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On the Support of the Measures in a Symmetric Convolution Semigroup

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The aim of the present paper is to prove the following result: Let $(\mu_t)_{t>0}$ be a vaguely continuous convolution semigroup consisting of symmetric probability measures on a locally compact abelian group G. Then there exists a closed subgroup H of G such that $\operatorname{supp}(\mu_t)=H$ for all $t>0$.

As a consequence of this result we get that the support of a symmetric infinitely divisible distribution in $\mathbb{R}^n$ is a closed subgroup of $\mathbb{R}^n$.

Preliminaries. For a locally compact space X we denote by $C_c(X)$ the set of continuous complex-valued functions on X with compact support. For a Radon measure μ on X we denote by $\operatorname{supp}(\mu)$ the support of μ and sometimes we use the more detailed notation $\operatorname{supp}_X(\mu)$ in order to avoid confusion.

The restriction of a Radon measure μ on X to a Borel subset Y of X is denoted $\mu \mid Y$.

A net $(\mu_i)_{i\in I}$ of Radon measures on X is said to converge *vaguely* to a Radon measure μ on X if

$$\lim_I \langle \mu_i, f\rangle = \langle \mu, f\rangle \qquad \text{for all } f\in C_c(X).$$

In the following G denotes a locally compact abelian group and the dual group of G is denoted Γ. Concerning the Fourier analysis on G we use the notation from the book of Rudin [7]. It is convenient to put $G'=G\setminus\{0\}$, and for a subset $A\subseteq G$ we denote by $G(A)$ the smallest closed subgroup of G which contains A.

For probability Radon measures μ_1 and μ_2 on G we have

$$\operatorname{supp}(\mu_1 * \mu_2)=\operatorname{cl}(\operatorname{supp}(\mu_1)+\operatorname{supp}(\mu_2)).$$

By a (vaguely continuous) *convolution semigroup* on G we mean a family $(\mu_t)_{t>0}$ of Radon probability measures on G satisfying

$$\mu_t * \mu_s = \mu_{t+s} \qquad \text{for } t, s>0, \tag{1}$$

$$\lim_{t\to 0} \mu_t = \varepsilon_0 \qquad \text{vaguely.} \tag{2}$$

To every convolution semigroup $(\mu_t)_{t>0}$ on G is associated a continuous negative definite function[1] $\psi: \Gamma \to \mathbb{C}$ satisfying $\psi(0)=0$ such that

$$\hat{\mu}_t(\gamma)=e^{-t\psi(\gamma)} \quad \text{for } t>0 \text{ and } \gamma\in\Gamma, \tag{3}$$

cf. [2] or [4], and to every continuous negative definite function $\psi: \Gamma \to \mathbb{C}$ satisfying $\psi(0)=0$ is associated a convolution semigroup $(\mu_t)_{t>0}$ on G such that (3) holds.

A convolution semigroup $(\mu_t)_{t>0}$ is *symmetric* (i.e. μ_t is symmetric for all $t>0$) if and only if the associated negative definite function ψ is real.

For a convolution semigroup $(\mu_t)_{t>0}$ on G the *Lévy measure* μ is the non-negative measure on $G'=G\setminus\{0\}$ defined as the vague limit

$$\mu=\lim_{t\to 0}\frac{1}{t}(\mu_t \mid G'). \tag{4}$$

The Lévy measure has the following properties, cf. [2],

$$\int_{G'} \operatorname{Re}(1-(x,\gamma))\,d\mu(x)<\infty \quad \text{for all } \gamma\in\Gamma, \tag{5}$$

and

$$\mu(G\setminus V)<\infty \tag{6}$$

for all compact neighborhoods V of 0 in G.

Proposition 1. (Cf. [2], Lemma 18.18). *Let μ be a non-negative symmetric measure on G' such that*

$$\psi(\gamma):=\int \operatorname{Re}(1-(x,\gamma))\,d\mu(x)<\infty \quad \textit{for all } \gamma\in\Gamma.$$

Then ψ is a continuous negative definite function on Γ and the associated convolution semigroup $(\mu_t)_{t>0}$ on G has Lévy measure μ.

A convolution semigroup is said to be of *local type*, cf. [5], if the Lévy measure vanishes.

