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Potential Theory on the Infinite Dimensional Torus*

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1. Introduction

During the last 15 years there has been developped an extensive potential theory for harmonic spaces. The main motivation has been to unify the study of the solutions to various elliptic and parabolic partial differential equations of second order on an open subset of $\mathbb{R}^n$ or more generally on a differentiable manifold.

Classical potential theory in $\mathbb{R}^n$ is translation invariant and it is therefore natural to consider harmonic spaces $(G, \mathscr{H})$, where the underlying space G is a group and the harmonic sheaf $\mathscr{H}$ is invariant under the translations of G. We say then that $(G, \mathscr{H})$ is a harmonic group. Harmonic groups were studied by Bliedtner in [3]. The only existing examples of harmonic groups seem to be defined on a Lie group G and the sheaf $\mathscr{H}$ is the sheaf of solutions to an invariant second order differential operator of elliptic or parabolic type.

In [4] there was an attempt to prove that the base space of a harmonic group is a Lie group, but as pointed out in [17] the proof was not correct although the result might still be true.

The purpose of this paper is to prove that such a result can not be true, because we construct a harmonic group with base space T^∞ (countable product of circle groups). More precisely we construct a translation invariant sheaf $\mathscr{H}$ on T^∞ such that $(T^\infty, \mathscr{H})$ is a $\mathfrak{P}$-Brelot space in the sense of Constantinescu and Cornea [7]. A similar construction may be carried out for the groups $\mathbb{R}^n \times T^\infty$.

In [9] Forst considered the problem of constructing a harmonic group from a special Dirichlet space in the case of the base space being a locally compact abelian group G. The Dirichlet space is given in terms of a symmetric convolution semigroup on G, which in turn is the transition semigroup of a Hunt process on G. Then Forst proves that if the convolution semigroup satisfies certain axioms (cf. Theorem 1.12 below), then the harmonic functions defined in terms of the Hunt process satisfy the axioms of a harmonic group.

However, no examples of convolution semigroups were given which satisfied the axioms, except that it was clear that the Brownian semigroup in $\mathbb{R}^n$ satisfied the axioms.

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One possible way of proving the existence of a harmonic sheaf $\mathscr{H}$ on T^∞ such that $(T^\infty, \mathscr{H})$ is a harmonic group would then be to construct a convolution semigroup $(\mu_t)_{t>0}$ on T^∞ satisfying the axioms of Forst. This will be done in the first part of this work.

It is interesting however to have a construction of the harmonic sheaf $\mathscr{H}$ on T^∞ which does not depend on the Hunt process, and a proof of the main properties of $\mathscr{H}$ which does not depend on the results of Forst. In the second part of this paper we carry this out: The harmonic functions on T^∞ are constructed as the solutions to the equation $Af=0$, where A is the infinitesimal generator of the (Brownian) convolution semigroup on T^∞ constructed in the first part.

The two parts of the paper are to some extent independent of each other, but some of the estimates obtained in the first part will be used in the second part.

Contents

In §1 we recall some fundamental concepts which will be used throughout the paper. Although we are only going to deal with the compact groups T^p and T^∞ we have preferred to present the material in §1 for arbitrary locally compact abelian groups.

In §2 we define the Brownian convolution semigroup on T. The measure μ_t to time t has a density $g_t(\theta)$ which is equal to the theta function $\vartheta_3\left(\frac{\theta}{2}, e^{-t}\right)$. It is essential for the following to have thorough knowledge of this function, in particular to have estimates of the function when t is small or large. All the necessary estimates are proved in this paragraph.

In §3 we use the estimates of §2 to give information about the Brownian semigroup on T^p. This paragraph is only a prelude to §4 where the main topic of part I is developed.

For an arbitrary sequence $\mathscr{A}=(a_1, a_2, \cdots)$ of positive numbers we define a Brownian semigroup $(\mu_t^{\mathscr{A}})_{t>0}$ on T^∞ as

$$\mu_t^{\mathscr{A}} = \bigotimes_{k=1}^{\infty} \mu_{t a_k} \quad \text{for } t>0,$$

and we use the estimates of §2 to examine how $\mu_t^{\mathscr{A}}$ depends on $\mathscr{A}$. It is proved that $\mu_t^{\mathscr{A}}$ is absolutely continuous with respect to Haar measure on T^∞ if and only if $\sum_{k=1}^{\infty} e^{-2ta_k}<\infty$ (cf. 4.3), and that $\mu_t^{\mathscr{A}}$ has a continuous density if and only if $\sum_{k=1}^{\infty} e^{-ta_k}<\infty$. Furthermore, if $\sum_{k=1}^{\infty} \frac{1}{\sqrt{a_k}}<\infty$ we prove an estimate for the density $g_t^{\mathscr{A}}$ for $\mu_t^{\mathscr{A}}$ which implies that the resolvent (cf. 1.8) has densities which are finite and continuous on $T^\infty \setminus \{0\}$. This shows that $(\mu_t^{\mathscr{A}})_{t>0}$ satisfies the axioms of Forst, cf. 1.12 and 1.13.

In §6 we introduce the harmonic functions on T^∞ as solutions in a distribution sense to the equation

$$\sum_{k=1}^{\infty} a_k \frac{\partial^2 h}{\partial \theta_k^2} - \lambda h = 0, \tag{1}$$

where $\mathscr{A}=(a_1, a_2, \ldots)$ is as above and $\lambda \geqq 0$, and we prove that the solutions form a sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$. An important approximation lemma in §6 states that harmonic functions on T^∞ can be approximated by solutions to (1) on T^p for p sufficiently large. (When we use (1) for functions on T^p we take only terms in the series with indices $\leqq p$.) This approximation technique permits us to obtain information about $\mathscr{H}_\lambda^{\mathscr{A}}$ from known results on "harmonic functions" on T^p and therefore we state some useful facts about harmonic functions on T^p in §5. A few results from §5 deserve being mentioned. Let V be a bounded domain in $\mathbb{R}^p$ with smooth boundary and let $P_\lambda(x, \xi)$ denote the Poisson kernel for V for the differential operator

$$\sum_{k=1}^{p} a_k \frac{\partial^2}{\partial \theta_k^2} - \lambda, \tag{2}$$

where $\lambda \geqq 0$, and $x \in V$, $\xi \in \partial V$.

1) For every compact subset $K \subseteq V$ there exist constants $A, B>0$ such that

$$P_\lambda(x, \xi) \leqq A e^{-B\sqrt{\lambda}} \quad \text{for } x \in K, \xi \in \partial V \quad \text{and} \quad \lambda \geqq 1.$$

2) For $x \in V$ and $\xi \in \partial V$ the mapping $\lambda \to P_\lambda(x, \xi)$ is completely monotone.

In §7 we prove that if $U \subseteq T^p$ is a regular subset of T^p with respect to (2) then $U \times T^\infty$ is a regular subset of T^∞ with respect to $\mathscr{H}_\lambda^{\mathscr{A}}$, thus establishing the existence of a base of regular sets for the sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$.

In §6 and §7 the only assumptions on the sequence $\mathscr{A}$ is that $a_k>0$ for all $k \in \mathbb{N}$. In order to prove that $\mathscr{H}_\lambda^{\mathscr{A}}$ has the Brelot convergence property we must impose a growth condition on $\mathscr{A}$. In §8 we prove the Brelot convergence property under the assumption on $\mathscr{A}$ that

$$\sum_{k=1}^{\infty} \frac{(\sqrt{2})^k}{\sqrt{a_k}} < \infty.$$

The proof depends on explicit knowledge of the Poisson kernel for special domains in T^∞ of the form $U \times T^\infty$, and this permits us to prove Harnack-type inequalities for positive harmonic functions. In the proof of the existence of a Poisson kernel for $U \times T^\infty$ we make use of the results 1) and 2) mentioned above.

In §9 we finally prove that if $\mathscr{A}=(1, 1, \cdots)$ then the Brelot convergence property is not satisfied.

In the reading of this paper we recommend the reader to start with §§6–8 in order to get familiar with the main ideas, and then go back to the previous paragraphs when it is needed.

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Part I. Convolution Semigroups on T^∞

1. Convolution Semigroups on Locally Compact Abelian Groups

In the following we will recall parts of the theory of convolution semigroups on locally compact abelian groups. A detailed exposition can be found in [2].

1.1. Let G denote a locally compact abelian group with Haar measure dx. The neutral element in G is always denoted 0. The dual group of G is denoted Γ and the dual Haar measure on Γ is denoted $d\gamma$.

The set of continuous functions $f\colon G \to \mathbb{R}$ is denoted $C(G)$. The set of functions $f \in C(G)$ which tend to zero at infinity, resp. which have compact support, is denoted $C_0(G)$ resp. $C_c(G)$. Under the uniform norm $C_0(G)$ is a Banach space.

1.2. A family $(\mu_t)_{t>0}$ of positive measures on G is called a *convolution semigroup* on G if

$$\mu_t(G) \leqq 1 \qquad \text{for } t>0, \tag{1}$$

$$\mu_t * \mu_s = \mu_{t+s} \qquad \text{for } t, s>0, \tag{2}$$

$$\lim_{t\to 0} \mu_t = \varepsilon_0 \text{ vaguely.} \tag{3}$$

The condition (3) simply means that $\lim_{t\to 0} \langle \mu_t, f\rangle = f(0)$ for all $f \in C_c(G)$.

A continuous function $\psi\colon \Gamma \to \mathbb{C}$ is called *negative definite* if the following condition is satisfied:

For every $n \in \mathbb{N}$ and for every n-tuple $(\gamma_1, \ldots, \gamma_n)$ of elements from Γ the $n \times n$ matrix

$$(\psi(\gamma_i) + \overline{\psi(\gamma_j)} - \psi(\gamma_i - \gamma_j))$$

is non-negative hermitian.

To every convolution semigroup $(\mu_t)_{t>0}$ on G is *associated* a continuous negative definite function $\psi\colon \Gamma \to \mathbb{C}$ such that

$$\hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)} \qquad \text{for } \gamma \in \Gamma \quad \text{and} \quad t>0. \tag{4}$$

Conversely, if $\psi\colon \Gamma \to \mathbb{C}$ is a continuous negative definite function on Γ there exists a uniquely determined convolution semigroup $(\mu_t)_{t>0}$ on G such that (4) holds.

1.3 A convolution semigroup $(\mu_t)_{t>0}$ on G is called *symmetric* if all the measures $(\mu_t)_{t>0}$ are symmetric. This is true if and only if the associated negative definite function ψ is real-valued.

For a symmetric convolution semigroup $(\mu_t)_{t>0}$ on G we have (cf. [1])

$$\operatorname{supp}(\mu_t)=\{\gamma\in\Gamma \mid \psi(\gamma)=\psi(0)\}^{\perp} \quad \text{for all } t>0. \tag{5}$$

1.4. A continuous function $f\colon \Gamma\to\mathbb{R}$ is called a *quadratic form* if

$$q(\gamma+\delta)+q(\gamma-\delta)=2q(\gamma)+2q(\delta) \quad \text{for } \gamma,\delta\in\Gamma.$$

A quadratic form q satisfies $q(0)=0$ and $q(n\gamma)=n^2 q(\gamma)$ for $n\in\mathbb{Z}$ and $\gamma\in\Gamma$.

A non-negative quadratic form is negative definite.

1.5. A convolution semigroup $(\mu_t)_{t>0}$ on G induces a strongly continuous contraction semigroup $(P_t)_{t>0}$ on $C_0(G)$, namely

$$P_t f=\mu_t * f \quad \text{for } t>0 \quad \text{and} \quad f\in C_0(G).$$

The infinitesimal generator is denoted (A, D_A) and defined as follows

$$D_A=\left\{f\in C_0(G) \mid \lim_{t\to 0}\frac{1}{t}(P_t f-f) \text{ exists in } C_0(G)\right\}$$

$$Af=\lim_{t\to 0}\frac{1}{t}(P_t f-f) \quad \text{for } f\in D_A.$$

The convolution semigroup $(\mu_t)_{t>0}$ is said to be of *local type* if the infinitesimal generator (A, D_A) is a local operator in the following sense:

For every $f\in D_A$ we have $\operatorname{supp}(Af)\subseteq\operatorname{supp}(f)$.

The following theorem concerning convolution semigroups of local type is a special case of Theorem 18.27 in [2].

1.6. **Theorem.** *A symmetric convolution semigroup $(\mu_t)_{t>0}$ on G is of local type if and only if the associated negative definite function ψ on Γ is of the form $\psi(\gamma)=c+q(\gamma)$ for $\gamma\in\Gamma$, where $c\geqq 0$ and q is a non-negative quadratic form.*

1.7. **Theorem.** *Let $(\mu_t)_{t>0}$ be a symmetric*[1] *convolution semigroup on G with associated negative definite function ψ on Γ. For each $t>0$ the following conditions are equivalent:*

(i) *μ_t has a continuous density g_t with respect to Haar measure on G.*

(ii) $e^{-t\psi}\in L^1(\Gamma)$.

If (ii) *holds for every $t>0$ then $g_t\in D_A$ and*

$$A g_t(x)=\frac{d}{dt}g_t(x) \quad \textit{for } t>0 \quad \textit{and} \quad x\in G.$$

Furthermore, the function $g\colon]0,\infty[\times G\to\mathbb{R}$ defined by $g(t,x)=g_t(x)$ is continuous.

Proof. Each measure μ_t is positive definite because $\hat{\mu}_t=e^{-t\psi}$ is positive since ψ is real. If μ_t has a continuous density g_t then g_t is necessarily a positive definite

[1] The symmetry is only needed in the proof of (i) $\Rightarrow$ (ii).

function, so (ii) follows from Bochner's theorem and the inversion theorem, cf. [15].

On the other hand (ii)$\Rightarrow$(i) follows from the inversion theorem which implies that

$$g_t(x)=\int_\Gamma (x,\gamma)\, e^{-t\psi(\gamma)}\, d\gamma \qquad \text{for } x\in G \tag{6}$$

is a continuous density for μ_t if (ii) holds. By the Riemann-Lebesgue lemma we get that $g_t\in G_0(G)$.

Suppose now that (ii) holds for all $t>0$. Since $\sup_{x\geqq 0} x e^{-tx}=\frac{1}{t}\, e^{-1}$ for $t>0$ we get that

$$\int_\Gamma \psi(\gamma)\, e^{-t\psi(\gamma)}\, d\gamma \leqq \frac{2}{t}\, e^{-1}\int_\Gamma e^{-\frac{1}{2}t\psi(\gamma)}\, d\gamma<\infty \qquad \text{for } t>0.$$

For t and $s>0$ and $x\in G$ we have

$$\begin{aligned}\frac{1}{s}\,(P_s\, g_t(x)-g_t(x))&=\frac{1}{s}\,(g_{t+s}(x)-g_t(x))\\ &=\int_\Gamma (x,\gamma)\, e^{-t\psi(\gamma)}\left(\frac{1}{s}\,(e^{-s\psi(\gamma)}-1)\right) d\gamma\end{aligned}$$

which converges uniformly for $x\in G$ to

$$-\int_\Gamma (x,\gamma)\,\psi(\gamma)\, e^{-t\psi(\gamma)}\, d\gamma.$$

The dominated convergence theorem can be applied on account of the inequality

$$\frac{1}{s}\,|e^{-s\psi(\gamma)}-1|\leqq\psi(\gamma) \qquad \text{for } s>0 \quad \text{and} \quad \gamma\in\Gamma.$$

It follows that $g_t\in D_A$ and

$$A\, g_t(x)=\frac{d}{dt}\, g_t(x)=-\int (x,\gamma)\,\psi(\gamma)\, e^{-t\psi(\gamma)}\, d\gamma$$

for $t>0$ and $x\in G$.

Using (6) it is straightforward to prove that $g(t,x)=g_t(x)$ is continuous. ▯

1.8. Let $(\mu_t)_{t>0}$ be a convolution semigroup on G. The family $(\rho_\lambda)_{\lambda>0}$ of positive bounded measures defined by

$$\langle\rho_\lambda, f\rangle=\int_0^\infty e^{-\lambda t}\langle\mu_t, f\rangle\, dt \qquad \text{for } \lambda>0 \quad \text{and} \quad f\in C_c(G) \tag{7}$$

is called the *resolvent* for $(\mu_t)_{t>0}$.

The Fourier transform of ρ_λ is $\hat\rho_\lambda=1/(\psi+\lambda)$.

The convolution semigroup is called *transient* if

$$\int_0^\infty \langle\mu_t, f\rangle\, dt<\infty \qquad \text{for all } f\in C_c^+(G).$$

If $(\mu_t)_{t>0}$ is transient there exists a positive measure κ called the potential kernel for $(\mu_t)_{t>0}$ and defined by

$$\langle\kappa, f\rangle = \int_0^\infty \langle\mu_t, f\rangle\, dt \qquad \text{for } f\in C_c(G).$$

Notice that for $\lambda>0$ $(e^{-\lambda t}\mu_t)_{t>0}$ is a transient convolution semigroup with potential kernel ρ_λ.

1.9. **Proposition.** *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G. With the notation from above we have*

$$\langle\rho_\lambda, Af-\lambda f\rangle = -f(0) \qquad \text{for } f\in D_A.$$

Proof. It is well known that

$$\rho_\lambda * (Af-\lambda f) = -f \qquad \text{for } f\in D_A.$$

Since $(\mu_t)_{t>0}$ is symmetric $f\in D_A$ implies $\check{f}\in D_A$ and $A(\check{f})=(Af)\check{}$, and we therefore get for $f\in D_A$

$$\langle\rho_\lambda, Af-\lambda f\rangle = \rho_\lambda * (A(\check{f})-\lambda\check{f})(0) = -\check{f}(0) = -f(0). \quad \square$$

1.10. Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G and suppose that each measure μ_t has a continuous density g_t. In this case we consider the integrals

$$\tilde{\rho}_\lambda(x) = \int_0^\infty e^{-\lambda t} g_t(x)\, dt \qquad \text{for } \lambda\geqq 0 \quad \text{and} \quad x\in G. \tag{8}$$

It is clear that $\tilde{\rho}_\lambda$ is a density for ρ_λ with respect to Haar measure when $\lambda>0$, and in the transient case that $\tilde{\rho}_0$ is a density for the potential kernel κ.

By the last part of Theorem 1.7 follows that $\tilde{\rho}_\lambda$ is a lower semicontinuous function on G for $\lambda\geqq 0$. In the cases we will be dealing with we have $\tilde{\rho}_\lambda(0)=\infty$ for $\lambda>0$. We are interested in cases where $\tilde{\rho}_\lambda$ satisfy further regularity conditions. Due to the result 1.12 below it is interesting to know if $\tilde{\rho}_\lambda$ is a continuous function on $G\setminus\{0\}$.

1.11. In the theory of harmonic spaces we follow the terminology of Constantinescu and Cornea [7].

A harmonic sheaf $\mathscr{H}$ on G is called *translation invariant* if for every open subset $U\subseteq G$, every $a\in G$ and every $h\in\mathscr{H}(U)$ the translated function $\tau_a h$ belongs to $\mathscr{H}(a+U)$, where $\tau_a h(x)=h(x-a)$.

We say that $\mathscr{H}$ is *symmetric* if for every open subset $U\subseteq G$ and every $h\in\mathscr{H}(U)$ the reflected function $\check{h}$ belongs to $\mathscr{H}(-U)$.

1.12. **Theorem.** (Forst [9]). *Let G be a non-discrete second countable locally compact abelian group and let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup on G satisfying the following axioms*[2]

(i) $(\mu_t)_{t>0}$ *is of local type.*

(ii) $(\mu_t)_{t>0}$ *is transient with potential kernel κ.*

(iii) κ *has a lower semicontinuous density N which is finite and continuous on $G\setminus\{0\}$.*

[2] One can prove that (i)–(iii) implies that the connected component of the neutral element in G is open, cf. [1] Proposition 9.

Then there exists a translation invariant harmonic sheaf $\mathscr{H}$ on G such that $(G, \mathscr{H})$ is a $\mathfrak{P}$-Brelot space.

The harmonic sheaf $\mathscr{H}$ is constructed as the harmonic functions of the Hunt process associated with $(\mu_t)_{t>0}$ via the (sub) markov transition semigroup

$$P_t(x, B) = \mu_t(B - x),$$

where $x \in G$, $t > 0$ and B is a Borel subset of G.

As an application of Theorem 1.12 to the compact group T^∞ we find the following result.

1.13. **Theorem.** *Let $(\mu_t)_{t>0}$ be a symmetric convolution semigroup of probability measures on T^∞ with the following properties:*

(i) *$(\mu_t)_{t>0}$ is of local type.*

(ii) *μ_t has a continuous density g_t for every $t > 0$.*

(iii) *$\tilde{\rho}_\lambda$ is finite and continuous on $T^\infty \setminus \{0\}$ for every $\lambda > 0$.*

Then for every $\lambda > 0$ there exists a translation invariant harmonic sheaf $\mathscr{H}_\lambda$ on T^∞ such that $(T^\infty, \mathscr{H}_\lambda)$ is a $\mathfrak{P}$-Brelot space.

In §4 we will construct a convolution semigroup $(\mu_t)_{t>0}$ on T^∞ verifying the hypotheses of Theorem 1.13.

1.14. *Remark.* A convolution semigroup $(\mu_t)_{t>0}$ of probability measures on T^∞ is never transient and we are therefore forced to consider the convolution semigroup $(e^{-\lambda t}\mu_t)_{t>0}$ with potential kernel ρ_λ for $\lambda > 0$. Under the hypotheses of Theorem 1.13 it is easy to see that each of the convolution semigroups $(e^{-\lambda t}\mu_t)_{t>0}$ satisfies the axioms from Theorem 1.12.

