * f \quad \text{for } f \in L^2(G). \quad \square$$

3.8. *Remarks.* Let (β, V) be a translation invariant Dirichlet form on $L^2(G)$ with associated continuous, negative definite function ψ. Let (α, V) be the hermitian part of (β, V).

1. It is seen from the proof of Theorem 3.7, that the continuous, negative definite function associated with (α, V) is simply $\operatorname{Re}\psi$, and it follows that the strongly continuous contraction semigroup $(P_t^\alpha)_{t>0}$ associated with (α, V) (cf. 1.5) is given by

$$P_t^\alpha = P_{t/2}\, P_{t/2}^* \qquad \text{for } t>0,$$

where $(P_t)_{t>0}$ is the semigroup associated with (β, V).

2. Let $f \in D_A$ and $g \in V$ be real functions. Then

$$\beta(f,g) = (-Af, g) \qquad \text{is real,}$$

because the semigroup (P_t) is given as convolution with positive measures on G. For real $f, g \in V$ we have that $\lambda R_\lambda f \in D_A$ and $\lambda R_\lambda f \to f$ in $\|\cdot\|_\alpha$, and hence that

$$|\beta(\lambda R_\lambda f, g) - \beta(f,g)| \leqq C\, \|\lambda R_\lambda f - f\|_\alpha\, \|g\|_\alpha,$$

which implies that $\beta(f,g)$ is real.

3. From Deny [2], p. 190, we know that (α, V) is positive definite if $(\operatorname{Re}\psi)^{-1} \in L^1_{\mathrm{loc}}$, and in this case the completion $\hat{V}^\alpha$ of V with respect to the *norm* $\|\cdot\|_\alpha$ is a regular, translation invariant Dirichlet space. The form β can be extended to a positive closed form (also denoted β) on $\hat{V}^\alpha$. It can be seen (cf. a proof in Bliedtner [1], p. 34) that the unit contraction operates on $\hat{V}^\alpha$ with respect to β and β^*.

4. Suppose that $(\operatorname{Re}\psi)^{-1} \in L^1_{\mathrm{loc}}$. Then ψ^{-1} is $d\gamma$-locally integrable, and the measure $\psi^{-1}\, d\gamma$ is positive definite. Let κ be the (generalized) Fouriertransform (cf. Godement [3]) of $\psi^{-1}\, d\gamma$, i.e. the measure κ on G determined by

$$\int f * \tilde{g}(\gamma)(\psi(\gamma))^{-1} d\gamma = \int \mathscr{F}f(x)\, \overline{\mathscr{F}g(x)}\, d\kappa(x)$$

for $f, g \in \mathscr{K}(\Gamma)$ (where $\tilde{g}(\gamma) = \overline{g(-\gamma)}$). It is easy to see that

$$\kappa = \int_0^\infty \mu_t\, dt \qquad \text{vaguely,}$$

and it follows that

$$\rho_\lambda \to \kappa \qquad \text{vaguely as } \lambda \to 0.$$

For $f \in \mathscr{K}(G)$ and $\lambda > 0$ we find

$$\begin{aligned} \|\rho_\lambda * f\|_\alpha^2 &= \int |\hat{f}|^2\, |\lambda + \psi|^{-2} \operatorname{Re}\psi\, d\gamma \\ &\leqq \int |\hat{f}|^2 (\operatorname{Re}\psi)^{-1}\, d\gamma, \end{aligned}$$

and the right hand side is finite since it is equal to $\kappa^\alpha(f * \tilde{f})$ where κ^α is the Fouriertransform of the measure $(\operatorname{Re}\psi)^{-1}\, d\gamma$. It follows that there exists an element $u \in \hat{V}^\alpha$ and a sequence (λ_n) such that $\lambda_n \to 0$ and

$$\rho_{\lambda_n} * f \to u \quad \text{weakly in } \hat{V}^\alpha.$$

On the other hand

$$\rho_{\lambda_n} * f \to \kappa * f \quad \text{uniformly on compact sets},$$

and it follows that

$$\mu = \kappa * f \quad \text{loc.-p.p. in } G,$$

and that

$$\begin{aligned}
\beta(\kappa * f, v) &= \lim_{n\to\infty} \beta(\rho_{\lambda_n} * f, v) \\
&= \lim_{n\to\infty} \int \widehat{\rho_{\lambda_n} * f} \cdot \bar{\hat{v}}\, \psi\, d\gamma \\
&= \lim_{n\to\infty} \int \hat{f}\bar{\hat{v}}\, \psi (\lambda_n + \psi)^{-1}\, d\gamma \\
&= \int \hat{f}\, \bar{\hat{v}}\, d\gamma \\
&= \int f \bar{v}\, dx,
\end{aligned}$$

for all $v \in V$, and $\kappa * f$ is thus the *β-potential generated by* $f \in \mathscr{K}(G)$.

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