A convolution semigroup of the type encountered in Proposition 1 is said to be *without local component.*

A continuous function $q: \Gamma \to \mathbb{R}$ is called a *quadratic form* (cf. [6]) if

$$q(\gamma+\delta)+q(\gamma-\delta)=2q(\gamma)+2q(\delta) \quad \text{for } \gamma,\delta\in\Gamma. \tag{7}$$

A quadratic form is easily seen to have the following properties

$$q(0)=0 \quad \text{and} \quad q(n\gamma)=n^2 q(\gamma) \quad \text{for } n\in\mathbb{Z} \text{ and } \gamma\in\Gamma. \tag{8}$$

A non-negative quadratic form is negative definite and the associated convolution semigroup is symmetric and of local type. Conversely, if $(\mu_t)_{t>0}$ is a symmetric convolution semigroup of local type, then the associated continuous negative definite function is a non-negative quadratic form.

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[1] A function $\psi: G\to\mathbb{C}$ defined on an abelian group G is called *negative definite* if for every natural number n and for every n-tuple $(x_1,\dots,x_n)$ of elements from G the matrix $(\psi(x_i)+\overline{\psi(x_j)}-\psi(x_i-x_j))$ is non-negative hermitian

Proposition 2. (Cf. [2], Corollary 18.20, or [6]). *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G with associated continuous negative definite function ψ on Γ. Then there exists a non-negative quadratic form q on Γ and a non-negative symmetric measure μ on G' such that*

$$\psi(\gamma)=q(\gamma)+\int_{G'} \mathrm{Re}\,(1-(x,\gamma))\,d\mu(x) \qquad \textit{for } \gamma\in\Gamma.$$

The function q is determined as

$$q(\gamma)=\lim_{n\to\infty}\frac{\psi(n\gamma)}{n^2} \qquad \textit{for } \gamma\in\Gamma,$$

and μ is the Lévy measure for $(\mu_t)_{t>0}$.

Corollary 3. *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G. Then there exists a symmetric convolution semigroup $(\mu_t')_{t>0}$ of local type and a symmetric convolution semigroup $(\mu_t'')_{t>0}$ without local component such that*

$$\mu_t=\mu_t' * \mu_t'' \qquad \textit{for } t>0.$$

Proof. With the notation from Proposition 2 we let $(\mu_t')_{t>0}$ be the convolution semigroup associated with q and $(\mu_t'')_{t>0}$ the convolution semigroup associated with

$$\psi_1(\gamma)=\int_{G'} \mathrm{Re}\,(1-(x,\gamma))\,d\mu(x) \qquad \text{for } \gamma\in\Gamma. \quad \square$$

The Main Result. The aim of this paper is to prove the following result.

Theorem 4. *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on a locally compact abelian group G. Then there exists a closed subgroup H of G such that* $\mathrm{supp}\,(\mu_t)=H$ *for all $t>0$.*

The symmetry condition is essential for the validity of the result as the example of a translation semigroup shows.

Corollary 5. *Let μ be a symmetric infinitely divisible distribution on $\mathbb{R}^n$. Then* $\mathrm{supp}\,(\mu)$ *is a closed subgroup of $\mathbb{R}^n$.*

The corollary follows immediately from Theorem 4, because there exists a symmetric convolution semigroup $(\mu_t)_{t>0}$ on $\mathbb{R}^n$ such that $\mu=\mu_1$, cf. e.g. [1], p. 278. $\square$

Proof of Theorem 4. By Corollary 3 it suffices to prove the theorem for convolution semigroups without local component and for convolution semigroups of local type, and this is done in Propositions 8 and 9 below. $\square$

We need the following lemmas.

Lemma 6. *Let X be a locally compact space and $(\mu_i)_{i\in I}$ a net of Radon measures on X converging vaguely to a Radon measure μ on X. Then*

$$\mathrm{supp}\,(\mu)\subseteq\overline{\bigcup_{i\geqq i_0}\mathrm{supp}\,(\mu_i)} \qquad \textit{for all } i_0\in I.$$

The proof is straightforward.

Lemma 7. *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G with Lévy measure μ. Then we have*

(i) $0 \in \operatorname{supp}(\mu_t)$ *for all* $t>0$,

(ii) $\operatorname{supp}(\mu_t) \subseteq \operatorname{supp}(\mu_s)$ *for* $t \leqq s$,

(iii) $G(\operatorname{supp}_{G'}(\mu)) \subseteq \operatorname{supp}(\mu_t)$ *for all* $t>0$.