1.15. A function $\varphi\colon]0, \infty[\to \mathbb{R}$ is called *completely monotone* if it is C^∞ and satisfies $(-1)^n \varphi^{(n)}(x) \geqq 0$ for all $x > 0$ and $n \geqq 0$.

We shall make use of the following result.

1.16. **Proposition.** *Let $\psi\colon \Gamma \to \mathbb{R}$ be a real-valued continuous negative definite function and let $\varphi\colon [0, \infty[\to \mathbb{R}$ be continuous and completely monotone. Then $\varphi \circ \psi\colon \Gamma \to \mathbb{R}$ is continuous and positive definite.*

Proof. By Bernstein's theorem there exists a positive bounded measure μ on $[0, \infty[$ such that

$$\varphi(x) = \int_0^\infty e^{-tx}\, d\mu(t) \qquad \text{for } x \geqq 0,$$

(cf. e.g. [13]), and therefore

$$\varphi(\psi(\gamma)) = \int_0^\infty e^{-t\psi(\gamma)}\, d\mu(t) \qquad \text{for } \gamma \in \Gamma.$$

The composition makes sense because $\psi(\gamma) \geqq 0$ for all $\gamma \in \Gamma$. From 1.2 follows that $e^{-t\psi}$ is positive definite for all $t \geqq 0$ and therefore $\varphi \circ \psi$ is positive definite. □

Using the canonical extension of φ to the half-plane $\operatorname{Re} z \geqq 0$, (cf. [2]), Proposition 1.16 remains true for complex-valued negative definite functions.

2. The Brownian Semigroup on T

2.1. The Brownian convolution semigroup $(\mu_t)_{t>0}$ on $\mathbb{R}$ is the family of probability measures $\mu_t = p_t(x)\,dx$ on $\mathbb{R}$, where

$$p_t(x) = (4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right) \quad \text{for } x \in \mathbb{R}. \tag{1}$$

The Fourier transform of p_t is equal to

$$\hat{p}_t(y) = \int_{\mathbb{R}} e^{-ixy}\, p_t(x)\, dx = e^{-ty^2} \quad \text{for } y \in \mathbb{R}. \tag{2}$$

2.2. The Brownian convolution semigroup on $T(=\{z \in \mathbb{C} \mid |z|=1\})$ is perhaps less well-known. It is the family $(\mu_t)_{t>0}$ of probability measures on T given by the densities g_t with respect to normalized Haar measure on T, where

$$g_t(\theta) = \sum_{n \in \mathbb{Z}} e^{-tn^2} e^{in\theta} = 1 + 2\sum_{n=1}^{\infty} e^{-tn^2} \cos(n\theta) \quad \text{for } \theta \in \mathbb{R}. \tag{3}$$

(Here and in the following we describe functions on T as functions on $\mathbb{R}$ which are periodic with period 2π.)

The Fourier transform (coefficients) of g_t is equal to

$$\hat{g}_t(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta}\, g_t(\theta)\, d\theta = e^{-tn^2} \quad \text{for } n \in \mathbb{Z}, \tag{4}$$

and it follows that $g_t(\theta) \geq 0$ because the function $n \to e^{-tn^2}$ is positive definite on $\mathbb{Z}$, cf. (2). It also follows from (4) that $(\mu_t)_{t>0}$ is a convolution semigroup on T. The negative definite function (cf. 1.2) associated with $(\mu_t)_{t>0}$ is $q(n) = n^2$ for $n \in \mathbb{Z}$ which is a non-negative quadratic form on $\mathbb{Z}$, so $(\mu_t)_{t>0}$ is of local type. Note that every non-negative quadratic form on $\mathbb{Z}$ has the form $n \to an^2$ where $a \geq 0$, so $(\mu_t)_{t>0}$ is essentially the only symmetric convolution semigroup of probability measures on T which is of local type.

The following expression for g_t is very important.

2.3. **Proposition.** *For $\theta \in \mathbb{R}$ and $t>0$ we have*

$$g_t(\theta) = \sqrt{\frac{\pi}{t}} \sum_{k \in \mathbb{Z}} \exp\left(-\frac{(\theta + 2\pi k)^2}{4t}\right).$$

Proof. Let ω denote the discrete measure on $\mathbb{R}$ which has the mass 1 in each of the points $2\pi n$, $n \in \mathbb{Z}$. The function $G_t = \omega * p_t$, where p_t is given by (1), is a continuous periodic function, and the Fourier coefficients of G_t are given for $n \in \mathbb{Z}$ by

$$\begin{aligned}\hat{G}_t(n) &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_0^{2\pi} p_t(\theta + 2\pi k)\, e^{-in\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} p_t(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\, \hat{p}_t(n) = \frac{1}{2\pi}\, e^{-tn^2}.\end{aligned}$$

This shows that $2\pi G_t$ and g_t have the same Fourier coefficients, and the formula follows. $\square$

2.4. The function $g_t(\theta)$ is essentially a theta function. With the notation of Hille [11], p. 156 we put

$$\vartheta_3(x,q)=\sum_{n\in\mathbb{Z}} q^{n^2} e^{2inx}$$

and then $g_t(\theta)=\vartheta_3\left(\frac{\theta}{2}, e^{-t}\right)$. Proposition 2.3 is a special case of a well-known formula for ϑ_3. The function $g_t(\theta)$ is called the *heat-kernel* for T because it satisfies the heat-equation

$$\frac{\partial^2}{\partial\theta^2} g_t(\theta)=\frac{\partial}{\partial t} g_t(\theta),$$

and $g_t(\theta)$ has the physical interpretation of representing the temperature at the point $e^{i\theta}$ to time t in a ring of radius $1(\approx T)$, when a "unit of heat" is put at the point 1 to time 0. It is therefore to expect that $g_t(\theta)$ for fixed $t>0$ is an even function, decreasing for $\theta\in[0,\pi]$. It is clearly even, but it is not at all obvious by the previous formulas that it is decreasing on $[0,\pi]$. This follows however from the classical product formula of Jacobi for ϑ_3 (cf. [11], p. 163) which in this context reads

$$g_t(\theta)=\prod_{n=1}^{\infty}(1-e^{-2tn})\prod_{n=1}^{\infty}(1+2e^{-(2n-1)t}\cos\theta+e^{-(4n-2)t}).$$

In fact all the factors are decreasing for $\theta\in[0,\pi]$.

The formula in Proposition 2.3 shows that the heat-kernel $g_t(\theta)$ for T is obtained by summing equidistant shifts of the heat-kernel p_t for $\mathbb{R}$, as could be expected physically.

2.5. We introduce the following notation

$$\varphi(t,\theta)=\sum_{n=1}^{\infty} e^{-tn^2}\cos(n\theta) \quad \text{for } t>0 \text{ and } \theta\in\mathbb{R}, \tag{5}$$

$$\varphi(t)=\varphi(t,0)=\sum_{n=1}^{\infty} e^{-tn^2} \quad \text{for } t>0. \tag{6}$$

Clearly $|\varphi(t,\theta)|\leqq\varphi(t)$ for $t>0$ and $\theta\in\mathbb{R}$.

2.6. **Lemma.** *The function $\varphi(t)$ has the following properties*

(i) $\varphi(t)\leqq\frac{1}{2}\sqrt{\frac{\pi}{t}}$ *for* $t>0$,

(ii) $\varphi(t)\sim\frac{1}{2}\sqrt{\frac{\pi}{t}}$ *for* $t\to 0$,

(iii) $\varphi(t)\sim e^{-t}$ *for* $t\to\infty$.

Proof. We have

$$\varphi(t)\leqq\int_0^{\infty} e^{-tx^2}\,dx=\frac{1}{\sqrt{t}}\int_0^{\infty} e^{-u^2}\,du=\frac{1}{2}\sqrt{\frac{\pi}{t}}$$

and

$$\varphi(t) \geqq \int_1^\infty e^{-tx^2}\,dx = \frac{1}{\sqrt{t}} \int_{\sqrt{t}}^\infty e^{-u^2}\,du$$

hence

$$\lim_{t\to 0} \sqrt{t}\,\varphi(t) = \tfrac{1}{2}\sqrt{\pi}.$$

The property (iii) is obvious. □

2.7. **Proposition.** *The following estimate holds*

$$g_t(\theta) \leqq \left(1 + 2\sqrt{\frac{\pi}{t}}\right) \exp\left(-\frac{\theta^2}{4t}\right) \quad \textit{for } \theta \in [-\pi, \pi],\ t>0.$$

In particular

$$\lim_{t\to 0} g_t(\theta) = 0 \quad \textit{for } \theta \in [-\pi,\pi] \setminus \{0\}.$$

We have also $g_t(0) \sim \sqrt{\frac{\pi}{t}}$ *for* $t \to 0$.

Proof. By Proposition 2.3 we get

$$g_t(\theta) = \sqrt{\frac{\pi}{t}} \exp\left(-\frac{\theta^2}{4t}\right)\left\{1 + \sum_{n=1}^\infty \left[\exp\left(-\frac{\pi n}{t}(\pi n + \theta)\right) + \exp\left(-\frac{\pi n}{t}(\pi n - \theta)\right)\right]\right\}.$$

It is clearly enough to prove the inequality for $\theta \in [0, \pi]$ and for such θ we find

$$\begin{aligned} g_t(\theta) &\leqq \sqrt{\frac{\pi}{t}} \exp\left(-\frac{\theta^2}{4t}\right)\left\{1 + \sum_{n=1}^\infty \left[\exp\left(-\frac{\pi^2 n^2}{t}\right) + \exp\left(-\frac{\pi^2 n(n-1)}{t}\right)\right]\right\} \\ &\leqq 2\sqrt{\frac{\pi}{t}} \exp\left(-\frac{\theta^2}{4t}\right)\left\{1 + \sum_{n=1}^\infty \exp\left(-\frac{\pi^2 n^2}{t}\right)\right\} \\ &= 2\sqrt{\frac{\pi}{t}} \exp\left(-\frac{\theta^2}{4t}\right)\left(1 + \varphi\left(\frac{\pi^2}{t}\right)\right). \end{aligned}$$

Using Lemma 2.6(i) the inequality follows. By Lemma 2.6(ii) we finally get that $g_t(0) \sim \sqrt{\frac{\pi}{t}}$ for $t \to 0$. □

For later use we put

$$\rho(t) = \frac{1}{\pi}\int_0^\pi \sqrt{g_t(\theta)}\,d\theta \qquad \text{for } t>0, \tag{7}$$

and the following result about the behavior of ρ will turn out to be important.

2.8. **Proposition.** *The function* ρ *has the properties*

(i) $0 < \rho(t) < 1$ *for* $t > 0$,

(ii) $\lim_{t\to 0} \rho(t) = 0$,

(iii) $1 - \rho(t) \sim \frac{1}{4} e^{-2t}$ *for* $t \to \infty$.

Proof. The inequality $\rho(t)<1$ follows from the Cauchy-Schwarz inequality. By Proposition 2.7 we get

$$\rho(t)\leqq\left(1+2\sqrt{\frac{\pi}{t}}\right)^{\frac{1}{2}}\frac{1}{\pi}\int_0^\pi e^{-\frac{\theta^2}{8t}}d\theta\leqq(8t+16\sqrt{t\pi})^{\frac{1}{2}}\frac{1}{\pi}\int_0^\infty e^{-u^2}du$$

which proves (ii).

For $t>\pi$ we have

$$|\varphi(t,\theta)|\leqq\varphi(t)\leqq\tfrac{1}{2}\sqrt{\frac{\pi}{t}}<\tfrac{1}{2},$$

and therefore

$$\sqrt{g_t(\theta)}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}2^n\varphi^n(t,\theta)\qquad\text{uniformly for }\theta\in\mathbb{R}$$

so that

$$\rho(t)-1=\sum_{n=2}^{\infty}\binom{\frac{1}{2}}{n}2^n\frac{1}{\pi}\int_0^\pi\varphi^n(t,\theta)d\theta.$$

Since

$$\lim_{t\to\infty}e^{tn}\varphi^n(t,\theta)=\cos^n\theta\qquad\text{uniformly for }\theta\in\mathbb{R}$$

we finally get

$$\lim_{t\to\infty}e^{2t}(\rho(t)-1)=\binom{\frac{1}{2}}{2}2^2\frac{1}{\pi}\int_0^\pi\cos^2\theta\,d\theta=-\tfrac{1}{4}.\quad\square$$

2.9. The resolvent $(\rho_\lambda)_{\lambda>0}$ for the Brownian semigroup on T has the following densities (cf. 1.8 and 1.10)

$$\tilde{\rho}_\lambda(\theta)=\int_0^\infty e^{-\lambda t}g_t(\theta)dt\qquad\text{for }\theta\in\mathbb{R}.\tag{8}$$

2.10. **Proposition.** *For each $\lambda>0$ the function $\tilde{\rho}_\lambda$ is a continuous function with the absolutely convergent Fourier series*

$$\tilde{\rho}_\lambda(\theta)=\sum_{n\in\mathbb{Z}}\frac{1}{n^2+\lambda}e^{in\theta}.\tag{9}$$

For $\theta\in[-\pi,\pi]$ we have

$$\tilde{\rho}_\lambda(\theta)=\frac{\pi}{\sqrt{\lambda}}\frac{\cosh((\pi-|\theta|)\sqrt{\lambda})}{\sinh(\pi\sqrt{\lambda})}.\tag{10}$$

Proof. By Lemma 2.6(i) it follows that

$$g_t(\theta)\leqq1+\sqrt{\frac{\pi}{t}}\qquad\text{for }\theta\in\mathbb{R},$$

so the dominated convergence theorem implies that $\tilde{\rho}_\lambda$ is continuous. We also have (cf. 1.8) $\hat{\rho}_\lambda(n)=1/(n^2+\lambda)$ for $n\in\mathbb{Z}$ and $\sum_{n\in\mathbb{Z}}(n^2+\lambda)^{-1}<\infty$, so the right hand side of (9) is a continuous density for ρ_λ and hence equal to $\tilde{\rho}_\lambda$.

By Proposition 2.3 we get

$$\tilde{\rho}_\lambda(\theta)=\sqrt{\pi}\sum_{n\in\mathbb{Z}}\int_0^\infty e^{-\lambda t}t^{-\frac{1}{2}}\exp\left(-\frac{(\theta+2\pi n)^2}{4t}\right)dt \qquad \text{for } \theta\in\mathbb{R},$$

and using

$$\int_0^\infty e^{-\lambda t}t^{-\frac{1}{2}}\exp\left(-\frac{a}{4t}\right)dt=\sqrt{\pi}\lambda^{-\frac{1}{2}}\exp(-\sqrt{a\lambda}) \qquad \text{for } a\geq 0,\ \lambda>0,$$

cf. [8] p. 146, we get

$$\tilde{\rho}_\lambda(\theta)=\pi\lambda^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}}\exp(-\sqrt{\lambda}\,|\theta+2\pi n|).$$

For $\theta\in[-\pi,\pi]$ it is easy to reduce this sum to the expression in (10). □

2.11. **Corollary.** *Let* $a\in[0,1]$. *Then the following functions are completely monotone on* $]0,\infty[$

(i) $\lambda\to\dfrac{1}{\sqrt{\lambda}}\dfrac{\cosh(a\sqrt{\lambda})}{\sinh(\sqrt{\lambda})}$,

(ii) $\lambda\to\dfrac{\sinh(a\sqrt{\lambda})}{\sinh(\sqrt{\lambda})}$.

Proof. The function $\lambda\to\tilde{\rho}_\lambda(\theta)$ is completely monotone for fixed θ because it is the Laplace transform of the positive function $t\to g_t(\theta)$. It follows that (i) is completely monotone.

For $\theta\in[0,\pi]$ we find

$$\tilde{\rho}'_\lambda(\theta)=-\pi\frac{\sinh((\pi-\theta)\sqrt{\lambda})}{\sinh(\pi\sqrt{\lambda})}=\int_0^\infty e^{-\lambda t}g'_t(\theta)\,dt,$$

and since $g_t(\theta)$ is decreasing for $\theta\in[0,\pi]$ it follows that $\lambda\to-\tilde{\rho}'_\lambda(\theta)$ is completely monotone and hence (ii) is completely monotone. □

2.12. *Remark.* The classical infinite product expansion for $\sinh x$ can also be used to prove that the function (ii) of Corollary 2.11 is completely monotone. In the same way it can also be proved that the function

$$\lambda\to\frac{\cosh(a\sqrt{\lambda})}{\cosh(\sqrt{\lambda})}$$

is completely monotone when $a\in[0,1]$.

The infinitesimal generator (A,D_A) for the semigroup $(P_t)_{t>0}$ on $C(T)$ induced by the Brownian semigroup, cf. 1.5, can easily be identified.

2.13. **Proposition.** *The domain* D_A *contains* $C^2(T)$ *and for* $f\in C^2(T)$ *we have* $Af=\dfrac{d^2f}{d\theta^2}$.

Proof. We know by Theorem 1.7 that $g_t \in D_A$ and that $A g_t = \frac{d}{dt} g_t$, but since $\frac{d^2}{d\theta^2} g_t = \frac{d}{dt} g_t$ we find

$$A g_t = \frac{d^2}{d\theta^2} g_t.$$

It is easy to see that $D_A * C(T) \subseteq D_A$ and $A(f*g) = (Af)*g$ for $f \in D_A$ and $g \in C(T)$. For $f \in C^2(T)$ we get in particular

$$A(g_t * f) = (A g_t) * f = \left(\frac{d^2}{d\theta^2} g_t\right) * f = g_t * \frac{d^2 f}{d\theta^2},$$

hence

$$\lim_{t \to 0} A(g_t * f) = \frac{d^2 f}{d\theta^2} \quad (\text{and } \lim_{t \to 0} g_t * f = f)$$

in $C(T)$. The operator (A, D_A) being closed we get $f \in D_A$ and $Af = \frac{d^2 f}{d\theta^2}$. $\square$

3. *The Brownian Semigroup on T^p*

3.1. The Brownian semigroup $(\mu_t)_{t>0}$ on T^p, $p \geq 1$, is defined as the direct product of p copies of the Brownian semigroup on T. The measure μ_t has therefore a density $g_t^{[p]}$ with respect to Haar measure on T^p given as

$$g_t^{[p]}(\theta) = \prod_{i=1}^{p} g_t(\theta_i) \qquad \text{for } \theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p,$$

where g_t is defined in 2.2. Here and in the following we describe functions on T^p as functions on $\mathbb{R}^p$ periodic in each variable with period 2π.

By reasons which become clear later we want to consider a slight generalization of this semigroup.

Let $a = (a_1, \ldots, a_p)$ be a p-tuple of positive numbers and consider the convolution semigroup $(\mu_t^a)_{t>0}$ on T^p defined by

$$\mu_t^a = \mu_{ta_1} \otimes \cdots \otimes \mu_{ta_p} \qquad \text{for } t > 0, \tag{1}$$

where $(\mu_t)_{t>0}$ is the Brownian semigroup on T. The measure μ_t^a has the following density with respect to Haar measure on T^p

$$g_t^a(\theta) = \prod_{i=1}^{p} g_{ta_i}(\theta_i) \qquad \text{for } \theta \in \mathbb{R}^p. \tag{2}$$

The Fourier series for g_t^a is

$$g_t^a(\theta) = \sum_{n \in \mathbb{Z}^p} \exp(-t(a_1 n_1^2 + \cdots + a_p n_p^2))\, e^{i\langle n, \theta\rangle} \tag{3}$$

which converges absolutely and uniformly. Here we use the standard notation

$$\langle n, \theta \rangle = n_1 \theta_1 + \cdots + n_p \theta_p \qquad \text{for } n \in \mathbb{Z}^p \quad \text{and} \quad \theta \in \mathbb{R}^p.$$

The negative definite function associated with $(\mu_t^a)_{t>0}$ is the non-negative quadratic form

$$q(n)=a_1 n_1^2+\cdots+a_p n_p^2, \tag{4}$$

so $(\mu_t^a)_{t>0}$ is of local type, cf. 1.6.

Introducing $\alpha=\min(a_1,\ldots,a_p)$ and $\beta=\max(a_1,\ldots,a_p)$ we get from 2.6 and 2.7 that

$$g_t^a(\theta)\leqq\left(1+\sqrt{\frac{\pi}{t\alpha}}\right)^p \quad \text{for } \theta\in\mathbb{R}^p \text{ and } t>0 \tag{5}$$

and

$$g_t^a(\theta)\leqq\left(1+2\sqrt{\frac{\pi}{t\alpha}}\right)^p \exp\left(-\frac{\|\theta\|^2}{4t\beta}\right) \quad \text{for } \theta\in[-\pi,\pi]^p \text{ and } t>0, \tag{6}$$

so in particular

$$\lim_{t\to 0} g_t^a(\theta)=0 \quad \text{for } \theta\in[-\pi,\pi]^p\setminus\{0\}.$$

The resolvent $(\rho_\lambda^a)_{\lambda>0}$ for $(\mu_t^a)_{t>0}$ has densities $(\tilde{\rho}_\lambda^a)_{\lambda>0}$ given by (cf. 1.10)

$$\tilde{\rho}_\lambda^a(\theta)=\int_0^\infty e^{-\lambda t} g_t^a(\theta)\,dt \quad \text{for } \theta\in\mathbb{R}^p. \tag{7}$$

3.2. **Proposition.** *For each $\lambda>0$ the function $\tilde{\rho}_\lambda^a$ is lower semicontinuous on T^p and continuous on $T^p\setminus\{0\}$, where 0 denotes the neutral element in T^p. For $p\geqq 2$ we have $\tilde{\rho}_\lambda^a(0)=\infty$. Furthermore $\tilde{\rho}_\lambda^a\in L^2(T^p)$ if and only if $p\leqq 3$.*

Proof. From (5) it follows that

$$\theta\to\int_1^\infty e^{-\lambda t} g_t^a(\theta)\,dt$$

is finite and continuous on $\mathbb{R}^p$, and from (6) it follows that

$$\theta\to\int_0^1 e^{-\lambda t} g_t^a(\theta)\,dt$$

is finite and continuous on $[-\pi,\pi]^p\setminus\{0\}$. Therefore $\tilde{\rho}_\lambda^a$ is finite and continuous on $T^p\setminus\{0\}$. In the case $p=1$ $\tilde{\rho}_\lambda^a$ is finite and continuous at every point of T (cf. 2.10), but since

$$g_t^a(0)\sim\prod_{i=1}^p\sqrt{\frac{\pi}{t a_i}} \quad \text{for } t\to 0,$$

cf. 2.7, it follows that $\tilde{\rho}_\lambda^a(0)=\infty$ for $p\geqq 2$.