Proof. We have $\mu_t = \mu_{t/2} * \mu_{t/2}$ and $\operatorname{supp}(\mu_{t/2})$ is symmetric, and it follows that

$$\operatorname{supp}(\mu_t) \supseteq \operatorname{supp}(\mu_{t/2}) + \operatorname{supp}(\mu_{t/2}) = \operatorname{supp}(\mu_{t/2}) - \operatorname{supp}(\mu_{t/2}) \ni 0,$$

and therefore

$$\operatorname{supp}(\mu_{t+s}) \supseteq \operatorname{supp}(\mu_t) + \operatorname{supp}(\mu_s) \supseteq \operatorname{supp}(\mu_s),$$

thus proving (i) and (ii). Using (4) it follows from Lemma 6 that

$$\operatorname{supp}_{G'}(\mu) \subseteq \operatorname{cl}_{G'}\Big(\bigcup_{s \leqq t} \operatorname{supp}_{G'}(\mu_s \mid G')\Big) \quad \text{for all } t>0,$$

but since $\operatorname{supp}_{G'}(\mu_s \mid G') \subseteq \operatorname{supp}(\mu_s)$ for $s>0$, we get by (ii) that

$$\operatorname{supp}_{G'}(\mu) \subseteq \operatorname{supp}(\mu_t) \quad \text{for all } t>0.$$

For $n \in \mathbb{N}$ and $t>0$ we then find

$$(\operatorname{supp}_{G'}(\mu))^n \subseteq (\operatorname{supp}(\mu_{t/n}))^n \subseteq \operatorname{supp}(\mu_t),$$

and finally

$$G(\operatorname{supp}_{G'}(\mu)) = \operatorname{cl}\Big(\{0\} \cup \bigcup_{n=1}^{\infty} (\operatorname{supp}_{G'}(\mu))^n\Big) \subseteq \operatorname{supp}(\mu_t)$$

for all $t>0$. □

Proposition 8. *Let μ be a non-negative symmetric measure on G' such that*

$$\psi(\gamma) := \int_{G'} \operatorname{Re}(1-(x,\gamma))\, d\mu(x) < \infty \quad \textit{for all } \gamma \in \Gamma,$$

and let $(\mu_t)_{t>0}$ be the symmetric convolution semigroup on G associated with ψ. Then we have

$$\operatorname{supp}(\mu_t) = G(\operatorname{supp}_{G'}(\mu)) \quad \textit{for all } t>0.$$

Proof. Let $\check{V}$ denote the set of compact symmetric neighborhoods of 0 in G. For $V \in \check{V}$ we put

$$\psi_V(\gamma) = \int_{G \setminus V} \operatorname{Re}(1-(x,\gamma))\, d\mu(x) \quad \text{for } \gamma \in \Gamma,$$

and from Proposition 1 we know that ψ_V is a continuous negative definite function. The associated convolution semigroup is denoted $(\mu_t^V)_{t>0}$.

It is clear that $\psi_V \leqq \psi$ and for $V_1, V_2 \in \check{V}$ such that $V_1 \subseteq V_2$ we have $\psi_{V_2} \leqq \psi_{V_1}$. For every $\gamma \in \Gamma$ we have

$$\sup_{\check{V}} \psi_V(\gamma) = \psi(\gamma),$$

(cf. [3], Théorème 1, p. 107), and it follows by Dini's Theorem that the net $(\psi_V)_{V\in\check{V}}$ converges to ψ uniformly over compact subsets of Γ. For $t>0$ we then have

$$\lim_{\check{V}} \hat{\mu}_t^V(\gamma)=\lim_{\check{V}} e^{-t\psi_V(\gamma)}=e^{-t\psi(\gamma)}=\hat{\mu}_t(\gamma)$$

uniformly over compact subsets of Γ, and by the continuity theorem for the Fourier transformation, cf. [2], Theorem 3.13, it follows that

$$\lim_{\check{V}} \mu_t^V=\mu_t \quad \text{vaguely for } t>0. \tag{9}$$

For $V\in\check{V}$ we define $\mu_V=\mu\,|\,(G\setminus V)$, and it follows by (6) that the total mass a_V of μ_V is finite. We therefore get

$$\psi_V(\gamma)=\int \operatorname{Re}(1-(x,\gamma))\,d\mu_V(x)=a_V-\hat{\mu}_V(\gamma) \quad \text{for } \gamma\in\Gamma,$$

and consequently

$$e^{-t\psi_V(\gamma)}=e^{-ta_V}\,e^{t\hat{\mu}_V(\gamma)} \quad \text{for } \gamma\in\Gamma \quad \text{and} \quad t>0,$$

and it follows that

$$\mu_t^V=e^{-ta_V}\sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_V^n \quad \text{for } t>0, \tag{10}$$

where μ_V^n is the n'th convolution power of μ_V and $\mu_V^0=\varepsilon_0$.