The Fourier transform of the measure ρ_λ^a is given by $\hat{\rho}_\lambda^a=1/(q+\lambda)$ where q is defined in (4), and this function is square summable on $\mathbb{Z}^p$ if and only if $p\leqq 3$, so the last assertion follows from Plancherel's theorem. □

By the same method of proof as in Proposition 2.13 it is easy to obtain the following result:

3.3. **Proposition.** *Let (A, D_A) denote the infinitesimal generator for the semigroup on $C(T^p)$ induced by $(\mu_t^a)_{t>0}$. Then $C^2(T^p) \subseteq D_A$ and*

$$Af = \sum_{k=1}^{p} a_k \frac{\partial^2 f}{\partial \theta_k^2} \quad \textit{for } f \in C^2(T^p).$$

4. Brownian Semigroups on T^∞.

4.1. The product of countably many copies of a set X is denoted X^∞. The set T^∞ is a compact abelian group with respect to the ordinary product structures. The neutral element of T^∞ is denoted 0. The normalized Haar measure on T^∞ is the product of the normalized Haar measures on the countably many factors T.

The subgroup of $\mathbb{Z}^\infty$ consisting of all sequences in $\mathbb{Z}^\infty$ which are eventually zero is denoted $\mathbb{Z}^{(\infty)}$. The dual group of T^∞ can be identified with $\mathbb{Z}^{(\infty)}$ in the following way, cf. [15]: Each $n = (n_1, n_2, \ldots, n_p, 0, \ldots) \in \mathbb{Z}^{(\infty)}$ determines a character γ_n on T^∞ namely

$$\gamma_n(z) = \prod_{k=1}^{p} z_k^{n_k} \quad \text{for } z = (z_1, z_2, \ldots) \in T^\infty.$$

The mapping $n \to \gamma_n$ is an isomorphism of $\mathbb{Z}^{(\infty)}$ onto the dual group of T^∞.

We will often describe functions on T^∞ as functions on $\mathbb{R}^\infty$ which are periodic with $(2\pi\mathbb{Z})^\infty$ as periodicity group.

4.2. Let $(\mu_t)_{t>0}$ be the Brownian semigroup on T. One could define the Brownian semigroup $(\mu_t^{[\infty]})_{t>0}$ on T^∞ as the infinite product

$$\mu_t^{[\infty]} = \bigotimes_{k=1}^{\infty} \mu_t \quad \text{for } t > 0$$

of countably many copies of μ_t. However, by the theorem of Kakutani [12], $\mu_t^{[\infty]}$ is singular with respect to Haar measure for every $t > 0$, and therefore the convolution semigroup $(\mu_t^{[\infty]})_{t>0}$ does not lead to a satisfactory potential theory on T^∞.

We consider therefore a sequence $\mathscr{A} = (a_1, a_2, \ldots)$ of positive numbers ($a_i > 0$ for all i) and define for each $t > 0$ the product measure

$$\mu_t^{\mathscr{A}} = \bigotimes_{k=1}^{\infty} \mu_{ta_k} \tag{1}$$

and will study the convolution semigroup $(\mu_t^{\mathscr{A}})_{t>0}$ and its dependence on the sequence $\mathscr{A}$. We call all these semigroups on T^∞ for *Brownian semigroups* on T^∞.

In order to see that $(\mu_t^{\mathscr{A}})_{t>0}$ is indeed a convolution semigroup on T^∞ we find the Fourier transform of $\mu_t^{\mathscr{A}}$:

$$\hat{\mu}_t^{\mathscr{A}}(n) = \prod_{k=1}^{\infty} \hat{\mu}_{ta_k}(n_k) = \exp\left(-t \sum_{k=1}^{\infty} a_k n_k^2\right) \quad \text{for } n \in \mathbb{Z}^{(\infty)}. \tag{2}$$

Note that the infinite product and sum above are "finite" because $n \in \mathbb{Z}^{(\infty)}$ is eventually zero.

From (2) follows easily that $(\mu_t^{\mathscr{A}})_{t>0}$ is a symmetric convolution semigroup of probability measures on T^∞, and the associated negative definite function is

$$q(n) = \sum_{k=1}^{\infty} a_k n_k^2 \quad \text{for } n \in \mathbb{Z}^{(\infty)}.$$

Since q is a non-negative quadratic form on $\mathbb{Z}^{(\infty)}$ we get that $(\mu_t^{\mathscr{A}})_{t>0}$ is of *local type*, and since $q(n)>0$ for $n \neq 0$ we get by 1.3(5) that $\operatorname{supp}(\mu_t^{\mathscr{A}}) = T^\infty$ for all $t>0$.[3] (This contradicts of course not that $\mu_t^{\mathscr{A}}$ is concentrated on a set of Haar measure zero in the case $\mathscr{A} = (1, 1, \ldots)$.)

By means of Kakutani's theorem we can determine for which sequences $\mathscr{A}$, $\mu_t^{\mathscr{A}}$ is absolutely continuous with respect to Haar measure on T^∞.

4.3. **Theorem.** *Let $t>0$ be fixed. Then $\mu_t^{\mathscr{A}}$ defined by* (1) *is absolutely continuous with respect to Haar measure on T^∞ if and only if*

$$\sum_{k=1}^{\infty} e^{-2ta_k} < \infty.$$

In the affirmative case the infinite product

$$\prod_{k=1}^{\infty} g_{ta_k}(\theta_k)$$

converges almost everywhere to a density for $\mu_t^{\mathscr{A}}$.

Proof. By Kakutani's theorem [12] we know that $\mu_t^{\mathscr{A}}$ is absolutely continuous with respect to Haar measure on T^∞ if and only if

$$\prod_{k=1}^{\infty} \rho(t a_k) > 0,$$

where ρ is the function introduced in § 2 formula (7). This is however equivalent with

$$\sum_{k=1}^{\infty} (1 - \rho(t a_k)) < \infty,$$

which in turn is equivalent with

$$\sum_{k=1}^{\infty} e^{-2ta_k} < \infty$$

on account of the properties of ρ proved in Proposition 2.8. The rest of Theorem 4.3 follows from Kakutani's theorem. The n'th partial product

$$\prod_{k=1}^{n} g_{ta_k}(\theta_k)$$

shall be considered as a function on T^∞ which depends only on the first n variables. □

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[3] This follows also from $\operatorname{supp}(\mu_t) = T$ for all $t>0$.

4.4. *Remark.* Defining

$$t_0=\inf\left\{t\in]0,\infty[\;\middle|\;\sum_{k=1}^{\infty}e^{-2ta_k}<\infty\right\},$$

it follows from the above result and Kakutani's theorem that $\mu_t^{\mathscr{A}}$ is singular for $0<t<t_0$ and $\mu_t^{\mathscr{A}}$ is absolutely continuous for $t_0<t$. Whether $\mu_{t_0}^{\mathscr{A}}$ is singular or absolutely continuous in the case $0<t_0<\infty$ depends on $\mathscr{A}$.

If we put $a_k=\frac{1}{2\alpha}\log(k+1)$ for $k=1,2,\dots$, where $\alpha>0$, we get $t_0=\alpha$ and $\mu_{t_0}^{\mathscr{A}}$ is singular. If we put $a_k=\frac{1}{2\alpha}\log(k+1)+\frac{1}{\alpha}\log\log(k+2)$ for $k=1,2,\dots$, where $\alpha>0$, we get $t_0=\alpha$ and $\mu_{t_0}^{\mathscr{A}}$ is absolutely continuous.

Convolution semigroups $(\mu_t)_{t>0}$ on $\mathbb{R}$ with the property of being singular for $t<t_0$ and absolutely continuous for $t>t_0$ have been constructed by Rubin and Stratton, cf. [16].

It is also possible to determine the sequences $\mathscr{A}$ for which $\mu_t^{\mathscr{A}}$ has a continuous density. We begin with a lemma.

4.5. **Lemma.** *For every sequence $\mathscr{A}$ and every $t>0$ we have*

$$\sum_{n\in\mathbb{Z}^{(\infty)}}\hat{\mu}_t^{\mathscr{A}}(n)=\prod_{k=1}^{\infty}g_{ta_k}(0)\leqq\infty.$$

Proof. We have

$$\sum_{n_1,\dots,n_p\in\mathbb{Z}}\exp\left(-t(a_1n_1^2+\dots+a_pn_p^2)\right)=\prod_{k=1}^{p}g_{ta_k}(0)$$

and the supremum over $p\in\mathbb{N}$ of the left and right hand side of this equation is respectively the left and right hand side of the equation of the lemma. □

4.6. **Theorem.** *For fixed $t>0$ the following conditions are equivalent,*

(i) *$\mu_t^{\mathscr{A}}$ has a continuous density $g_t^{\mathscr{A}}$ with respect to Haar measure on T^∞,*

(ii) $\sum_{n\in\mathbb{Z}^{(\infty)}}\hat{\mu}_t^{\mathscr{A}}(n)<\infty$,

(iii) $\sum_{k=1}^{\infty}e^{-ta_k}<\infty$.

If the condition (i)–(iii) *are satisfied we have*

$$g_t^{\mathscr{A}}(\theta)=\prod_{k=1}^{\infty}g_{ta_k}(\theta_k)\quad\text{for }\theta\in\mathbb{R}^\infty,\tag{3}$$

and the convergence is uniform for $\theta\in\mathbb{R}^\infty$.

Proof. The equivalence of (i) and (ii) is a special case of Theorem 1.7. By Lemma 4.5 (ii) is satisfied if and only if the infinite product

$$\prod_{k=1}^{\infty}g_{ta_k}(0)$$

is convergent, and since $g_{ta_k}(0)=1+2\varphi(ta_k)$ this is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \varphi(ta_k),$$

which by Lemma 2.6 is equivalent to (iii).

If the conditions (i)–(iii) are satisfied we know from Theorem 4.3 that

$$\prod_{k=1}^{\infty} g_{ta_k}(\theta_k)$$

converges almost everywhere to a density for $\mu_t^{\mathscr{A}}$, so it suffices to prove that this product converges uniformly for $\theta \in \mathbb{R}^{\infty}$. For $\theta \in \mathbb{R}^{\infty}$ we find

$$\begin{aligned}\left|\prod_{k=1}^{n+p} g_{ta_k}(\theta_k) - \prod_{k=1}^{n} g_{ta_k}(\theta_k)\right| &= \prod_{k=1}^{n} g_{ta_k}(\theta_k)\left|\prod_{k=n+1}^{n+p} g_{ta_k}(\theta_k) - 1\right| \\ &\leq \prod_{k=1}^{n} g_{ta_k}(0)\left|\prod_{k=n+1}^{n+p} (1+2\varphi(ta_k, \theta_k)) - 1\right| \\ &\leq \prod_{k=1}^{n} g_{ta_k}(0)\left(\prod_{k=n+1}^{n+p} (1+2\varphi(ta_k)) - 1\right) \qquad (*) \\ &= \prod_{k=1}^{n+p} g_{ta_k}(0) - \prod_{k=1}^{n} g_{ta_k}(0),\end{aligned}$$

where the inequality $(*)$ can be seen by carring out the multiplication. By Lemma 4.5 the infinite product

$$\prod_{k=1}^{\infty} g_{ta_k}(0)$$

is convergent, so the uniform convergence follows. □

4.7. *Remark.* With the notation of the Remark 4.4 we find that $\mu_t^{\mathscr{A}}$ is absolutely continuous but without a continuous density for $t \in]t_0, 2t_0[$, and $\mu_t^{\mathscr{A}}$ has a continuous density for $t > 2t_0$.

Since $g_t(0)$ is a decreasing function of t, the inequality in the proof of Theorem 4.6 also shows that the convergence of (3) is uniform in $\theta \in \mathbb{R}^{\infty}$ and $t \in [t_0 + \varepsilon, \infty[$ for every $\varepsilon > 0$. Therefore $g_t^{\mathscr{A}}(\theta)$ is a continuous function of $(t, \theta) \in]t_0, \infty[\times \mathbb{R}^{\infty}$. In particular we have the following Corollary which also follows from Theorem 1.7.

4.8. **Corollary.** *Suppose that* $\mathscr{A}$ *satisfies*

$$\sum_{k=1}^{\infty} e^{-ta_k} < \infty \qquad \textit{for all } t > 0.$$

Then the function

$$g_t^{\mathscr{A}}(\theta) = \prod_{k=1}^{\infty} g_{ta_k}(\theta_k)$$

is continuous for $(t, \theta) \in]0, \infty[\times \mathbb{R}^{\infty}$.

The sequence $\mathscr{A}$ given by $a_k = k^\varepsilon$ for $k=1,2,\dots$ satisfies the condition of Corollary 4.8 for every $\varepsilon>0$.

We shall now consider a condition on the sequence $\mathscr{A}$ which implies that all the measures $\mu_t^{\mathscr{A}}$ have continuous densities and which furthermore allows us to estimate these densities. The condition is the following

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{\sqrt{a_k}} < \infty. \tag{4}$$

4.9. **Theorem.** *Suppose that $\mathscr{A}$ verifies* (4). *Then every $\mu_t^{\mathscr{A}}$ has a continuous density $g_t^{\mathscr{A}}$ given as*

$$g_t^{\mathscr{A}}(\theta) = \prod_{k=1}^{\infty} g_{ta_k}(\theta_k) \quad \textit{for } \theta \in \mathbb{R}^\infty,$$

and the following estimates hold

$$g_t^{\mathscr{A}}(\theta) \leqq g_t^{\mathscr{A}}(0) \leqq \exp\left(\alpha \sqrt{\frac{\pi}{t}}\right) \quad \textit{for } \theta\in\mathbb{R}^\infty \quad \textit{and} \quad t>0. \tag{5}$$

$$g_t^{\mathscr{A}}(\theta) \leqq \exp\left(2\alpha\sqrt{\frac{\pi}{t}}\right) \exp\left(-\frac{1}{4t}\sum_{k=1}^{\infty}\frac{\theta_k^2}{a_k}\right) \quad \textit{for } \theta\in[-\pi,\pi]^\infty \quad \textit{and} \quad t>0. \tag{6}$$

In particular we have

$$\lim_{t\to 0} g_t^{\mathscr{A}}(\theta) = 0 \quad \textit{for } \theta\in[-\pi,\pi]^\infty \setminus \{0\}.$$

Proof. For $t>0$ there exists a constant $x_0 = x_0(t)$ such that $e^{-tx} \leqq 1/\sqrt{x}$ for $x \geqq x_0$, and therefore (4) implies that $\sum_{k=1}^{\infty} e^{-ta_k} < \infty$ for all $t>0$. It follows from Theorem 4.6 that $\mu_t^{\mathscr{A}}$ has a continuous density $g_t^{\mathscr{A}}$ given by (3), so it is clear that

$$g_t^{\mathscr{A}}(\theta) \leqq g_t^{\mathscr{A}}(0) = \prod_{k=1}^{\infty} g_{ta_k}(0) = \prod_{k=1}^{\infty}(1+2\varphi(ta_k)).$$

Using the elementary inequality $1+x \leqq e^x$ for $x \geqq 0$ we get

$$\prod_{k=1}^{\infty}(1+2\varphi(ta_k)) \leqq \exp\left(2\sum_{k=1}^{\infty}\varphi(ta_k)\right) \leqq \exp\left(\alpha\sqrt{\frac{\pi}{t}}\right),$$

where we have used Lemma 2.6(i) and the number α from (4). By Proposition 2.7 we have for $k=1,2,\dots$

$$g_{ta_k}(\theta_k) \leqq \left(1+2\left(\frac{\pi}{ta_k}\right)^{\frac{1}{2}}\right) \exp\left(-\frac{\theta_k^2}{4ta_k}\right) \quad \text{if } \theta_k\in[-\pi,\pi],$$

and for $\theta\in[-\pi,\pi]^\infty$ we therefore get

$$\begin{aligned} g_t^{\mathscr{A}}(\theta) &\leqq \prod_{k=1}^{\infty}\left(1+2\left(\frac{\pi}{ta_k}\right)^{\frac{1}{2}}\right)\exp\left(-\frac{\theta_k^2}{4ta_k}\right) \\ &\leqq \exp\left(2\alpha\sqrt{\frac{\pi}{t}}\right)\exp\left(-\frac{1}{4t}\sum_{k=1}^{\infty}\frac{\theta_k^2}{a_k}\right). \end{aligned}$$

Note that (4) implies the convergence of the series $\sum 1/a_k$. If $\theta\in[-\pi,\pi]^\infty\setminus\{0\}$ then

$$\sum_{k=1}^{\infty}\frac{\theta_k^2}{a_k}>0,$$

and since

$$\lim_{t\to 0}\exp\left(\frac{A}{\sqrt{t}}-\frac{B}{t}\right)=0 \qquad \text{for } A,B>0,$$

the last assertion follows. □

4.10. The resolvent of the Brownian semigroup $(\mu_t^{\mathscr{A}})_{t>0}$ is the family of measures $(\rho_\lambda^{\mathscr{A}})_{\lambda>0}$ defined by the vector integral (cf. 1.8)

$$\rho_\lambda^{\mathscr{A}}=\int_0^\infty e^{-\lambda t}\mu_t^{\mathscr{A}}\,dt \qquad \text{for } \lambda>0.$$

If the measure $\mu_t^{\mathscr{A}}$ is absolutely continuous for all $t>0$ and hence having a continuous density $g_t^{\mathscr{A}}$ for all $t>0$ by 4.3 and 4.6, we know by 1.10 that $\rho_\lambda^{\mathscr{A}}$ has a lower semicontinuous density $\tilde{\rho}_\lambda^{\mathscr{A}}$ given by

$$\tilde{\rho}_\lambda^{\mathscr{A}}(\theta)=\int_0^\infty e^{-\lambda t}g_t^{\mathscr{A}}(\theta)\,dt \qquad \text{for } \theta\in\mathbb{R}^\infty. \tag{7}$$

If $\mathscr{A}$ satisfies (4) even more can be proved.

4.11. **Theorem.** *Suppose that $\mathscr{A}$ verifies condition* (4) *and let $\lambda>0$ be fixed. Then the function $\tilde{\rho}_\lambda^{\mathscr{A}}$ given by* (7) *is finite and continuous on $T^\infty\setminus\{0\}$ and*

$$\lim_{\theta\to 0}\tilde{\rho}_\lambda^{\mathscr{A}}(\theta)=\tilde{\rho}_\lambda^{\mathscr{A}}(0)=\infty. \tag{8}$$

Proof. The inequality

$$g_t^{\mathscr{A}}(0)=\prod_{k=1}^{\infty}g_{ta_k}(0)\geqq g_{ta_1}(0)\,g_{ta_2}(0)$$

together with the asymptotic formula (cf. 2.7)

$$g_{ta_1}(0)\,g_{ta_2}(0)\sim\frac{\pi}{\sqrt{a_1a_2}}\frac{1}{t} \qquad \text{for } t\to 0$$

show that $\tilde{\rho}_\lambda^{\mathscr{A}}(0)=\infty$, and since $\tilde{\rho}_\lambda^{\mathscr{A}}$ is lower semicontinuous (8) follows. For the rest of the proof it is convenient to put

$$h_1(\theta)=\int_0^1 e^{-\lambda t}g_t^{\mathscr{A}}(\theta)\,dt \quad\text{and}\quad h_2(\theta)=\int_1^\infty e^{-\lambda t}g_t^{\mathscr{A}}(\theta)\,dt.$$

By (5) we have

$$g_t^{\mathscr{A}}(\theta)\leqq\exp(\alpha\sqrt{\pi}) \qquad \text{for } \theta\in\mathbb{R}^\infty \quad\text{and}\quad t\geqq 1,$$

so it follows by the dominated convergence theorem that h_2 is finite and continuous on T^∞.

We next prove that h_1 is finite and continuous on $T^\infty \setminus \{0\}$. Let

$$\theta \in [-\pi, \pi]^\infty \setminus \{0\}$$

be given and let $\theta^{(n)} \in [-\pi, \pi]^\infty$ be a sequence such that $\lim_{n\to\infty} \theta^{(n)} = \theta$ in $[-\pi, \pi]^\infty$, i.e. $\lim_{n\to\infty} \theta_k^{(n)} = \theta_k$ for every $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} 1/a_k < \infty$ it is clear that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(\theta_k^{(n)})^2}{a_k} = \sum_{k=1}^{\infty} \frac{\theta_k^2}{a_k} = A > 0$$

so there exists a number $n_0 \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \frac{(\theta_k^{(n)})^2}{a_k} \geqq \tfrac{1}{2} A \qquad \text{for } n \geqq n_0.$$

By (6) we then have for $n \geqq n_0$ and $t > 0$

$$g_t^{\mathscr{A}}(\theta^{(n)}) \leqq \exp\left(2\alpha \sqrt{\frac{\pi}{t}}\right) \exp\left(-\frac{A}{8t}\right).$$

The expression on the right side of this inequality tends to zero for $t \to 0$, in particular it is integrable over $]0, 1[$, so by the dominated convergence theorem we get

$$\lim_{n\to\infty} h_1(\theta^{(n)}) = h_1(\theta) < \infty. \quad \square$$

4.12. *Conclusion.* In the previous sections we have proved the following: For every sequence $\mathscr{A}$ of positive numbers satisfying $\sum_{k=1}^{\infty} 1/\sqrt{a_k} < \infty$ the Brownian semigroup $(\mu_t^{\mathscr{A}})_{t>0}$ satisfies the conditions of Theorem 1.13. For every such $\mathscr{A}$ and every $\lambda > 0$ there exists a translation invariant harmonic sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ on T^∞ such that $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ is a $\mathfrak{P}$-Brelot space.

4.13. Let $\mathscr{A}$ be an arbitrary sequence of positive numbers, and let $(P_t)_{t>0}$ be the semigroup of operators on $C(T^\infty)$ induced by $(\mu_t^{\mathscr{A}})_{t>0}$, cf. 1.5. We shall now describe the infinitesimal generator (A, D_A) for $(P_t)_{t>0}$, and since $(\mu_t^{\mathscr{A}})_{t>0}$ is of local type we know that (A, D_A) is a local operator in the sense of 1.5, so it is to expect that A is a differential operator in some sense.

For $p \in \mathbb{N}$ we define $\pi_p\colon T^\infty \to T^p$ by

$$\pi_p(z) = (z_1, \ldots, z_p) \qquad \text{for } z = (z_1, z_2, \ldots) \in T^\infty.$$

For $f \in C(T^p)$ the function $f \circ \pi_p \in C(T^\infty)$ depends only on the first p variables.

4.14. **Proposition.** *For every $p \in \mathbb{N}$ and every $f \in C^2(T^p)$ we have $f \circ \pi_p \in D_A$ and*

$$A(f \circ \pi_p) = \sum_{k=1}^{p} a_k \frac{\partial^2 f}{\partial \theta_k^2} \circ \pi_p.$$

Proof. For $f \in C^2(T^p)$ we easily find

$$\mu_t^{\mathscr{A}} * (f \circ \pi_p) = \left[\left(\bigotimes_{k=1}^{p} \mu_{ta_k}\right) * f\right] \circ \pi_p,$$

and by Proposition 3.3 we have

$$\lim_{t\to 0}\frac{1}{t}\left[\left(\bigotimes_{k=1}^{p}\mu_{ta_k}\right)*f-f\right]=\sum_{k=1}^{p}a_k\frac{\partial^2 f}{\partial\theta_k^2}$$

uniformly on T^p, hence

$$\lim_{t\to 0}\frac{1}{t}\left[\mu_t^{\mathscr{A}}*(f\circ\pi_p)-f\circ\pi_p\right]=\sum_{k=1}^{p}a_k\frac{\partial^2 f}{\partial\theta_k^2}\circ\pi_p$$

uniformly on T^∞. □

4.15. In the preceding sections we have seen that certain growth conditions on the sequence $\mathscr{A}$ imply that the convolution semigroup $(\mu_t^{\mathscr{A}})_{t>0}$ has nice properties. In § 8 we need an estimate which essentially deals with another convolution semigroup on T^∞, namely the analogue of the Cauchy semigroup.

The function $\psi\colon \mathbb{Z}^{(\infty)}\to\mathbb{R}$ defined by

$$\psi(n)=\left(\sum_{k=1}^{\infty}a_k n_k^2\right)^{\frac{1}{2}}\qquad\text{for } n\in\mathbb{Z}^{(\infty)}\tag{9}$$

is negative definite as the square root of a negative definite function (cf. [2], p. 45). There exists consequently a symmetric convolution semigroup $(\sigma_t^{\mathscr{A}})_{t>0}$ on T^∞, called the Cauchy semigroup, such that

$$\hat{\sigma}_t^{\mathscr{A}}(n)=e^{-t\psi(n)}\qquad\text{for } t>0\quad\text{and}\quad n\in\mathbb{Z}^{(\infty)}.$$

We want to find a condition on $\mathscr{A}$ which ensures that all the measures $\sigma_t^{\mathscr{A}}$, $t>0$, have continuous densities. By Theorem 1.7 this is equivalent with finding a condition which ensures that

$$\sum_{n\in\mathbb{Z}^{(\infty)}}e^{-t\psi(n)}<\infty\qquad\text{for all } t>0,$$

and this is exactly what is needed in § 8.

4.16. **Proposition.** *If $\mathscr{A}$ satisfies the condition*

$$\sum_{k=1}^{\infty}\frac{(\sqrt{2})^k}{\sqrt{a_k}}<\infty,\tag{10}$$

then

$$\sum_{n\in\mathbb{Z}^{(\infty)}}e^{-t\psi(n)}<\infty\qquad\textit{for all } t>0,$$

where ψ is given by (9).

Proof. We put

$$\sigma(a_1,\dots,a_p;t)=\sum_{n\in\mathbb{Z}^p}\exp\left(-t(a_1 n_1^2+\cdots+a_p n_p^2)^{\frac{1}{2}}\right).\tag{11}$$

To estimate (11) we use the inequalities

$$\sqrt{a+b}\geqq\frac{1}{\sqrt{2}}(\sqrt{a}+\sqrt{b})\qquad\text{for } a,b\geqq 0,$$

and

$$\sum_{n\in\mathbb{Z}} e^{-t|n|} \leqq 1+\frac{2}{t} \quad \text{for } t>0,$$

and procede in the following way:

$$\begin{aligned}
\sigma(a_1, \ldots, a_p; t) &\leqq \sum_{n_1\in\mathbb{Z}} \exp\left(-\frac{t\sqrt{a_1}}{\sqrt{2}}|n_1|\right)\sigma\left(a_2, \ldots, a_p; \frac{t}{\sqrt{2}}\right)\\
&\leqq \left(1+\frac{2\sqrt{2}}{t\sqrt{a_1}}\right)\sigma\left(a_2, \ldots, a_p; \frac{t}{\sqrt{2}}\right)\\
&\leqq \left(1+\frac{2\sqrt{2}}{t\sqrt{a_1}}\right)\left(1+\frac{2(\sqrt{2})^2}{t\sqrt{a_2}}\right)\sigma\left(a_3, \ldots, a_p; \frac{t}{(\sqrt{2})^2}\right)\\
&\leqq \prod_{k=1}^{p-1}\left(1+\frac{2(\sqrt{2})^k}{t\sqrt{a_k}}\right)\sigma\left(a_p; \frac{t}{(\sqrt{2})^{p-1}}\right) \leqq \prod_{k=1}^{p}\left(1+\frac{2(\sqrt{2})^k}{t\sqrt{a_k}}\right).
\end{aligned}$$

We therefore get

$$\sum_{n\in\mathbb{Z}^{(\infty)}} e^{-t\psi(n)} = \sup_{p\in\mathbb{N}} \sigma(a_1, \ldots, a_p; t) \leqq \prod_{k=1}^{\infty}\left(1+\frac{2(\sqrt{2})^k}{t\sqrt{a_k}}\right)$$

which is finite because the infinite series in (10) is convergent. $\square$

4.17. **Corollary.** *If* $\mathscr{A}$ *satisfies* (10) *then for all* $t>0$, $\lambda\geqq 0$ *and* $p\geqq 0$

$$\sum_{n\in\mathbb{Z}^{(\infty)}} \exp\left(-t\left(\lambda+\sum_{k=1}^{\infty} a_{p+k} n_k^2\right)^{\frac{1}{2}}\right) < \infty$$

Proof. The Corollary follows because the sum in question is majorized by

$$\sum_{n\in\mathbb{Z}^{(\infty)}} e^{-t\psi(n)}. \quad \square$$

4.18. *Remark.* It is clear that (10) implies (4). We have not been able to decide whether (4) suffices to ensure the convergence of

$$\sum_{n\in\mathbb{Z}^{(\infty)}} e^{-t\psi(n)}$$

for all $t>0$.

Part II. Harmonic Functions on T^∞

5. Harmonic Functions on T^p

5.1. In this paragraph $p\in\mathbb{N}$ is fixed and $a=(a_1, a_2, \ldots, a_p)$ is a p-tuple of numbers all >0. For every $\lambda\geqq 0$ we define a differential operator L_λ on $\mathbb{R}^p$ by

$$L_\lambda = \sum_{k=1}^{p} a_k \frac{\partial^2}{\partial\theta_k^2} - \lambda. \tag{1}$$

Let V be an open subset of $\mathbb{R}^p$. A function $f \in C^\infty(V)$ is called L_λ-harmonic if $L_\lambda f = 0$ in V. The sheaf of L_λ-harmonic functions on $\mathbb{R}^p$ turns $\mathbb{R}^p$ into a Brelot space. We refer the reader to [10] for information about the properties of this harmonic space. We will often make use of the boundary minimum principle for L_λ-superharmonic functions.

We can also consider L_λ as a differential operator on the differentiable manifold T^p by the following device:

Let $\gamma: \mathbb{R}^p \to T^p$ be the mapping

$$\gamma(\theta_1, \ldots, \theta_p) = (e^{i\theta_1}, \ldots, e^{i\theta_p}) \quad \text{for } (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p.$$

For an open subset $U \subseteq T^p$ and a function $f \in C^\infty(U)$ we consider $L_\lambda(f \circ \gamma)$ on $\gamma^{-1}(U)$, and it is easy to see that there exists a uniquely determined function $L_\lambda f: U \to \mathbb{R}$ such that

$$L_\lambda(f \circ \gamma) = (L_\lambda f) \circ \gamma \quad \text{on} \quad \gamma^{-1}(U). \tag{2}$$

We will simply write

$$L_\lambda f = \sum_{k=1}^{p} a_k \frac{\partial^2 f}{\partial \theta_k^2} - \lambda f \quad \text{for } f \in C^\infty(U).$$

A function $f \in C^\infty(U)$ is called *L_λ-harmonic* in U if $L_\lambda f = 0$ in U. The set of L_λ-harmonic functions in U is denoted $\mathscr{H}_\lambda^p(U)$. Of course $\mathscr{H}_\lambda^p(U)$ depends also on $a_1, \ldots, a_p$ but these numbers will be fixed throughout the paragraph so we avoid them in the notation.

Since $f \in \mathscr{H}_\lambda^p(U)$ if and only if $L_\lambda(f \circ \gamma) = 0$ on $\gamma^{-1}(U)$, it is clear that $\mathscr{H}_\lambda^p$ is a harmonic sheaf on T^p; (we use the terminology of Constantinescu and Cornea [7]). From the fact that the sheaf of solutions to (1) on $\mathbb{R}^p$ turns $\mathbb{R}^p$ into a Brelot space, it is easy to see that $(T^p, \mathscr{H}_\lambda^p)$ is a Brelot space.

To construct $\mathscr{H}_\lambda^p$-regular sets on T^p one may proceed in the following way: Let $\Omega \subseteq \mathbb{R}^p$ be an open set such that $\gamma: \Omega \to T^p$ is a homeomorphism of Ω onto $\gamma(\Omega)$ (which is necessarily open in T^p). If V is a regular subset of Ω in the sense of classical potential theory then V is also regular with respect to the sheaf of solutions to (1) ([7], p. 79) and $U = \gamma(V)$ is regular on T^p with respect to $\mathscr{H}_\lambda^p$. In fact, for $f \in C(\partial U)$ let $H_{f \circ \gamma}$ be the solution to the L_λ-Dirichlet problem for V with boundary function $f \circ \gamma \in C(\partial V)$. Then $H_f = H_{f \circ \gamma} \circ \gamma^{-1} \in \mathscr{H}_\lambda^p(U)$ is the solution to the $\mathscr{H}_\lambda^p$-Dirichlet problem for U with boundary function f. By γ^{-1} we mean the inverse function of $\gamma \mid \Omega$.

The property of a subset $U \subseteq T^p$ being regular with respect to $\mathscr{H}_\lambda^p$ is independent of $\lambda \geq 0$ and the p-tuple $(a_1, \ldots, a_p)$ as is easily seen invoking the corresponding result in $\mathbb{R}^p$. We can therefore talk about regular subsets of T^p without specifying the sheaf.

We will later make use of regular sets with smooth boundaries and this concept is made precise in the following definition.

5.2. *Definition.* Let Ω be an open subset of $\mathbb{R}^p$ such that γ is a homeomorphism of Ω onto $\gamma(\Omega)$. If V is a bounded domain with C^∞-boundary such that $\bar{V} \subseteq \Omega$ then $U = \gamma(V)$ is called a *strongly regular* subset of T^p.

It is clear that T^p has a base of strongly regular domains.

For an open subset $U \subseteq T^p$ we define

$$C_c^\infty(U) = \{f \in C^\infty(T^p) \mid \operatorname{supp}(f) \subseteq U\}, \tag{3}$$

and using the normalized Haar measure on T^p, denoted $d\theta$ or dz, we can regard locally integrable functions on T^p as distributions and we can talk about distribution solutions to $L_\lambda f = 0$.

Along these lines we have the following result.

5.3. **Proposition.** *Let $U \subseteq T^p$ be open and suppose that $f \in C(U)$ satisfies $L_\lambda f = 0$ in the sense of distributions on U. Then $f \in \mathscr{H}_\lambda^p(U)$.*

Proof. Since both hypothesis and conclusion are of local nature the result reduces to an analogous statement for an open subset of $\mathbb{R}^p$, and this case is settled by the ellipticity of L_λ. □

5.4. **Proposition.** *Let $U \subseteq T^p$ be a domain and let $f \in C^\infty(U)$ satisfy $L_\lambda f \leqq 0$ in U. If there exists a point $x_0 \in U$ such that $f(x_0) = \inf_U f \leqq 0$, then f is constant.*

Proof. Putting

$$A = \{x \in U \mid f(x) = \inf_U f\}$$

we have that A is non-empty and closed in U. By the Hopf minimum principle (cf. [14], p. 64) it follows that A is open and the conclusion follows. □

The following boundary minimum principle will be crucial later:

5.5. **Proposition.** *Let $U \subseteq T^p$ be open, $U \neq T^p$, and let $f\colon \bar{U} \to \mathbb{R}$ be continuous and satisfying $L_\lambda f \leqq 0$ on U in the distribution sense.*

If $f \geqq 0$ on ∂U then $f \geqq 0$ on U.

Proof. The proof follows classical lines. Let $\delta\colon U \to]0, \infty[$ denote the distance to $T^p \setminus U$, i.e.

$$\delta(x) = \inf_{y \in T^p \setminus U} \|x - y\|,$$

where $\|\cdot\|$ is the ordinary distance in $\mathbb{C}^p$. By B_ε we denote the open ball in T^p with radius $\varepsilon > 0$ and center at $(1, 1, \ldots, 1)$. For all sufficiently small $\varepsilon > 0$ we choose a C^∞-function $\varphi_\varepsilon\colon T^p \to [0, \infty[$ such that $\operatorname{supp}(\varphi_\varepsilon) \subseteq B_\varepsilon$ and $\int_{T^p} \varphi_\varepsilon \, d\theta = 1$, and then $(\varphi_\varepsilon)_{\varepsilon > 0}$ is an approximate unit.

In $U_\varepsilon = \{x \in U \mid \delta(x) > \varepsilon\}$ we define $f_\varepsilon = f * \varphi_\varepsilon$, where the convolution $*$ is on the group T^p. Then $f_\varepsilon \in C^\infty(U_\varepsilon)$, $L_\lambda f_\varepsilon \leqq 0$ in U_ε and f_ε has a continuous extension to $\bar{U}_\varepsilon$.

Let $a > 0$ be arbitrary. Since f is uniformly continuous on $\bar{U}$ and $f \geqq 0$ on ∂U there exists a number $b > 0$ such that

$$x \in U, \quad \delta(x) \leqq b \Rightarrow f(x) + a \geqq 0.$$

For $2\varepsilon < b$ we get $f_\varepsilon + a \geqq 0$ on ∂U_ε and since $L_\lambda(f_\varepsilon + a) = L_\lambda f_\varepsilon - \lambda a \leqq 0$ in U_ε we get by Proposition 5.4 that $f_\varepsilon + a \geqq 0$ in $\bar{U}_\varepsilon$. In fact, if $\inf_{U_\varepsilon}(f_\varepsilon + a) < 0$ there would exist $x_0 \in U_\varepsilon$ such that

$$f_\varepsilon(x_0) + a = \inf_{U_\varepsilon}(f_\varepsilon + a) < 0,$$

and an application of Proposition 5.4 leads to a contradiction.

For $x\in U$ fixed we therefore have $f_\varepsilon(x)+a\geqq 0$ for all sufficiently small $\varepsilon>0$, and letting ε tend to zero we obtain $f(x)+a\geqq 0$ and hence $f(x)\geqq 0$. □

5.6. *Remark.* One can prove that a continuous function $f\colon U\to\mathbb{R}$ is superharmonic in the harmonic space $(T^p, \mathscr{H}_\lambda^p)$ if and only if $L_\lambda f\leqq 0$ in the sense of distributions, but we shall not need this result.

5.7. **Proposition.** (i) *If $\lambda=0$ then $\mathscr{H}_0^p(T^p)$ consists of the constant functions, and the only potential on T^p is zero.*

(ii) *If $\lambda>0$ then $\mathscr{H}_\lambda^p(T^p)=\{0\}$, and $(T^p, \mathscr{H}_\lambda^p)$ is a $\mathfrak{P}$-Brelot space. The function $\tilde\rho_\lambda^a$ defined in 3.1(7) is a strictly positive potential which is L_λ-harmonic in $T^p\setminus\{0\}$.*

Proof. (i) $\lambda=0$. It is clear that every constant function belongs to $\mathscr{H}_0^p(T^p)$. If $f\in\mathscr{H}_0^p(T^p)$ there exists a point $x_0\in T^p$ such that $f(x_0)=\inf_{T^p} f$, and it follows by Proposition 5.4 that $g=f-f(x_0)$ is constantly zero. If s is a potential on T^p there exists by the lower semicontinuity of s a point $x_0\in T^p$ such that $s(x_0)=\inf_{T^p} s$, and $s(x_0)=0$ because $s(x_0)$ is a non-negative harmonic minorant of s. Since $(T^p, \mathscr{H}_0^p)$ is a Brelot space we conclude that $s=0$, cf. [7], p. 138.

(ii) $\lambda>0$. Let $f\in\mathscr{H}_\lambda^p(T^p)$ and assume that $\inf_{T^p} f\leqq 0$ (if not we replace f by $-$f). There exists $x_0\in T^p$ such that $f(x_0)=\inf_{T^p} f$, and by Proposition 5.4 we get that f is constant, but then $L_\lambda f=-\lambda f=0$ implies that $f=0$.

Let $(\mu_t^a)_{t>0}$ be the convolution semigroup studied in § 3 and let $(\rho_\lambda^a)_{\lambda>0}$ be the resolvent. If (A, D_A) denotes the infinitesimal generator for the semigroup on $C(T^p)$ induced by $(\mu_t^a)_{t>0}$ we have by Proposition 1.9 that

$$\langle\rho_\lambda^a, Af-\lambda f\rangle=-f(0)\qquad\text{for } f\in D_A,$$

in particular by 3.3

$$\langle\rho_\lambda^a, L_\lambda f\rangle=-f(0)\qquad\text{for } f\in C^\infty(T^p),$$

and it follows that $L_\lambda\rho_\lambda^a=-\varepsilon_0$ in the distribution sense. Since $\tilde\rho_\lambda^a$ is continuous on $T^p\setminus\{0\}$, Proposition 5.3 implies that $\tilde\rho_\lambda^a$ is L_λ-harmonic in $T^p\setminus\{0\}$.

In the case $p\geqq 2$ we conclude that $\tilde\rho_\lambda^a$ is superharmonic because it is lower semicontinuous and has the value ∞ at 0 (cf. 3.2).

In the case $p=1$ $\tilde\rho_\lambda^a$ is continuous and has a global maximum at 0, and therefore it is clear that $\tilde\rho_\lambda^a$ is superharmonic.

Since 0 is the only L_λ-harmonic function on T^p every non-negative superharmonic function is a potential.

The potential $\tilde\rho_\lambda^a$ is >0 at every point. This follows either from [7] p. 138 or directly from the concrete expression for $\tilde\rho_\lambda^a$. □

5.8. *Remark.* The sheaf $\mathscr{H}_\lambda^p$ is translation invariant on T^p for every $\lambda\geqq 0$.