From (10) we get

$$\operatorname{supp}(\mu_t^V)=\operatorname{cl}\left(\bigcup_{n=0}^{\infty}\operatorname{supp}(\mu_V^n)\right)=G(\operatorname{supp}(\mu_V)) \quad \text{for } t>0,$$

and in particular

$$\operatorname{supp}(\mu_t^V)\subseteq G(\operatorname{supp}_{G'}(\mu)) \quad \text{for } t>0 \quad \text{and} \quad V\in\check{V}.$$

An application of Lemma 6 to (9) yields that

$$\operatorname{supp}(\mu_t)\subseteq G(\operatorname{supp}_{G'}(\mu)) \quad \text{for } t>0,$$

which together with Lemma 7 (iii) show that

$$\operatorname{supp}(\mu_t)=G(\operatorname{supp}_{G'}(\mu)) \quad \text{for } t>0. \quad \square$$

Proposition 9. $(\mu_t)_{t>0}$ *be a symmetric convolution semigroup of local type on G. Then there exists a closed connected subgroup* $H\subseteq G$ *such that* $\operatorname{supp}(\mu_t)=H$ *for all* $t>0$. *The subgroup H is equal to* $\{\gamma\in\Gamma\,|\,q(\gamma)=0\}^{\perp}$, *where q is the continuous negative definite function associated with* $(\mu_t)_{t>0}$.

Proof. Let H be the smallest closed subgroup of G which contains $\operatorname{supp}(\mu_t)$ for all $t>0$, and let q be the continuous negative definite function associated with $(\mu_t)_{t>0}$. We know that q is a non-negative quadratic form. By Proposition 8.27 in [2] we have

$$H^{\perp}=\{\gamma\in\Gamma\,|\,q(\gamma)=0\} \quad \text{and} \quad H=\{\gamma\in\Gamma\,|\,q(\gamma)=0\}^{\perp}.$$

Let $t>0$ be fixed. We claim that μ_t is a *Gaussian measure* on H in the sense of Urbanik [8], and by the results of [8] it then follows that $\operatorname{supp}(\mu_t)=H$, and that H is a connected group.

We shall prove:

1) There exists a convolution semigroup $(\sigma_s)_{s>0}$ on H such that $\sigma_1=\mu_t$.

This is clear by taking $\sigma_s=\mu_{ts}\,|\,H$ for $s>0$.

2) For every non-trivial character χ on H the image measure $\chi(\mu_t)$ of μ_t under $\chi\colon H\to\mathbb{T}$ is a normal measure on $\mathbb{T}$.

Let χ be a non-trivial character on H. Then there exists a character $\gamma\in\Gamma\setminus H^\perp$ such that the restriction of γ to H is equal to χ. The Fourier transform of the measure $\chi(\mu_t)$ is a function on $\mathbb{Z}$, which for $n\in\mathbb{Z}$ has the value

$$\widehat{\chi(\mu_t)}(n)=\widehat{\gamma(\mu_t)}(n)=\hat{\mu}_t(n\gamma)=e^{-tq(n\gamma)}=e^{-tq(\gamma)n^2},$$

and since $q(\gamma)>0$ for $\gamma\in\Gamma\setminus H^\perp$ it follows by [8] that $\chi(\mu_t)$ is a normal measure on $\mathbb{T}$. $\square$

Remark. Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G. It follows by the previous results that we have the following expression for $\operatorname{supp}(\mu_t)$:

$$\operatorname{supp}(\mu_t)=\operatorname{cl}(\{\gamma\in\Gamma\mid q(\gamma)=0\}^\perp+G(\operatorname{supp}_{G'}(\mu)))\qquad\text{for } t>0.$$

Here q is the quadratic form determined from the associated continuous negative definite function ψ by

$$q(\gamma)=\lim_{n\to\infty}\frac{\psi(n\gamma)}{n^2}\qquad\text{for } \gamma\in\Gamma,$$

and μ is the Lévy measure.

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