5.9. Let $\mathscr{H}$ be a harmonic sheaf on a locally compact space X and let $V\subseteq X$ be an $\mathscr{H}$-regular set ([7] p. 12). Then every $f\in C(\partial V)$ has a unique continuous extension H_f to $\bar V$ such that $H_f|V\in\mathscr{H}(V)$, and there exists a family of positive measures $(\omega_x^V)_{x\in V}$ such that

$$H_f(x)=\int f\,d\omega_x^V\qquad\text{for } f\in C(\partial V) \text{ and } x\in V.$$

Let σ be a privileged positive measure on ∂V such that $\mathrm{supp}(\sigma)=\partial V$. If there exists a continuous function $P\colon V\times\partial V\to[0,\infty[$ such that

$$\omega_x^V = P(x,\xi)\,d\sigma(\xi) \qquad \text{for all } x\in V,$$

i.e. such that

$$H_f(x)=\int_{\partial V} P(x,\xi)f(\xi)\,d\sigma(\xi) \qquad \text{for } f\in C(\partial V) \text{ and } x\in V,$$

then P is called the *Poisson kernel* for V with respect to $\mathscr{H}$ (and σ). If the Poisson kernel exists it is uniquely determined due to the requirement that $\mathrm{supp}(\sigma)=\partial V$.

5.10. **Proposition.** *Suppose that for each open set $U\subseteq X$ the vector space $\mathscr{H}(U)$ is closed in $C(U)$ in the topology of compact convergence. Suppose further that the Poisson kernel P exists for a regular set $V\subseteq X$. Then the function $x\to P(x,\xi)$ belongs to $\mathscr{H}(V)$ for every $\xi\in\partial V$.*

Proof. Let B' be a base of compact neighbourhoods of $\xi_0\in\partial V$ and choose for every $W\in B'$ a function $f_W\in C^+(\partial V)$ such that

$$\mathrm{supp}(f_W)\subseteq W \quad\text{and}\quad \int f_W\,d\sigma=1.$$

Then it is easy to see that

$$\lim_{B'} H_{f_W}(x)=\lim_{B'}\int P(x,\xi)f_W(\xi)d\sigma(\xi)=P(x,\xi_0)$$

uniformly for x in compact subsets of V, and the assertion follows due to the closedness assumption about $\mathscr{H}$. $\square$

5.11. Let $\mathscr{H}$ be the sheaf of C^∞-solutions to $L_\lambda f=0$ in $\mathbb{R}^p$ and let $V\subseteq\mathbb{R}^p$ be a bounded domain with C^∞-boundary. Then it is well known that V is $\mathscr{H}$-regular and there exists a Poisson kernel P_λ for V with respect to $\mathscr{H}$ and the surface measure σ_V on ∂V. In fact it follows by Green's formula that

$$P_\lambda(x,\xi)=\frac{\partial}{\partial\nu}G_\lambda(x,\xi) \qquad \text{for } x\in V \text{ and } \xi\in\partial V,$$

where G_λ is the Green function for V with respect to L_λ, and $\dfrac{\partial}{\partial\nu}$ is the inward conormal derivative applied above to the function $y\to G_\lambda(x,y)$ at the boundary point $\xi\in\partial V$, cf. [14] p. 88.

It follows by the maximum principle at the boundary, cf. [14] p. 68, that $P_\lambda(x,\xi)>0$ for $x\in V$ and $\xi\in\partial V$.

The following estimate of P_λ is crucial.

5.12. **Theorem.** *Let $V\subseteq\mathbb{R}^p$ be a bounded domain with C^∞-boundary and let P_λ be the Poisson kernel for V as defined in* 5.11.

For every compact subset $K\subseteq V$ there exist constants A and $B>0$ such that

$$P_\lambda(x,\xi)\leq A\,e^{-\sqrt{\lambda}B} \qquad \textit{for } x\in K,\ \xi\in\partial V \textit{ and } \lambda\geq 1.^4$$

4 There exist actually constants A and B such that the inequality holds for all $\lambda\geq 0$. This follows from the proof below when $p\geq 3$, but for $p=1,2$ special estimates are needed. Since we are only interested in large λ we do not develop this further.

Proof. Let p_t^a denote the probability density on $\mathbb{R}^p$ defined by

$$p_t^a(x) = p_{ta_1}(x_1) \dots p_{ta_p}(x_p) \quad \text{for } x \in \mathbb{R}^p \text{ and } t > 0,$$

where $p_t(x)$ is given by 2.1(1). Putting $\alpha = (a_1 \dots a_p)^{1/p}$ and

$$\|x\|_a = \left(\sum_{i=1}^{p} \frac{x_i^2}{a_i}\right)^{\frac{1}{2}} \quad \text{for } x \in \mathbb{R}^p$$

we have

$$p_t^a(x) = (4\pi t\alpha)^{-\frac{p}{2}} \exp\left(-\frac{\|x\|_a^2}{4t}\right).$$

For $\lambda > 0$ a fundamental solution to the differential operator L_λ in $\mathbb{R}^p$ is $-N_\lambda$ where

$$N_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t^a(x)\, dt \quad \text{for } x \in \mathbb{R}^p,$$

cf. 1.9. Introducing

$$\sigma_\lambda(r) = \int_0^\infty (4\pi t\alpha)^{-\frac{p}{2}} e^{-\frac{1}{2}\lambda t} e^{-\frac{r^2}{8t}}\, dt \quad \text{for } r \geqq 0 \text{ and } \lambda > 0$$

we get the following estimate

$$N_\lambda(x) \leqq \sigma_1(\|x\|_a) \exp(-\tfrac{1}{2}\sqrt{\lambda}\,\|x\|_a) \quad \text{for } x \in \mathbb{R}^p \text{ and } \lambda \geqq 1. \tag{4}$$

In fact, putting

$$\varphi(t) = e^{-\lambda t} \exp\left(-\frac{\|x\|_a^2}{4t}\right) \quad \text{for } x \in \mathbb{R}^p \text{ fixed,}$$

we find

$$\max_{t>0} \varphi(t) = \exp(-\sqrt{\lambda}\,\|x\|_a),$$

and hence

$$\varphi(t) \leqq \sqrt{\varphi(t)} \exp(-\tfrac{1}{2}\sqrt{\lambda}\,\|x\|_a),$$

so that

$$N_\lambda(x) \leqq \sigma_\lambda(\|x\|_a) \exp(-\tfrac{1}{2}\sqrt{\lambda}\,\|x\|_a)$$

and (4) follows.

We suppose from now on that $\lambda \geqq 1$.

The Green function G_λ for V is equal to

$$G_\lambda(x, y) = N_\lambda(x - y) - h_\lambda(x, y) \quad \text{for } x \in V \text{ and } y \in \overline{V},$$

where $y \to h_\lambda(x, y)$ is the solution to the L_λ-Dirichlet problem for V with boundary values $y \to N_\lambda(x - y)$. By the boundary minimum principle $h_\lambda(x, y) \geqq 0$ and $G_\lambda(x, y) \geqq 0$ and hence by (4)

$$0 \leqq G_\lambda(x, y) \leqq N_\lambda(x - y) \leqq \sigma_1(\|x - y\|_a) \exp(-\tfrac{1}{2}\sqrt{\lambda}\,\|x - y\|_a).$$

Let K be a compact subset of V. We define

$$2\delta = \inf\{\|x-y\|_a \mid x\in K, y\in \mathbb{R}^p \setminus V\}$$

and choose a domain V_1 with C^∞-boundary such that $K \subseteq V_1 \subseteq \overline{V}_1 \subseteq V$ and

$$\inf\{\|x-y\|_a \mid x\in K, y\in \mathbb{R}^p \setminus V_1\} \geqq \delta.$$

Let H be chosen as the continuous function on $\overline{V} \setminus V_1$ which is 1 on ∂V_1, 0 on ∂V and which satisfies $L_0 H = 0$ in $V \setminus \overline{V}_1$. The function H is even C^∞ up to the boundary and in particular we have

$$\sup_{\partial V} \frac{\partial H}{\partial \nu} < \infty$$

where $\frac{\partial}{\partial \nu}$ is the inward conormal derivative.

For $x \in K$ and $y \in \partial V_1$ we have $\|x-y\|_a \geqq \delta$ and hence

$$G_\lambda(x,y) \leqq \sigma_1(\delta)\exp(-\tfrac{1}{2}\sqrt{\lambda}\,\delta). \tag{5}$$

Let $x \in K$. The function

$$\varphi_x(y) = G_\lambda(x,y) - \sigma_1(\delta)\exp(-\tfrac{1}{2}\sqrt{\lambda}\,\delta)\, H(y)$$

defined for $y \in \overline{V} \setminus V_1$ has the following properties:

(i) $L_0 \varphi_x(y) = \lambda G_\lambda(x,y) \geqq 0$ for $y \in V \setminus \overline{V}_1$,
(ii) $\varphi_x(y) = 0$ for $y \in \partial V$,
(iii) $\varphi_x(y) \leqq 0$ for $y \in \partial V_1$ (on account of (5)).

By (i) φ_x is L_0-subharmonic, and by the boundary maximum principle for such functions we get $\varphi_x \leqq 0$ in $\overline{V} \setminus V_1$. The function φ_x being $\leqq 0$ and 0 on ∂V it is clear that $\frac{\partial}{\partial \nu}\varphi_x \leqq 0$ on ∂V, and hence

$$P_\lambda(x,\xi) = \frac{\partial}{\partial \nu} G_\lambda(x,\xi) \leqq \sigma_1(\delta)\exp(-\tfrac{1}{2}\sqrt{\lambda}\,\delta)\frac{\partial H}{\partial \nu}(\xi) \quad \text{for } \xi \in \partial V.$$

Putting $A = \sigma_1(\delta) \sup_{\partial V} \frac{\partial H}{\partial \nu}$ and $B = \frac{1}{2}\delta$ we finally have

$$P_\lambda(x,\xi) \leqq A e^{-B\sqrt{\lambda}} \quad \text{for } x\in K,\ \xi \in \partial V \text{ and } \lambda \geqq 1. \quad \square$$

5.13. **Theorem.** *Let $V \subseteq \mathbb{R}^p$ be a bounded domain with C^∞-boundary and let P_λ denote the Poisson kernel for V as defined in* 5.11.

(i) *For every $x \in V$ and $\xi \in \partial V$ the function*

$$\lambda \to P_\lambda(x,\xi)$$

is continuous on $[0,\infty[$ *and completely monotone.*

(ii) *For every* $x \in V$ *and* $f \in C^+(\partial V)$ *the function*

$$\lambda \to \int P_\lambda(x, \xi) f(\xi)\, d\sigma_V(\xi)$$

is continuous on $[0, \infty[$ *and completely monotone.*

Proof. For $f \in C^+(\partial V)$ we denote by $H_f(x, \lambda)$ the value at $x \in \bar{V}$ of the solution to the L_λ-Dirichlet problem for V with boundary function f.

Suppose first that $f \in C^+(\partial V)$ is the restriction to ∂V of a non-negative C^∞-function in $\mathbb{R}^p$. Then $H_f(x, \lambda)$ has the following regularity properties:

a) For each $x \in \bar{V}$ the function $H_f(x, \cdot)$ is continuous on $[0, \infty[$, C^∞ on $]0, \infty[$.

b) For $n \geqq 0$ and $\lambda > 0$ the function $\dfrac{\partial^n}{\partial \lambda^n} H_f(\cdot, \lambda)$ is continuous on $\bar{V}$.

c) $H_f \in C^\infty(V \times]0, \infty[)$.

Differentiating the equation

$$L_0 H_f(x, \lambda) = \lambda H_f(x, \lambda)$$

n times with respect to λ we get by c)

$$\frac{\partial^n}{\partial \lambda^n} L_0 H_f(x, \lambda) = L_0 \left(\frac{\partial^n}{\partial \lambda^n} H_f(x, \lambda) \right) = \lambda \frac{\partial^n}{\partial \lambda^n} H_f(x, \lambda) + n \frac{\partial^{n-1}}{\partial \lambda^{n-1}} H_f(x, \lambda),$$

and hence

$$L_\lambda \left(\frac{\partial^n}{\partial \lambda^n} H_f(x, \lambda) \right) = n \frac{\partial^{n-1}}{\partial \lambda^{n-1}} H_f(x, \lambda) \quad \text{for } x \in V \text{ and } \lambda > 0. \tag{6}$$

We have clearly $H_f(x, \lambda) \geqq 0$ for $x \in V$ and $\lambda > 0$, and suppose for the purpose of induction that for some $n \geqq 1$

$$(-1)^{n-1} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} H_f(x, \lambda) \geqq 0 \quad \text{for } x \in V \text{ and } \lambda > 0.$$

Let $\lambda > 0$ be fixed and put

$$\varphi(x) = (-1)^n \frac{\partial^n}{\partial \lambda^n} H_f(x, \lambda) \quad \text{for } x \in \bar{V}.$$

By (6) and the induction hypothesis we get $L_\lambda \varphi \leqq 0$ so φ is L_λ-superharmonic in V. For $x \in \partial V$ we have $H_f(x, \mu) = f(x)$ for $\mu > 0$ and therefore $\varphi(x) = 0$. Using b) and the boundary minimum principle for L_λ-superharmonic functions we get $\varphi \geqq 0$, and this completes the induction. We have now proved that $\lambda \to H_f(x, \lambda)$ is continuous and completely monotone on $[0, \infty[$ for fixed $x \in V$ and $f \in C^+(\partial V)$ being the restriction of a non-negative C^∞-function in $\mathbb{R}^p$.

To prove (i) let $x \in V$ and $\xi_0 \in \partial V$ be fixed and let $f_n \in C_+^\infty(\mathbb{R}^p)$ be a sequence of functions such that $f_n d\sigma_V$ converges vaguely to the Dirac measure ε_{ξ_0} at ξ_0. For every $\lambda \geqq 0$ we then have

$$\lim_{n \to \infty} H_{f_n}(x, \lambda) = \lim_{n \to \infty} \int P_\lambda(x, \xi) f_n(\xi) d\sigma_V(\xi) = P_\lambda(x, \xi_0),$$

so the function $\lambda \to P_\lambda(x, \xi_0)$ is pointwise limit of completely monotone functions hence itself completely monotone. It is somewhat tedious to verify that $\lambda \to P_\lambda(x, \xi_0)$ is continuous for $\lambda = 0$.

Finally it is easy to see that (i) implies (ii). $\square$

5.14. *Remark.* Theorem 5.13 is valid for more general linear second order elliptic operators and we shall treat this in a subsequent paper. Using the iterated differences characterization of completely monotone functions (cf. [13]) we can avoid the tedious verifications of differentiability needed in the above proof.

The result in Theorem 5.13 is also hidden in the following formula for hitting distributions, cf. [5] p. 61:

$$P_A^\lambda f(x) = E^x\{e^{-\lambda T_A} f(X_{T_A});\ T_A < \infty\}.$$

5.15. Let $\mathscr{H}_\lambda^p$ be the sheaf of L_λ-harmonic functions on T^p. Every strongly regular domain $U \subseteq T^p$ (cf. 5.2) has a Poisson kernel with respect to $\mathscr{H}_\lambda^p$ and the surface measure σ_U on the boundary of U.

Using the terminology of Definition 5.2 and denoting by P_λ^V the Poisson kernel for V defined in 5.11, then P_λ^U defined by

$$P_\lambda^U(x, \xi) = P_\lambda^V(\gamma^{-1}(x), \gamma^{-1}(\xi)) \quad \text{for } x \in U \text{ and } \xi \in \partial U,$$

where $\gamma^{-1} = (\gamma \mid \Omega)^{-1}$, is the Poisson kernel for U. To verify this one makes use of the fact that the Jacobian of γ is identically 1, and therefore γ maps the surface measure of V onto the surface measure of U. It is also clear that the results from Theorem 5.12 and 5.13 carry over, so we have the following theorem:

5.16. **Theorem.** *Let $U \subseteq T^p$ be a strongly regular domain. The Poisson kernel P_λ for U with respect to the sheaf $\mathscr{H}_\lambda^p$ and the surface measure σ_U exists and satisfies:*

(i) $P_\lambda(x, \xi) > 0$ *for* $x \in U$, $\xi \in \partial U$ *and* $\lambda \geqq 0$.

(ii) $\lambda \to P_\lambda(x, \xi)$ *is continuous and completely monotone on* $[0, \infty[$ *for* $x \in U$ *and* $\xi \in \partial U$.

(iii) *For every compact subset* $K \subseteq U$ *there exist constants* A *and* $B > 0$ *such that*

$$P_\lambda(x, \xi) \leqq A e^{-\sqrt{\lambda} B} \quad \textit{for } x \in K, \xi \in \partial U \textit{ and } \lambda \geqq 1.$$

6. *Definition of Harmonic Functions on T^∞*

In all of this paragraph $\mathscr{A} = (a_1, a_2, \ldots)$ is an arbitrary sequence of positive numbers.

6.1. For $p \in \mathbb{N}$ we define $\pi_p \colon T^\infty \to T^p$ by

$$\pi_p(z) = (z_1, \ldots, z_p) \quad \text{for } z = (z_1, z_2, \ldots) \in T^\infty. \tag{1}$$

For $p, q \in \mathbb{N}$ such that $p \leqq q$ we define $\pi_{p,q} \colon T^q \to T^p$ by

$$\pi_{p,q}(z) = (z_1, \ldots, z_p) \quad \text{for } z = (z_1, \ldots, z_q) \in T^q. \tag{2}$$

Then we have

$$\pi_{p,q} \circ \pi_q = \pi_p \quad \text{for } p \leqq q. \tag{3}$$

For $A \subseteq T^p$ we denote by $A \times T^\infty$ the following subset of T^∞

$$A \times T^\infty = \pi_p^{-1}(A) = \{z \in T^\infty \mid \pi_p(z) \in A\}.$$

This notation is very convenient although not quite correct. It leads to the following easily verified formulas

$$(A \times T^\infty)^- = \bar{A} \times T^\infty, \quad (A \times T^\infty)^\circ = \mathring{A} \times T^\infty, \quad \partial(A \times T^\infty) = \partial A \times T^\infty$$

where the topological operations $-, \circ, \partial$ (closure, interior, boundary) should be understood being with respect to T^∞ on the left-hand side and with respect to T^p on the right-hand side.

If for every $p \in \mathbb{N}$ we choose a base $\mathscr{B}_p$ for the topology of T^p, then the set of subsets of T^∞

$$\{U \times T^\infty \mid U \in \mathscr{B}_p,\ p \in \mathbb{N}\} \tag{4}$$

is a base for the topology of T^∞.

Let $X \subseteq T^\infty$ and consider a function $f\colon X \to \mathbb{R}$. We say that f *only depends on the first p variables*, if there exists a function $g\colon \pi_p(X) \to \mathbb{R}$ such that

$$f(z) = g(\pi_p(z)) \quad \text{for all } z \in X,$$

and we say that f *only depends on finitely many variables*, if there exists a $p \in \mathbb{N}$ such that f only depends on the first p variables.

The normalized Haar measure on T^p for $p \in \mathbb{N} \cup \{\infty\}$ is always denoted $d\theta, dx, dz, \ldots$, and it should be clear from the context which Haar measure these symbols are referring to.

6.2. For every $p \in \mathbb{N}$ we consider the set $C^\infty(T^p)$ of real-valued C^∞-functions on the differentiable manifold T^p. Every function f on T^p may be considered as a function on T^∞ depending only on the first p variables by composing f with π_p. We define

$$\mathscr{D}(T^\infty) = \bigcup_{p=1}^{\infty} \{f \circ \pi_p \mid f \in C^\infty(T^p)\}.$$

If $g \in \mathscr{D}(T^\infty)$ there exist $p \in \mathbb{N}$ and $f \in C^\infty(T^p)$ such that $g = f \circ \pi_p$. For any $q \geqq p$ there exists a function $f_q \in C^\infty(T^q)$ such that $g = f_q \circ \pi_q$. We only have to put $f_q = f \circ \pi_{p,q}$, cf. (3).

From this remark it is clear that $\mathscr{D}(T^\infty)$ is an algebra of functions on T^∞. The set $\mathscr{D}(T^\infty)$ is exactly the set of regular functions on T^∞ in the sense of Bruhat, cf. [6], but this paper is independent of [6].

For an open subset $\Omega \subseteq T^\infty$ we define

$$\mathscr{D}(\Omega) = \{f \in \mathscr{D}(T^\infty) \mid \operatorname{supp}(f) \subseteq \Omega\}.$$

6.3. Motivated by § 4 we consider an arbitrary sequence $\mathscr{A} = (a_1, a_2, \ldots)$ of positive numbers and a non-negative number λ and form the expression

$$L_\lambda^{\mathscr{A}} = \sum_{k=1}^{\infty} a_k \frac{\partial^2}{\partial \theta_k^2} - \lambda. \tag{5}$$

For every $p\in\mathbb{N}$ we define

$$L^p_\lambda=\sum_{k=1}^{p} a_k \frac{\partial^2}{\partial\theta_k^2}-\lambda, \tag{6}$$

which is the differential operator on T^p studied in § 5.

We will now define $L^{\mathscr{A}}_\lambda g$ for $g\in\mathscr{D}(T^\infty)$. For $g\in\mathscr{D}(T^\infty)$ there exist $p\in\mathbb{N}$ and $f\in C^\infty(T^p)$ such that $g=f\circ\pi_p$, and we then define

$$L^{\mathscr{A}}_\lambda g=(L^p_\lambda f)\circ\pi_p. \tag{7}$$

It is easy to see that the expression on the right-hand side of (7) is independent of the choice of $p\in\mathbb{N}$ and $f\in C^\infty(T^p)$ such that $g=f\circ\pi_p$.

We remark in passing that $L^{\mathscr{A}}_\lambda$ is a differential operator on T^∞ in the sense of Bruhat [6].

Using the idea of distributions we now define harmonic functions in the following way:

6.4. *Definition.* Let $\Omega\subseteq T^\infty$ be an open set. A continuous function $h\colon\Omega\to\mathbb{R}$ is called $L^{\mathscr{A}}_\lambda$*-harmonic* if

$$\int_\Omega h(\theta)L^{\mathscr{A}}_\lambda g(\theta)\,d\theta=0 \qquad \text{for all } g\in\mathscr{D}(\Omega).$$

The set of $L^{\mathscr{A}}_\lambda$-harmonic functions in Ω is denoted $\mathscr{H}^{\mathscr{A}}_\lambda(\Omega)$. In the terminology of [6] a continuous function $h\colon\Omega\to\mathbb{R}$ belongs to $\mathscr{H}^{\mathscr{A}}_\lambda(\Omega)$ if and only if $L^{\mathscr{A}}_\lambda h=0$ in the distribution sense.

It is clear that $\mathscr{H}^{\mathscr{A}}_\lambda(\Omega)$ is a closed subspace of $C(\Omega)$, when the vector space $C(\Omega)$ of continuous real-valued functions on Ω is equipped with the topology of uniform convergence on compact subsets of Ω. The following proposition exhibits a large class of $L^{\mathscr{A}}_\lambda$-harmonic functions.

6.5. **Proposition.** *Let $\Omega=U\times T^\infty$, where U is an open subset of T^p for some $p\in\mathbb{N}$. If $\varphi\in\mathscr{H}^p_\lambda(U)$ then $h=\varphi\circ\pi_p\in\mathscr{H}^{\mathscr{A}}_\lambda(\Omega)$.*

Proof. Let $g\in\mathscr{D}(\Omega)$. We shall prove that

$$\int_\Omega h(\theta)L^{\mathscr{A}}_\lambda g(\theta)\,d\theta=0.$$

There exists $q\geqq p$ such that $g=f\circ\pi_q$ where $f\in C^\infty(T^q)$, and using $\Omega=\pi_q(\Omega)\times T^\infty$ we find

$$\int_\Omega h(\theta)L^{\mathscr{A}}_\lambda g(\theta)\,d\theta=\int_\Omega \varphi(\pi_p(\theta))L^q_\lambda f(\pi_q(\theta))\,d\theta=\int_{\pi_q(\Omega)}\varphi(\pi_{p,q}(\vartheta))L^q_\lambda f(\vartheta)\,d\vartheta.$$

Using partial integration and the fact that $\operatorname{supp}(f)\subseteq\pi_q(\Omega)$ we see that the last integral is equal to

$$\int_{\pi_q(\Omega)} L^q_\lambda(\varphi\circ\pi_{p,q})(\vartheta)f(\vartheta)\,d\vartheta,$$

which is zero because

$$L^q_\lambda(\varphi\circ\pi_{p,q})(\vartheta)=L^p_\lambda\varphi(\pi_{p,q}(\vartheta))=0 \qquad \text{for } \vartheta\in\pi_q(\Omega)$$

by hypothesis. $\square$

6.6. **Theorem.** *The mapping $\mathcal{H}_\lambda^{\mathcal{A}}$ which to an open subset $\Omega\subseteq T^\infty$ associates the vector space $\mathcal{H}_\lambda^{\mathcal{A}}(\Omega)$ is a sheaf.*

Proof. If Ω_1 and Ω_2 are open subsets of T^∞ such that $\Omega_1\subseteq\Omega_2$, it is clear that $h\in\mathcal{H}_\lambda^{\mathcal{A}}(\Omega_2)$ implies that $h|\Omega_1\in\mathcal{H}_\lambda^{\mathcal{A}}(\Omega_1)$.

Suppose next that $\Omega\subseteq T^\infty$ is the union of a family $(\Omega_i)_{i\in I}$ of open subsets of T^∞ and that a function $h\colon\Omega\to\mathbb{R}$ satisfies $h|\Omega_i\in\mathcal{H}_\lambda^{\mathcal{A}}(\Omega_i)$ for all $i\in I$. We shall prove that $h\in\mathcal{H}_\lambda^{\mathcal{A}}(\Omega)$.

Without loss of generality we may assume that each Ω_i has the form $\Omega_i=U_i\times T^\infty$, where U_i is an open subset of T^{p_i} for some $p_i\in\mathbb{N}$.

Let $g\in\mathcal{D}(\Omega)$ and put $K=\operatorname{supp}(g)$. There exists a finite set $I_0\subseteq I$ such that $K\subseteq\bigcup_{i\in I_0}\Omega_i$. Choosing $p\geqq\max\{p_i\,|\,i\in I_0\}$ such that $g=f\circ\pi_p$ for $f\in C^\infty(T^p)$, it is possible for each $i\in I_0$ to write $\Omega_i=V_i\times T^\infty$ where $V_i=U_i\times T^{p-p_i}$ is an open subset of T^p. We then have

$$\operatorname{supp}(f)=\pi_p(K)\subseteq\bigcup_{i\in I_0}V_i,$$

and it is well-known that there exists a partition of unity $(\varphi_i)_{i\in I_0}$, i.e. for each $i\in I_0$ $\varphi_i\in C^\infty(T^p)$ and $\operatorname{supp}(\varphi_i)\subseteq V_i$, and furthermore

$$\sum_{i\in I_0}\varphi_i(z)=1\qquad\text{for } z\in\pi_p(K).$$

We therefore have

$$f(z)=\sum_{i\in I_0}f(z)\,\varphi_i(z)\qquad\text{for } z\in T^p,$$

and hence

$$g(\theta)=\sum_{i\in I_0}g_i(\theta)\qquad\text{for } \theta\in T^\infty,$$

where $g_i=g\cdot(\varphi_i\circ\pi_p)$. Since $g_i\in\mathcal{D}(\Omega_i)$ for each $i\in I_0$ we finally get

$$\begin{aligned}\int_\Omega h(\theta)\,E_\lambda^{\mathcal{A}}\,g(\theta)\,d\theta&=\sum_{i\in I_0}\int_\Omega h(\theta)\,E_\lambda^{\mathcal{A}}\,g_i(\theta)\,d\theta\\&=\sum_{i\in I_0}\int_{\Omega_i}h(\theta)\,E_\lambda^{\mathcal{A}}\,g_i(\theta)\,d\theta=0.\quad\square\end{aligned}$$

6.7. In the following we shall often approximate functions on T^∞ with functions on T^∞ which only depend on finitely many variables. We will use the following notation:

Let $f\colon U\times T^\infty\to\mathbb{R}$ be a given function, where U is a subset of T^p for some $p\in\mathbb{N}$.

For $r=p,p+1,\dots$ we define $T_rf\colon U\times T^{r-p}\to\mathbb{R}$ by

$$T_rf(x)=\int_{T^\infty}f(x,\theta)\,d\theta\qquad\text{for } x\in U\times T^{r-p}.\tag{8}$$

Eq. (8) shall be understood in the following way. For $x\in U\times T^{r-p}$ and $\theta\in T^\infty$ we identify (x,θ) with the point in $U\times T^\infty$ whose first r coordinates are given by x and the following coordinates by θ, and therefore $f(x,\theta)$ is well-defined. In the case $r=p$ we of course define $U\times T^0=U$. We also assume that f is sufficiently regular so the integral in (8) makes sense.

6.8. **Lemma of Approximation.** *Let U be a subset of T^p and let $f\colon U\times T^\infty\to\mathbb{R}$ be continuous.*

(i) *For each $r=p, p+1, \ldots$ the function $T_r f$ is continuous in $U\times T^{r-p}$ and*

$$\lim_{r\to\infty} T_r f(\pi_r(z))=f(z) \quad \text{for all } z\in U\times T^\infty. \tag{9}$$

If U is open the convergence in (9) is uniform on compact subsets of $U\times T^\infty$.

(ii) *If f is $L_\lambda^{\mathscr{A}}$-harmonic in $U\times T^\infty$, where U is open in T^p, then $T_r f$ is a C^∞-function in $U\times T^{r-p}$ and satisfies $L_\lambda^r(T_r f)=0$, i.e. $T_r f\in\mathscr{H}_\lambda^r(U\times T^{r-p})$. Furthermore $T_r f\circ\pi_r$ is $L_\lambda^{\mathscr{A}}$-harmonic in $U\times T^\infty$.*

Proof. (i) Let $r\geqq p$ be fixed and let $x\in U\times T^{r-p}$ be given. The continuity of f on the product space $U\times T^\infty$, where T^∞ is compact, implies that

$$\lim_{y\to x} f(y,\theta)=f(x,\theta) \quad \text{uniformly for } \theta\in T^\infty,$$

where y tends to x in $U\times T^{r-p}$. This clearly implies that $T_r f$ is a continuous function.

Let $z\in U\times T^\infty$ and $\varepsilon>0$ be given. By the continuity of f there exists $r_0\geqq p$ such that $|f(z)-f(w)|\leqq\varepsilon$ for all $w\in U\times T^\infty$ for which $\pi_{r_0}(w)=\pi_{r_0}(z)$, and therefore for $r\geqq r_0$

$$|T_r f(\pi_r(z))-f(z)|\leqq \int_{T^\infty} |f(\pi_r(z),\theta)-f(z)|\,d\theta\leqq\varepsilon.$$

Let now K be an arbitrary compact subset of U supposed open in T^p. We will prove that

$$\lim_{r\to\infty} T_r f(\pi_r(z))=f(z) \quad \text{uniformly for } z\in K\times T^\infty.$$

Let $\varepsilon>0$ be given. For every $z\in K\times T^\infty$ there exists an open neighbourhood Ω_z of z, $\Omega_z\subseteq U\times T^\infty$, such that

$$|f(w_1)-f(w_2)|\leqq\varepsilon \quad \text{for all } w_1, w_2\in\Omega_z,$$

and we may assume that Ω_z has the form $\Omega_z=U_z\times T^\infty$, where U_z is an open subset of T^{p_z} for some $p_z\in\mathbb{N}$. By compactness there exist finitely many points $z_1,\ldots,z_n$ in $K\times T^\infty$ such that

$$K\times T^\infty\subseteq\bigcup_{i=1}^n \Omega_{z_i},$$

and putting $r_0=\max\{p_{z_i}\mid i=1,\ldots,n\}$ we can write

$$\Omega_{z_i}=V_i\times T^\infty \quad \text{for } i=1,\ldots,n,$$

where $V_i=U_{z_i}\times T^{r_0-p_{z_i}}$ is an open subset of T^{r_0}.

For $w_1, w_2\in K\times T^\infty$ such that $\pi_{r_0}(w_1)=\pi_{r_0}(w_2)$ there exists $i\in\{1,\ldots,n\}$ such that $w_1, w_2 \in \Omega_{z_i}$, and therefore

$$|f(w_1)-f(w_2)| \leqq \varepsilon.$$

For $z \in K \times T^\infty$ and $r \geqq \max(p, r_0)$ we then find

$$|T_r f(\pi_r(z)) - f(z)| \leqq \int_{T^\infty} |f(\pi_r(z), \theta) - f(z)| \, d\theta \leqq \varepsilon.$$

Since every compact subset of $U \times T^\infty$ is contained in a compact set of the form $K \times T^\infty$, where K is a compact subset of U, we have proved (i).

(ii). Suppose that $f \in \mathscr{H}_\lambda^{\mathscr{A}}(U \times T^\infty)$ and $r \geqq p$. We will show that $L_\lambda^r(T_r f) = 0$ in the distribution sense on $U \times T^{r-p}$. Once this is done, it follows from Proposition 5.3 that $T_r f \in \mathscr{H}_\lambda^r(U \times T^{r-p})$, and by Proposition 6.5 we finally get that $T_r f \circ \pi_r \in \mathscr{H}_\lambda^{\mathscr{A}}(U \times T^\infty)$.

Let $\varphi \in C_c^\infty(U \times T^{r-p})$. We shall prove that

$$\int_{U \times T^{r-p}} T_r f(x) \, L_\lambda^r \, \varphi(x) \, dx = 0.$$

Since $g = \varphi \circ \pi_r \in \mathscr{D}(U \times T^\infty)$ we have

$$\int_{U \times T^\infty} f(z) \, L_\lambda^{\mathscr{A}} \, g(z) \, dz = \int_{U \times T^\infty} f(z) \, L_\lambda^r \, \varphi(\pi_r(z)) \, dz = 0,$$

and splitting $z \in U \times T^\infty$ as $z = (x, \theta)$ where $x \in U \times T^{r-p}$ and $\theta \in T^\infty$, and using that $dz = dx \otimes d\theta$ with dx and $d\theta$ being normalized Haar measure on T^r and T^∞ respectively, we get

$$\begin{aligned} 0 &= \int_{U \times T^{r-p}} \int_{T^\infty} f(x, \theta) \, L_\lambda^r \, \varphi(x) \, d\theta \, dx \\ &= \int_{U \times T^{r-p}} T_r f(x) \, L_\lambda^r \, \varphi(x) \, dx. \quad \square \end{aligned}$$

6.9. **Corollary.** *If $\lambda = 0$ the set $\mathscr{H}_0^{\mathscr{A}}(T^\infty)$ of $L_0^{\mathscr{A}}$-harmonic functions on T^∞ is the constant functions.*

If $\lambda > 0$ then $\mathscr{H}_\lambda^{\mathscr{A}}(T^\infty) = \{0\}$.

Proof. Every constant function belongs to $\mathscr{H}_0^{\mathscr{A}}(T^\infty)$, e.g. because of Proposition 6.5. If $h \in \mathscr{H}_\lambda^{\mathscr{A}}(T^\infty)$ we have by Lemma 6.8 that $T_r h \in \mathscr{H}_\lambda^r(T^r)$ for $r \geqq 1$. By Proposition 5.7 follows that $T_r h$ is constant in the case $\lambda = 0$ and zero in the case $\lambda > 0$. Letting $r \to \infty$ the assertion follows. $\square$

The approximation Lemma 6.8 leads to the characterization of $L_\lambda^{\mathscr{A}}$-harmonic functions as functions which are locally approximable by "ordinary harmonic" functions.

6.10. **Theorem.** *Let Ω be an open subset of T^∞. A function $h: \Omega \to \mathbb{R}$ is $L_\lambda^{\mathscr{A}}$-harmonic if and only if the following holds:*

For each $z \in \Omega$ there exist an open neighbourhood $\Omega_z \subseteq \Omega$ of z and a sequence $h_n: \Omega_z \to \mathbb{R}$ of functions satisfying.

(i) *For each $n \in \mathbb{N}$ there exist a number $p_n \in \mathbb{N}$ and a function $f_n \in \mathscr{H}_\lambda^{p_n}(\pi_{p_n}(\Omega_z))$ such that $h_n = f_n \circ \pi_{p_n}$,*

(ii) $\lim_{n \to \infty} h_n = h$ *uniformly on compact subsets of Ω_z.*

Proof. The "if-part". By Proposition 6.5 we know that $h_n \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega_z)$ for all n, hence by (ii) that $h \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega_z)$. The sheaf property finally assures that $h \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$.

The "only if-part". Suppose that $h \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$ and that $z \in \Omega$. There exist $p \in \mathbb{N}$ and an open set $U \subseteq T^p$ such that $z \in U \times T^\infty \subseteq \Omega$ and we put $\Omega_z = U \times T^\infty$. The sequence of functions $h_n = (T_{n+p} h) \circ \pi_{n+p}$ defined on Ω_z for $n \geqq 1$ satisfies (i) and (ii) on account of Lemma 6.8. $\square$

7. *Construction of Regular Sets*

In this paragraph $\mathscr{A} = (a_1, a_2, \ldots)$ is an arbitrary sequence of positive numbers and λ is an arbitrary non-negative number.

7.1. *Definition.* Let Ω be an open subset of T^∞. A continuous function $f: \Omega \to \mathbb{R}$ is called $L_\lambda^{\mathscr{A}}$*-superharmonic* if

$$\int_\Omega f(z)\, L_\lambda^{\mathscr{A}} g(z)\, dz \leqq 0 \qquad \text{for all } g \in \mathscr{D}^+(\Omega).$$

The set of continuous $L_\lambda^{\mathscr{A}}$-superharmonic functions in Ω is a convex cone containing $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$. Every non-negative constant function $h: \Omega \to \mathbb{R}$ is $L_\lambda^{\mathscr{A}}$-superharmonic:

If $g \in \mathscr{D}^+(\Omega)$ has the form $g = f \circ \pi_p$ with $f \in C^\infty(T^p)$, we get

$$\begin{aligned}\int_\Omega h(z)\, L_\lambda^{\mathscr{A}} g(z)\, dz &= h \int_{\operatorname{supp}(f) \times T^\infty} L_\lambda^p f(\pi_p(z))\, dz \\ &= h \int_{\operatorname{supp}(f)} L_\lambda^p f(x)\, dx = -\lambda h \int_{\operatorname{supp}(f)} f(x)\, dx \leqq 0,\end{aligned}$$

and the assertion follows.

7.2. **Lemma.** *Let U be an open subset of T^p such that $U \neq T^p$ and put $\Omega = U \times T^\infty$. For a continuous function $f: \bar{\Omega} \to \mathbb{R}$ such that $f \mid \Omega$ is $L_\lambda^{\mathscr{A}}$-superharmonic the boundary minimum principle holds, i.e.*

If $f \geqq 0$ on $\partial\Omega$ then $f \geqq 0$ in Ω.

Proof. For $r \geqq p$ the function $T_r f: \bar{U} \times T^{r-p} \to \mathbb{R}$ is continuous and satisfies $L_\lambda^r(T_r f) \geqq 0$ in the distribution sense in $U \times T^{r-p}$. Actually, the same calculation as in the proof of Lemma 6.8 gives for $\varphi \in C_c^\infty(U \times T^{r-p})$ that

$$\int_{U \times T^{r-p}} T_r f(x)\, L_\lambda^r \varphi(x)\, dx = \int_\Omega f(z)\, L_\lambda^{\mathscr{A}}(\varphi \circ \pi_r)(z)\, dz,$$

which by Definition 7.1 is $\leqq 0$ if $\varphi \geqq 0$, because then $\varphi \circ \pi_r \in \mathscr{D}^+(\Omega)$.

Furthermore $T_r f(x) \geqq 0$ for $x \in \partial(U \times T^{r-p})$ (boundary in T^r) because

$$T_r f(x) = \int_{T^\infty} f(x, \theta)\, d\theta$$

and for $x \in \partial(U \times T^{r-p})$ and $\theta \in T^\infty$ we have $\pi_p(x) \in \partial U$ (boundary in T^p) and hence $(x, \theta) \in \partial U \times T^\infty = \partial\Omega$, so by hypothesis $f(x, \theta) \geqq 0$.

From the boundary minimum principle in Proposition 5.5 we get $T_r f \geqq 0$ in $U \times T^{r-p}$, and hence

$$f(z) = \lim_{r\to\infty} T_r f(\pi_r(z)) \geqq 0$$

for all $z \in \Omega$. □

7.3. **Corollary.** *Let U be an open subset of T^p such that $U \neq T^p$ and put $\Omega = U \times T^\infty$. For a continuous function $h: \bar{\Omega} \to \mathbb{R}$ such that $h \mid \Omega \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$ we have*

$$\sup_{\bar{\Omega}} |h| = \sup_{\partial\Omega} |h|.$$

Proof. If we put $a = \sup_{\partial\Omega} |h|$ both of the functions $a \pm h$ are continuous on $\bar{\Omega}$, $L_\lambda^{\mathscr{A}}$-superharmonic in Ω and $\geqq 0$ on $\partial\Omega$. By Lemma 7.2 follows that $a \pm h \geqq 0$ in Ω, i.e. $\sup_{\Omega} |h| \leqq a$, and the assertion follows. □

We are now able to construct open sets in T^∞ which are regular with respect to $\mathscr{H}_\lambda^{\mathscr{A}}$.

7.4. **Theorem.** *Let U be a regular subset of T^p. Then $\Omega = U \times T^\infty$ is regular with respect to the sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ on T^∞.*

Proof. For a continuous function $f: \partial\Omega \to \mathbb{R}$ there exists at most one solution to the $L_\lambda^{\mathscr{A}}$-Dirichlet problem for Ω, i.e. at most one continuous function $F: \bar{\Omega} \to \mathbb{R}$ for which $F \mid \Omega \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$ and $F \mid \partial\Omega = f$. If namely F_1 and F_2 denote two such functions, we know that $G = F_1 - F_2$ is continuous on $\bar{\Omega}$, $G = 0$ on $\partial\Omega$, and that $G \mid \Omega \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$, and hence by Corollary 7.3 that $G \equiv 0$.

To prove the existence of a solution to the $L_\lambda^{\mathscr{A}}$-Dirichlet problem we introduce

$$A = \{f \in C(\partial\Omega) \mid \exists F \in C(\bar{\Omega}) \colon F \mid \partial\Omega = f,\ F \mid \Omega \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)\}.$$

We shall prove that $A = C(\partial\Omega)$ which will be accomplished by the following steps:

a) A is a closed subspace of $C(\partial\Omega)$.

b) Construction of a subset $B \subseteq A$ which is dense in $C(\partial\Omega)$.

Proof of a). It is clear that A is a subspace of $C(\partial\Omega)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence from A converging uniformly on $\partial\Omega$ to a function $f \in C(\partial\Omega)$. If $(F_n)_{n\in\mathbb{N}}$ denotes the corresponding sequence of "solutions", we have by Corollary 7.3 that

$$\sup_{\bar{\Omega}} |F_n - F_m| = \sup_{\partial\Omega} |f_n - f_m| \qquad \text{for all } n, m \in \mathbb{N}.$$

This implies that $(F_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C(\bar{\Omega})$, and there exists consequently a function $F \in C(\bar{\Omega})$ such that $F_n \to F$ uniformly on $\bar{\Omega}$. Clearly $F \mid \partial\Omega = f$, and since $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$ is closed in the topology of uniform convergence on compact subsets of Ω, we also have $F \mid \Omega \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$. This proves that $f \in A$.

b) For each $n \in \mathbb{Z}^{(\infty)}$ we consider the character γ_n on T^∞ defined in 4.1. For $n \in \mathbb{Z}^{(\infty)}$ and $\varphi \in C(\partial U)$ the tensor product $\varphi \otimes \gamma_n$ determines a continuous function on $\partial U \times T^\infty = \partial\Omega$, namely

$$\varphi \otimes \gamma_n(x, z) = \varphi(x)\, \gamma_n(z) \qquad \text{for } x \in \partial U \quad \text{and} \quad z \in T^\infty. \tag{1}$$

The subspace of $C(\partial\Omega)$ generated by the functions (1) when $\varphi\in C(\partial U)$ and $n\in\mathbb{Z}^{(\infty)}$ is denoted B, i.e.

$$B=\operatorname{span}\{\varphi\otimes\gamma_n\mid\varphi\in C(\partial U), n\in\mathbb{Z}^{(\infty)}\}. \tag{2}$$

B is a subalgebra of $C(\partial\Omega)$ and it is easy to see that the Stone-Weierstrass theorem can be applied to the effect that B is dense in $C(\partial\Omega)$. In order to see that $B\subseteq A$ it suffices to prove that $f=\varphi\otimes\gamma_n\in A$ for every $\varphi\in C(\partial U)$ and $n\in\mathbb{Z}^{(\infty)}$, and this is done in Theorem 7.5 below.

We finally have to prove that if $f\in C^+(\partial\Omega)$ and if F is the solution to the $L^{\mathscr{A}}_\lambda$-Dirichlet problem with boundary function f, then $F\geqq 0$ on Ω. This however follows from Lemma 7.2. $\square$

7.5. **Theorem.** *Let U be a regular subset of T^p and put $\Omega=U\times T^\infty$. Let $f=\varphi\otimes\gamma_n\in C(\partial\Omega)$ be given, where $\varphi\in C(\partial U)$ and $n\in\mathbb{Z}^{(\infty)}$.*

The solution to the $L^{\mathscr{A}}_\lambda$-Dirichlet problem for Ω with boundary function f is $F=\Phi\otimes\gamma_n\in C(\bar\Omega)$, where Φ is the solution to the L^p_c-Dirichlet problem for U with boundary function φ.

The constant c is given by

$$c=\lambda+\sum_{k=1}^{q} a_{p+k}\, n_k^2, \tag{3}$$

where $n\in\mathbb{Z}^{(\infty)}$ is equal to $n=(n_1, n_2, \ldots, n_q, 0, 0, \ldots)$.

Proof. Since $U\subseteq T^p$ is regular, in particular $\mathscr{H}^p_c$-regular, there exists a uniquely determined function $\Phi\in C(\bar U)$ such that $\Phi\mid\partial U=\varphi$ and $\Phi\mid U\in\mathscr{H}^p_c(U)$. Therefore $\Phi\in C^\infty(U)$ and $L^p_c\,\Phi=0$ in U. The function $F=\Phi\otimes\gamma_n$, i.e. the function defined by

$$F(x,z)=\Phi(x)\,\gamma_n(z) \qquad \text{for } (x,z)\in\bar\Omega=\bar U\times T^\infty,$$

is continuous on $\bar\Omega$ and $F\mid\partial\Omega=\varphi\otimes\gamma_n=f$. Furthermore we have $F\mid\Omega\in\mathscr{H}^{\mathscr{A}}_\lambda(\Omega)$. To see this we define $\tilde F\colon U\times T^q\to\mathbb{R}$ by

$$\tilde F(x,z)=\Phi(x)\prod_{k=1}^{q} z_k^{n_k} \qquad \text{for } x\in U \quad\text{and}\quad z\in T^q,$$

so that $F=\tilde F\circ\pi_{p+q}$ on Ω, and by Proposition 6.5 it therefore suffices to prove that the C^∞-function $\tilde F$ satisfies $L^{p+q}_\lambda\tilde F=0$ in $U\times T^q$. This means by definition that $L^{p+q}_\lambda(G)=0$ in $\gamma^{-1}(U)\times\mathbb{R}^q$, where $\gamma\colon\mathbb{R}^p\to T^p$ is the mapping defined in 5.1 and $G\colon\gamma^{-1}(U)\times\mathbb{R}^q\to\mathbb{R}$ is defined by

$$G(\theta)=\Phi(\gamma(\theta_1,\ldots,\theta_p))\prod_{k=1}^{q} e^{in_k\theta_{p+k}}$$

for

$$\theta=(\theta_1,\ldots,\theta_p,\theta_{p+1},\ldots,\theta_{p+q})\in\gamma^{-1}(U)\times\mathbb{R}^q.$$

We find

$$\begin{aligned}
L_\lambda^{p+q} G(\theta) &= \left(\sum_{k=1}^{p} a_k \frac{\partial^2 \Phi\circ\gamma}{\partial\theta_k^2}(\theta_1,\dots,\theta_p)\right)\prod_{k=1}^{q} e^{in_k\theta_{p+k}} \\
&\quad -\Phi\circ\gamma(\theta_1,\dots,\theta_p)\left(\sum_{k=1}^{q} a_{p+k} n_k^2\right)\prod_{k=1}^{q} e^{in_k\theta_{p+k}} - \lambda G(\theta) \\
&= \left(\sum_{k=1}^{p} a_k \frac{\partial^2 \Phi\circ\gamma}{\partial\theta_k^2}(\theta_1,\dots,\theta_p) - c\Phi\circ\gamma(\theta_1,\dots,\theta_p)\right)\prod_{k=1}^{q} e^{in_k\theta_{p+k}} \\
&= L_c^p(\Phi\circ\gamma)(\theta_1,\dots,\theta_p)\prod_{k=1}^{q} e^{in_k\theta_{p+k}} = 0
\end{aligned}$$

because $L_c^p(\Phi\circ\gamma)=0$ in $\gamma^{-1}(U)$. □

7.6. **Corollary.** *There exists a basis for the topology of T^∞ consisting of $\mathscr{H}_\lambda^{\mathscr{A}}$-regular domains.*

7.7. Summarizing the results of §6 and §7 we have for an arbitrary sequence $\mathscr{A}$ of positive numbers and an arbitrary $\lambda \geqq 0$ constructed a harmonic sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ on T^∞ and proved the existence of a base of $\mathscr{H}_\lambda^{\mathscr{A}}$-regular domains.

Our goal is to prove that $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ is a Brelot space and this will be done in the next paragraph, but for the result to hold we must impose a growth condition on $\mathscr{A}$.

8. Existence of the Poisson Kernel and Its Consequences

In this paragraph we will assume that the sequence $\mathscr{A}=(a_1, a_2, \dots)$ of positive numbers tends to infinity so rapidly that

$$\sum_{k=1}^{\infty} \frac{(\sqrt{2})^k}{\sqrt{a_k}} < \infty. \tag{1}$$

This is the condition (10) of 4.16.

As before λ is an arbitrary non-negative number.

8.1. Let $\Omega \subseteq T^\infty$ be an open set of the form $\Omega = U \times T^\infty$, where U is a strongly regular domain in T^p. We know that Ω is $\mathscr{H}_\lambda^{\mathscr{A}}$-regular (7.4) and will now show that there exists a Poisson kernel for Ω with respect to $\mathscr{H}_\lambda^{\mathscr{A}}$ and the measure $\sigma_U \otimes dw$ on $\partial\Omega = \partial U \times T^\infty$, where σ_U is the surface measure on the boundary of U and dw is the Haar measure on T^∞.

We first define a function $\Lambda\colon \mathbb{Z}^{(\infty)} \to [0, \infty[$ by

$$\Lambda(n) = \lambda + \sum_{k=1}^{\infty} a_{p+k} n_k^2 \qquad \text{for } n \in \mathbb{Z}^{(\infty)}. \tag{2}$$

Note that the sum is finite since $n \in \mathbb{Z}^{(\infty)}$ is eventually zero. The function Λ depends on $\mathscr{A}$ and λ and furthermore on p equal to the dimension of U (or T^p). The function Λ is negative definite as sum of a non-negative constant λ and a non-negative quadratic form. The Poisson kernel for U with respect to $\mathscr{H}_\lambda^p$ and σ_U is denoted P_λ, cf. 5.15 and 5.16.

8.2. **Lemma.** *With the above notation the Fourier series on* T^∞

$$\mathscr{P}(x,\xi,z)=\sum_{n\in\mathbb{Z}^{(\infty)}} P_{\Lambda(n)}(x,\xi)\,\gamma_n(z) \tag{3}$$

where $x\in U$, $\xi\in\partial U$ *and* $z\in T^\infty$, *converges absolutely and uniformly on compact subsets of* $U\times\partial U\times T^\infty$ *to a continuous function* $\mathscr{P}\colon U\times\partial U\times T^\infty\to[0,\infty[$.

Proof. Since $0<a_k$ and $a_k\to\infty$ for $k\to\infty$ there exists a finite subset $D\subseteq\mathbb{Z}^{(\infty)}$ such that

$$\Lambda(n)\geqq 1 \qquad \text{for all } n\in\mathbb{Z}^{(\infty)}\setminus D.$$

Let K be a compact subset of U. By Theorem 5.16 there exist constants A and $B>0$ such that

$$|P_{\Lambda(n)}(x,\xi)\,\gamma_n(z)|=P_{\Lambda(n)}(x,\xi)\leqq A\,e^{-B\Lambda(n)^{\frac{1}{2}}}$$

for $n\in\mathbb{Z}^{(\infty)}\setminus D$, $x\in K$, $\xi\in\partial U$ and $z\in T^\infty$. Furthermore

$$\sum_{n\in\mathbb{Z}^{(\infty)}} e^{-B\Lambda(n)^{\frac{1}{2}}}<\infty$$

because of Corollary 4.17, where the hypothesis (1) is used. It follows now by classical arguments that the series in (3) converges absolutely and uniformly on $K\times\partial U\times T^\infty$, and consequently the series in (3) converges absolutely and uniformly on compact subsets of $U\times\partial U\times T^\infty$ to a continuous function $\mathscr{P}$.

For fixed $x\in U$ and $\xi\in\partial U$ the function

$$n\to P_{\Lambda(n)}(x,\xi) \qquad \text{for } n\in\mathbb{Z}^{(\infty)} \tag{4}$$

is positive definite on the group $\mathbb{Z}^{(\infty)}$. In fact it is equal to $h\circ\Lambda$, where h is the function $h(\lambda)=P_\lambda(x,\xi)$, i.e. equal to the composition of the completely monotone function h (cf. 5.16) and the negative definite function Λ, and Proposition 1.16 can be applied.

The function (4) being positive definite and summable, it follows by Bochner's theorem and the inversion theorem that $\mathscr{P}\geqq 0$. □

8.3. **Theorem.** *Let* $\Omega\subseteq T^\infty$ *be of the form* $\Omega=U\times T^\infty$, *where* U *is a strongly regular domain in* T^p. *Let points in* $\Omega=U\times T^\infty$ *and* $\partial\Omega=\partial U\times T^\infty$ *be represented respectively as* (x,z) *and* (ξ,w) *where* $x\in U$, $\xi\in\partial U$ *and* $z,w\in T^\infty$.

Then the Poisson kernel for Ω *with respect to* $\mathscr{H}_\lambda^{\mathscr{A}}$ *and the measure* $\sigma_U\otimes dw$ *is the continuous function* $P\colon\Omega\times\partial\Omega\to[0,\infty[$ *given by*

$$P((x,z);(\xi,w))=\mathscr{P}(x,\xi,z\bar{w})=\sum_{n\in\mathbb{Z}^{(\infty)}} P_{\Lambda(n)}(x,\xi)\,\gamma_n(z\bar{w}). \tag{5}$$

Proof. We know by Lemma 8.2 that the function $P\colon\Omega\times\partial\Omega\to[0,\infty[$ defined by (5) is continuous, and we shall prove that

$$H_f(x,z)=\int_{\partial U}\int_{T^\infty} P((x,z);(\xi,w))\,f(\xi,w)\,dw\,d\sigma_U(\xi) \tag{6}$$

holds for all $(x,z)\in\Omega$ and all $f\in C(\partial\Omega)$, where H_f denotes the solution to the $L_\lambda^{\mathscr{A}}$-Dirichlet problem for Ω with boundary function f.

By the properties of the set B defined in the proof of Theorem 7.4 formula (2) it clearly suffices to prove (6) for functions of the form $f=\varphi\otimes\gamma_{n_0}$, where $\varphi\in C(\partial U)$ and $n_0\in\mathbb{Z}^{(\infty)}$.

With the terminology of Theorem 7.5 this amounts to prove that

$$\int_{\partial U}\int_{T^\infty} P((x,z);(\xi,w))\,\varphi(\xi)\,\gamma_{n_0}(w)\,dw\,d\sigma_U(\xi)=\Phi(x)\,\gamma_{n_0}(z). \tag{7}$$

Because of the uniform convergence in w of the series in (5) we get

$$\int_{T^\infty} P((x,z);(\xi,w))\,\gamma_{n_0}(w)\,dw=P_{\Lambda(n_0)}(x,\xi)\,\gamma_{n_0}(z),$$

so the left-hand side of (7) is equal to

$$\gamma_{n_0}(z)\int P_{\Lambda(n_0)}(x,\xi)\,\varphi(\xi)\,d\sigma_U(\xi),$$

which by Theorem 7.5 is equal to $\gamma_{n_0}(z)\,\Phi(x)$, because $\Lambda(n_0)$ is equal to the constant c in 7.5(3). □

8.4. *Remark.* The preceding proof shows that (6) holds even without knowledge of P being non-negative. On the other hand, once (6) is established, it follows from (6) that P must be non-negative because $f\in C^+(\partial\Omega)$ implies that $H_f\geqq 0$. Therefore the non-negativity of P may be established without making use of Theorem 5.16(ii).

We now deduce some important consequences of Theorem 8.3.

8.5. **Theorem.** *Let Ω be a domain in T^∞. Every function $h\in\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)^+$ is either identically zero or positive at every point of Ω.*

Proof. For $h\in\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)^+$ we put

$$A=\{\omega\in\Omega\mid h(\omega)=0\},$$

and it is clear that A is closed relatively to Ω. We will prove that A is also open relatively to Ω, and the statement then follows since Ω is supposed connected.

Assume that $\omega\in A$. There exist $p\in\mathbb{N}$ and a domain $U_1\subseteq T^p$ such that $\omega\in U_1\times T^\infty\subseteq\Omega$. We next choose a strongly regular domain U in T^p such that $\bar{U}\subseteq U_1$ and $\omega\in U\times T^\infty$. The Poisson kernel for $\Omega'=U\times T^\infty$ is denoted P. Due to the uniqueness of the solution to the Dirichlet problem for Ω' we have

$$0=h(\omega)=\int_{\partial U}\int_{T^\infty} P(\omega;(\xi,w))\,h(\xi,w)\,dw\,d\sigma_U(\xi)$$

and therefore

$$P(\omega,(\xi,w))\,h(\xi,w)=0 \qquad \text{for all } (\xi,w)\in\partial U\times T^\infty. \tag{8}$$

Let φ be a strictly positive L_λ^p-harmonic function defined in a neighbourhood of $\bar{U}$ in T^p (such a function clearly exists) and put $h_1=\varphi\circ\pi_p$. Then h_1 is $L_\lambda^{\mathscr{A}}$-harmonic in a neighbourhood of $\overline{\Omega'}$ in T^∞, and as above we find

$$0<h_1(\omega)=\int_{\partial U}\int_{T^\infty} P(\omega;(\xi,w))\,h_1(\xi,w)\,dw\,d\sigma_U(\xi),$$

and it follows that there exists a non-empty open subset G of $\partial U\times T^\infty$ such that $P(\omega;(\xi,w))>0$ for $(\xi,w)\in G$, hence by (8) that $h(\xi,w)=0$ for $(\xi,w)\in G$. We can choose

G of the form $G=V_1\times V_2\times T^\infty$ where V_1 is an open subset of ∂U and V_2 is an open subset of T^q for some $q\in\mathbb{N}$.

Since $U_1\times T^\infty\subseteq\Omega$ we may consider the function $T_r h$ for $r\geqq p$, cf. 6.7. We have $T_r h\in\mathscr{H}_\lambda^r(U_1\times T^{r-p})$ and $T_r h\geqq 0$ in $U_1\times T^{r-p}$.

Let $\xi\in V_1$ and $\eta\in V_2$ be chosen. Then $(\xi,\eta)\in U_1\times T^q$. Let r be arbitrary $>p+q$ and let τ be chosen in $T^{r-(p+q)}$. Considering $x=(\xi,\eta,\tau)$ as a point in $U_1\times T^{r-p}$ in the obvious way, we have

$$T_r h(x)=\int_{T^\infty} h(x,z)\,dz=0$$

because $(x,z)=(\xi,\eta,\tau,z)\in G$ for all $z\in T^\infty$.

Since $(T^r,\mathscr{H}_\lambda^r)$ is a Brelot space we conclude that $T_r h$ is identically zero in the domain $U_1\times T^{r-p}$. However, r was arbitrary $>p+q$, so letting $r\to\infty$ we get by Lemma 6.8 that h is identically zero in $U_1\times T^\infty$. This shows that $\omega\in U_1\times T^\infty\subseteq A$ and we have proved that A is open. □

8.6. **Corollary.** *Let P denote the Poisson kernel for a domain $\Omega=U\times T^\infty$, where U is a strongly regular domain in T^p.*

For every $(\xi,w)\in\partial\Omega$ the function $P(\cdot;(\xi,w))$ is strictly positive and belongs to $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$.

Proof. Proposition 5.10 can be applied to the effect that the function in question belongs to $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$, and it is also known to be non-negative, hence either identically zero or strictly positive.

If $P((x,z);(\xi,w))=0$ for all $(x,z)\in\Omega$ we get by (5) putting $z=w$

$$\sum_{n\in\mathbb{Z}^{(\infty)}} P_{A(n)}(x,\xi)=0,$$

which is impossible because $P_{A(n)}(x,\xi)>0$ for all $n\in\mathbb{Z}^{(\infty)}$. □

From the existence of a strictly positive and continuous Poisson kernel it is possible to deduce all the classical "Harnack-type results" for positive $L_\lambda^{\mathscr{A}}$-harmonic functions.

8.7. **Theorem.** *Let Ω be an open subset of T^∞ and let $\omega_0\in\Omega$. For every $\varepsilon>0$ there exists a neighbourhood $\Omega(\omega_0)$ of ω_0 contained in Ω such that*

$$(1-\varepsilon)h(\omega_0)\leqq h(\omega)\leqq(1+\varepsilon)h(\omega_0)$$

for all $\omega\in\Omega(\omega_0)$ and all $h\in\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)^+$.

Proof. Let Ω' be a neighbourhood of ω_0 of the form $\Omega'=U\times T^\infty$, where U is a strongly regular domain in T^p for some $p\in\mathbb{N}$, and such that $\overline{\Omega'}\subseteq\Omega$. Let P denote the Poisson kernel for Ω'. By the uniqueness of the solution to the $L_\lambda^{\mathscr{A}}$-Dirichlet problem for Ω' we have

$$h(\omega)=\int_{\partial\Omega'} P(\omega,\xi)h(\xi)\,d\xi \qquad \text{for } \omega\in\Omega' \text{ and } h\in\mathscr{H}_\lambda^{\mathscr{A}}(\Omega),$$

where we write $d\xi$ for the measure on $\partial\Omega'$ equal to $\sigma_U\otimes dw$. Since $P(\omega,\xi)>0$ for

$\omega \in \Omega'$ and $\xi \in \partial\Omega'$ the continuity of $P\colon \Omega' \times \partial\Omega' \to \mathbb{R}$ implies that

$$\lim_{\omega \to \omega_0} \frac{P(\omega, \xi)}{P(\omega_0, \xi)} = 1$$

uniformly for $\xi \in \partial\Omega'$. To every $\varepsilon > 0$ there exists then a neighbourhood $\Omega(\omega_0)$ of ω_0 contained in Ω' such that

$$\frac{P(\omega, \xi)}{P(\omega_0, \xi)} \in [1-\varepsilon, 1+\varepsilon] \quad \text{for } \omega \in \Omega(\omega_0) \text{ and } \xi \in \partial\Omega'.$$

For $h \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)^+$ and $\omega \in \Omega(\omega_0)$ we then get

$$h(\omega) = \int_{\partial\Omega'} \frac{P(\omega, \xi)}{P(\omega_0, \xi)} P(\omega_0, \xi) h(\xi) d\xi \begin{cases} \leqq (1+\varepsilon) h(\omega_0), \\ \geqq (1-\varepsilon) h(\omega_0). \; \square \end{cases}$$

8.8. **Theorem.** *Let Ω be a domain in T^∞ and let $\mathscr{F}$ be a family of positive $L_\lambda^{\mathscr{A}}$-harmonic functions in Ω. Then the function $h = \sup \mathscr{F}$ is either identically infinite in Ω or finite and continuous everywhere in Ω.*

Proof. Defining $A = \{\omega \in \Omega \mid h(\omega) < \infty\}$, it follows from Theorem 8.7 that A is open and closed relatively to Ω and furthermore that $\mathscr{F}$ is equicontinuous at every point of A. $\square$

8.9. **Theorem.** *The sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ has the Brelot convergence property, i.e. the limit function of an increasing sequence of $L_\lambda^{\mathscr{A}}$-harmonic functions in a domain is either identically infinite or a $L_\lambda^{\mathscr{A}}$-harmonic function in the domain.*

Proof. Let Ω be a domain in T^∞ and let $(h_n)_{n \in \mathbb{N}}$ be an increasing sequence of functions from $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$. Putting $\mathscr{F} = \{h_n - h_1 \mid n \geqq 2\}$, it follows by Theorem 8.8 that $h = \sup \mathscr{F}$ is either identically infinite in Ω or finite and continuous in Ω. In the latter case Dini's theorem implies that $\lim_{n \to \infty} (h_n - h_1) = h$ uniformly on compact subsets of Ω and therefore h and $h + h_1$ belong to $\mathscr{H}_\lambda^{\mathscr{A}}(\Omega)$. $\square$

We have now proved that $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ is a Brelot space. Like in Proposition 5.7 the existence of positive potentials depends on λ.

8.10. **Theorem.** *Suppose $\mathscr{A}$ satisfies* (1) *and let $\lambda \geqq 0$. Then $\mathscr{H}_\lambda^{\mathscr{A}}$ is a translation invariant, symmetric harmonic sheaf, and $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ is a Brelot space.*

(i) *If $\lambda = 0$ then every superharmonic function on T^∞ is constant and every potential is zero.*

(ii) *If $\lambda > 0$ then $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ is a $\mathfrak{P}$-Brelot space. The function $\tilde{\rho}_\lambda^{\mathscr{A}}$ defined in* 4.10(7) *is a strictly positive potential which is $L_\lambda^{\mathscr{A}}$-harmonic in $T^\infty \setminus \{0\}$.*

Proof. It is clear that $\mathscr{H}_\lambda^{\mathscr{A}}$ is symmetric and translation invariant. The proof of (i) and (ii) is similar to that of Proposition 5.7. In particular we have that

$$\langle \rho_\lambda^{\mathscr{A}}, Ag - \lambda g \rangle = -g(0) \quad \text{for } g \in D_A,$$

where (A, D_A) is the infinitesimal generator for the semigroup on $C(T^\infty)$ induced by $(\mu_t^{\mathscr{A}})_{t>0}$. By Proposition 4.14 we then have

$$\langle \rho_\lambda^{\mathscr{A}}, L_\lambda^{\mathscr{A}} g \rangle = -g(0) \quad \text{for } g \in \mathscr{D}(T^\infty),$$

hence by Theorem 4.11 that $\tilde{p}_\lambda^{\mathscr{A}}$ is $L_\lambda^{\mathscr{A}}$-harmonic in $T^\infty \setminus \{0\}$. Since $\tilde{p}_\lambda^{\mathscr{A}}$ is lower semicontinuous and $\tilde{p}_\lambda^{\mathscr{A}}(0)=\infty$ we get that $\tilde{p}_\lambda^{\mathscr{A}}$ is superharmonic, and it is a potential because $\mathscr{H}_\lambda^{\mathscr{A}}(T^\infty)=\{0\}$. □

8.11. *Remarks.* (i) The axiom of domination is verified for the Brelot space $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ when $\lambda>0$. In fact the domination axiom is true for any symmetric strong harmonic group, cf. Forst [9].

(ii). Let Ω be an open subset of T^∞ and let $f\colon \Omega \to \mathbb{R}$ be a continuous function. Then f is superharmonic in the harmonic space $(T^\infty, \mathscr{H}_\lambda^{\mathscr{A}})$ if and only if f is $L_\lambda^{\mathscr{A}}$-superharmonic in the sense of Definition 7.1.

(iii). Let $\Omega \subseteq T^\infty$ be a domain, let $\omega \in \Omega$ and let K be a compact subset of Ω. Then there exists a constant $a>0$ such that

$$\frac{1}{a} h(\omega) \leqq \inf_K h \leqq \sup_K h \leqq a h(\omega)$$

for all $h \in \mathscr{H}_\lambda^{\mathscr{A}}(\Omega)^+$.

8.12. The preceding constructions show that there exist harmonic groups $(G, \mathscr{H})$ where the base space is $G=T^\infty$. It is easy to modify the construction to the case $G=\mathbb{R}^n \times T^\infty$. This shows that the base space G of a harmonic group need not be a Lie group, (T^∞ is not a Lie group in the classical sense, neither is it a Lie group of infinite dimension), which settles a problem studied in [4], cf. also [17].

9. A Counterexample

In this paragraph we will assume that the sequence $\mathscr{A}$ is $a_1=a_2=\cdots=1$ and that $\lambda=0$. The sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ is simply denoted $\mathscr{H}$ in this case.

The sequence $\mathscr{A}$ does of course not satisfy the hypothesis (1) of § 8 and therefore the proof of the Brelot convergence property of the sheaf breaks down. It is the purpose of this paragraph to prove that the Brelot convergence property is really violated in this case.

9.1. We consider the open interval $V=]0,1[$. The Poisson kernel for V with respect to the differential operator

$$L_\lambda = \frac{d^2}{dx^2} - \lambda, \qquad \lambda \geqq 0,$$

and the measure $\sigma=\varepsilon_0+\varepsilon_1$ on ∂V is given by

$$P_\lambda(x,\xi)=\begin{cases} \dfrac{\sinh(\sqrt{\lambda}(1-x))}{\sinh(\sqrt{\lambda})} & \text{for } x\in V \text{ and } \xi=0, \\[2ex] \dfrac{\sinh(\sqrt{\lambda}\,x)}{\sinh(\sqrt{\lambda})} & \text{for } x\in V \text{ and } \xi=1. \end{cases}$$

For $\lambda=0$ this should be interpreted as

$$P_0(x,\xi)=\begin{cases}1-x & \text{for } x\in V \text{ and } \xi=0,\\ x & \text{for } x\in V \text{ and } \xi=1.\end{cases}$$

As a special case of Theorem 5.13 we get that the function

$$\lambda\to\frac{\sinh(\sqrt{\lambda}x)}{\sinh(\sqrt{\lambda})} \tag{1}$$

is continuous and completely monotone on $[0,\infty[$ for every $x\in V$. This should also be compared with Corollary 2.11.

9.2. **Theorem.** *For every $p\in\mathbb{N}$ the Fourier series*

$$h_p(x,\theta)=\sum_{n\in\mathbb{Z}^p}\frac{\sinh(\|n\|x)}{\sinh(\|n\|)}e^{i\langle n,\theta\rangle}, \tag{2}$$

where $x\in V$ and $\theta\in\mathbb{R}^p$, converges in the Fréchet space $C^\infty(V\times\mathbb{R}^p)$ to a C^∞-function h_p which satisfies the differential equation

$$\frac{\partial^2 h_p}{\partial x^2}+\frac{\partial^2 h_p}{\partial\theta_1^2}+\cdots+\frac{\partial^2 h_p}{\partial\theta_p^2}=0. \tag{3}$$

Furthermore h is strictly positive.

Proof. For $x\in V$ and $n\neq 0$ (hence $\|n\|\geqq 1$) we have

$$\frac{\sinh(\|n\|x)}{\sinh(\|n\|)}=e^{-\|n\|(1-x)}\frac{1-e^{-2\|n\|x}}{1-e^{-2\|n\|}}\leqq e^{-\|n\|(1-x)}\frac{1}{1-e^{-2}}.$$

Since $\sum_{n\in\mathbb{Z}^p}e^{-t\|n\|}<\infty$ (cf. 9.4 below) for $t>0$, it follows that the series in (2) converges uniformly on $[\varepsilon,1-\varepsilon]\times\mathbb{R}^p$ for every $\varepsilon\in]0,\frac{1}{2}[$.

Along the same lines it is easy to see that any of the series

$$\sum_{n\in\mathbb{Z}^p}D^\alpha\frac{\sinh(\|n\|x)}{\sinh(\|n\|)}e^{i\langle n,\theta\rangle}, \tag{4}$$

where D^α is an arbitrary partial derivative with respect to $x,\theta_1,\ldots,\theta_p$, converges uniformly on $[\varepsilon,1-\varepsilon]\times\mathbb{R}^p$ for $\varepsilon\in]0,\frac{1}{2}[$, and therefore h_p is a C^∞-function and $D^\alpha h_p$ is the sum of (4). It is then easy to check that h_p satisfies (3).

For each $x\in V$ the function

$$n\to\frac{\sinh(\|n\|x)}{\sinh(\|n\|)}\quad\text{for } n\in\mathbb{Z}^p$$

is positive definite on $\mathbb{Z}^p$ as the composition of the completely monotone function (1) and the negative definite function $n\to\|n\|^2$ on $\mathbb{Z}^p$, cf. 1.16. It follows that h_p is $\geqq 0$, and since h_p is harmonic in the ordinary sense and obviously not identically zero it must be strictly positive. $\square$

9.3. Let $s:]0,1[\to T$ be the mapping $s(\theta)=e^{i\theta}$. Then $U=s(]0,1[)=s(V)$ is a domain in T and $\Omega=U\times T^\infty$ is a domain in T^∞. Since h_p is periodic with period 2π in each of the last p variables we will consider h_p as a function on $V\times T^p$ without further comment. We then define a function $\varphi_p: \Omega\to\mathbb{R}$ by

$$\varphi_p(x,z)=h_p(s^{-1}(x),\pi_p(z)) \quad \text{for } x\in U \text{ and } z\in T^\infty. \tag{5}$$

Then φ_p depends only on the first $p+1$ variables, and by Proposition 6.5 $\varphi_p\in\mathscr{H}^+(\Omega)$. We will use the sequence $(\varphi_p)_{p\in\mathbb{N}}$ to construct an increasing sequence which violates the Brelot convergence property, but we first need some estimates of φ_p.

For $t>0$ and $p\in\mathbb{N}$ we define

$$\sigma_p(t)=\sum_{n\in\mathbb{Z}^p} e^{-t\|n\|}. \tag{6}$$

9.4. **Lemma.** *Let $s>0$ be arbitrary. The infinite series*

$$\sum_{p=1}^{\infty}\frac{\sigma_p(t)}{\sigma_p(s)}$$

converges for every $t>s$ and uniformly for $t\geqq s+\varepsilon$ for every $\varepsilon>0$. In particular we have

$$\lim_{p\to\infty}\frac{\sigma_p(t)}{\sigma_p(s)}=0 \quad \textit{for } t>s$$

and

$$\lim_{p\to\infty}\frac{\sigma_p(t)}{\sigma_p(s)}=\infty \quad \textit{for } t<s.$$

Proof. We will compare the quantity $\sigma_p(t)$ with the quantity

$$\tau_p(t)=\int_{\mathbb{R}^p} e^{-t\|x\|}\,dx \quad \text{where } t>0 \text{ and } p\in\mathbb{N}.$$

One easily finds $\tau_p(t)=C_p t^{-p}$ where C_p is a constant depending only on p.

For $n\in\mathbb{Z}^p$ we put $A_n=n+[0,1[^p$ and we then find

$$\sum_{n\in\mathbb{Z}^p}\exp\left(-t\sup_{A_n}\|x\|\right)\leqq\tau_p(t)\leqq\sum_{n\in\mathbb{Z}^p}\exp\left(-t\inf_{A_n}\|x\|\right).$$

However, we clearly have

$$\|n\|-\sqrt{p}\leqq\|x\|\leqq\|n\|+\sqrt{p} \quad \text{for } x\in A_n,$$

and hence

$$e^{-t\sqrt{p}}\sigma_p(t)\leqq\tau_p(t)\leqq e^{t\sqrt{p}}\sigma_p(t) \quad \text{for } p\in\mathbb{N} \text{ and } t>0.$$

For $s>0$, $\varepsilon>0$ and $t\geqq s+\varepsilon$ we then find

$$\frac{\sigma_p(t)}{\sigma_p(s)}\leqq\frac{\sigma_p(s+\varepsilon)}{\sigma_p(s)}\leqq e^{(2s+\varepsilon)\sqrt{p}}\,\frac{\tau_p(s+\varepsilon)}{\tau_p(s)}=e^{(2s+\varepsilon)\sqrt{p}}\left(\frac{s}{s+\varepsilon}\right)^p,$$

and putting $b_p = e^{(2s+\varepsilon)\sqrt{p}}$ we have $\lim_{p\to\infty} b_{p+1}/b_p = 1$, and hence

$$\sum_{p=1}^{\infty} e^{(2s+\varepsilon)\sqrt{p}} \left(\frac{s}{s+\varepsilon}\right)^p < \infty,$$

so the assertion follows. $\square$

9.5. For the sequence $\varphi_p \in \mathscr{H}^+(\Omega)$ defined by (5) we have

$$\varphi_p(e^{i\frac{1}{2}}, 0) = h_p(\tfrac{1}{2}, 0) = \sum_{n\in\mathbb{Z}^p} \frac{\sinh(\frac{1}{2}\|n\|)}{\sinh(\|n\|)},$$

where $0 \in T^\infty$ as usual denotes the neutral element. The functions

$$\psi_p = \varphi_p / \varphi_p(e^{i\frac{1}{2}}, 0), \qquad p \in \mathbb{N}, \tag{7}$$

belong to $\mathscr{H}^+(\Omega)$ and are all equal to 1 at the point $(e^{i\frac{1}{2}}, 0) \in \Omega$. We finally define

$$f_n = \sum_{p=1}^{n} \psi_p, \qquad n \in \mathbb{N}, \tag{8}$$

and $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{H}^+(\Omega)$.

9.6. **Proposition.** *Let Ω be the domain in T^∞ defined by $\Omega = s(]0,1[) \times T^\infty$ and let $(f_n)_{n\in\mathbb{N}}$ be the increasing sequence of functions from $\mathscr{H}^+(\Omega)$ defined in* (8).

(i) *For points ω in the subdomain $\Omega' = s(]0, \frac{1}{2}[) \times T^\infty$ of Ω the limit $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ exists and the convergence is uniform over compact subsets of Ω'. In particular $f \in \mathscr{H}(\Omega')$.*

(ii). *For points $\omega \in \Omega$ of the form $\omega = (e^{ix}, 0)$ where $x \in [\frac{1}{2}, 1[$ we have $\lim_{n\to\infty} f_n(\omega) = \infty$.*

Proof. For points $\omega \in \Omega$ of the form $\omega = (e^{ix}, 0)$ where $x \in]0,1[$ we have

$$\varphi_p(\omega) = \sum_{n\in\mathbb{Z}^p} \frac{\sinh(\|n\| x)}{\sinh(\|n\|)}.$$

For $n \in \mathbb{Z}^p \setminus \{0\}$ and $x \in]0,1[$ we have

$$(1-e^{-2x})e^{-\|n\|(1-x)} \leqq \frac{\sinh(\|n\| x)}{\sinh(\|n\|)} \leqq \frac{1}{1-e^{-2}} e^{-\|n\|(1-x)}$$

which together with

$$\tfrac{1}{2}(1-e^{-2x}) \leqq x \leqq \frac{1}{1-e^{-2}}$$

implies that (cf. (6))

$$\tfrac{1}{2}(1-e^{-2x})\, \sigma_p(1-x) \leqq \varphi_p(\omega) \leqq \frac{1}{1-e^{-2}} \sigma_p(1-x).$$

For $x = \frac{1}{2}$ we have in particular

$$\tfrac{1}{2}(1-e^{-1})\sigma_p(\tfrac{1}{2}) \leqq \varphi_p(e^{i\frac{1}{2}}, 0) \leqq \frac{1}{1-e^{-2}} \sigma_p(\tfrac{1}{2}),$$

so there exist positive constants c_1, c_2 such that

$$c_1(1-e^{-2x})\frac{\sigma_p(1-x)}{\sigma_p(\frac{1}{2})} \leqq \psi_p(e^{ix},0) \leqq c_2 \frac{\sigma_p(1-x)}{\sigma_p(\frac{1}{2})} \quad \text{for } x \in]0,1[. \tag{9}$$

(i). Suppose now that $\omega=(e^{ix},z)$ with $x\in]0,\frac{1}{2}-\varepsilon[$ for some $\varepsilon\in]0,\frac{1}{2}[$ and $z\in T^\infty$. By (9) we find

$$0\leqq \psi_p(\omega)\leqq \psi_p(e^{ix},0)\leqq c_2\frac{\sigma_p(1-x)}{\sigma_p(\frac{1}{2})},$$

and it follows by Lemma 9.4 that $\lim_{n\to\infty} f_n(\omega)$ exists uniformly for $x\in]0,\frac{1}{2}-\varepsilon]$ and $z\in T^\infty$ and this proves (i).

(ii). Suppose next that $\omega=(e^{ix},0)$ with $x\in]\frac{1}{2},1[$. By (9) and Lemma 9.4 we get $\lim_{p\to\infty}\psi_p(\omega)=\infty$ and a fortiori $\lim_{n\to\infty} f_n(\omega)=\infty$.

For $\omega=(e^{i\frac{1}{2}},0)$ we have $f_n(\omega)=n$. □

9.7. Proposition 9.6 shows that the sheaf $\mathscr{H}=\mathscr{H}_\lambda^{\mathscr{A}}$ with $\mathscr{A}=(1,1,\ldots)$ and $\lambda=0$ does not have the Brelot convergence property. The proof above can be modified to show that the Brelot convergence property also fails for the sheaf $\mathscr{H}_\lambda^{\mathscr{A}}$ where $\mathscr{A}=(1,1,\ldots)$ and $\lambda>0$.

Addendum

a) Let $\mathscr{A}$ be an arbitrary sequence of positive numbers and let $\lambda\geqq 0$.

We know from § 7 that T^∞ has a base of $\mathscr{H}_\lambda^{\mathscr{A}}$-regular domains, and we claim that the associated sweeping is elliptic.

To be precise, let U be a regular subset of T^p for some $p\geqq 1$, put $\Omega=U\times T^\infty$ and let μ_ω^Ω denote the harmonic measure associated with $\omega=(x,z)\in\Omega$. We will prove that

$$\operatorname{supp}\mu_\omega^\Omega=\partial\Omega=\partial U\times T^\infty,$$

which implies the ellipticity.

Let $\varphi\in C^+(\partial U)$. With the terminology of the proof of Theorem 5.13 we denote by $H_\varphi(x,\lambda)$ the value at $x\in U$ of the solution to the L_λ^p-Dirichlet problem for U with boundary function φ. By Theorem 5.13(ii) there exists a positive bounded measure σ_x on $[0,\infty[$ such that

$$H_\varphi(x,\lambda)=\int_0^\infty e^{-\lambda s}\,d\sigma_x(s) \quad \text{for } \lambda\geqq 0. \tag{1}$$

The function Λ defined in (2) § 8 is negative definite, and the associated convolution semigroup on T^∞ is

$$\tau_t=e^{-\lambda t}\bigotimes_{k=1}^\infty \mu_{ta_{p+k}} \quad \text{for } t>0,$$

where $(\mu_t)_{t>0}$ is the Brownian semigroup on T.

Putting

$$\nu_x = \int_0^\infty \tau_t \, d\sigma_x(t),$$

we get a positive bounded measure ν_x or T^∞ the Fourier transform of which is given by

$$\hat{\nu}_x(n) = H_\varphi(x, \Lambda(n)) \quad \text{for } n \in \mathbb{Z}^{(\infty)}.$$

The solution to the $L^{\mathscr{A}}_\lambda$-Dirichlet problem for Ω with boundary function $f = \varphi \otimes \gamma_n$ is by Theorem 7.5 given as

$$F(x, z) = H_\varphi(x, \Lambda(n)) \gamma_n(z) = \hat{\nu}_x(n) \gamma_n(z) = \gamma_n * \nu_x(z)$$

for $x \in U$ and $z \in T^\infty$.

From this formula follows immediately that the solution to the $L^{\mathscr{A}}_\lambda$-Dirichlet problem for Ω with boundary function $f = \varphi \otimes \chi$, where $\chi \in C(T^\infty)$, is

$$F(x, z) = \chi * \nu_x(z).$$

Now, if $\operatorname{supp} \mu_\omega^\Omega \neq \partial\Omega$, there exist $\varphi \in C^+(\partial U)$ and $\chi \in C^+(T^\infty)$ with $\varphi, \chi \neq 0$ such that

$$F(x, z) = \int \varphi \otimes \chi \, d\mu_\omega^\Omega = \chi * \nu_x(z) = 0. \tag{2}$$

Since $\varphi \neq 0$ the measure σ_x in (1) is non-zero, and using that $\operatorname{supp} \tau_t = T^\infty$ for all $t > 0$ (cf. 4.2), we get $\operatorname{supp} \nu_x = T^\infty$, which contradicts (2) because $\chi \geqq 0$, $\neq 0$.

Remark. It is easy to deduce Theorem 8.5 from the ellipticity. It follows that the result in Theorem 8.5 is valid without any growth condition on the sequence $\mathscr{A}$.

b) The method of § 9 applied to the sheaf $\mathscr{H}^{\mathscr{A}}_\lambda$ leads to the following result: Suppose $\inf_{k \in \mathbb{N}} a_k > 0$ and let $s_k = a_1 + \cdots + a_k$. If $\lim_{k \to \infty} (\sqrt{s_{k+1}} - \sqrt{s_k}) = 0$ the sheaf $\mathscr{H}^{\mathscr{A}}_\lambda$ does not have the Brelot convergence property. In particular the Brelot convergence property fails for the sheafs $\mathscr{H}^{\mathscr{A}}_\lambda$ with $a_k = k^\varepsilon$ for $0 \leqq \varepsilon < 1$.

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