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SHEPHARD'S APPROXIMATION THEOREM FOR CONVEX BODIES AND THE MILMAN THEOREM

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In [4] G. C. Shephard proved an interesting approximation theorem concerning indecomposable convex polyhedra in the q-dimensional space R^q (cf. p. 23 of the present paper).

The purpose of the present paper is to give a new proof of this theorem. First we find a correspondence between the set of homothety classes of convex bodies in R^q and a compact convex set in the Banach space $C(\Omega_q)$ of continuous functions on the unit sphere Ω_q in R^q such that the indecomposable classes correspond to the extreme points of this compact convex set. We next show that Shephard's approximation theorem is a consequence of Milman's theorem, valid for a compact convex set in a locally convex topological vector space [3, p. 9]. Our proof yields that Shephard's theorem is true not only for a indecomposable polyhedron but for any indecomposable convex body in R^q.

Chapter 15 in the monograph [2] deals with the notion of decomposable and indecomposable polyhedra and the approximation theorem of G. C. Shephard.

Let $\mathscr{C}_q$ denote the class of all convex bodies in R^q consisting of more than one point. If $K,L \in \mathscr{C}_q$ and $\lambda > 0$, we have

$$K+L \in \mathscr{C}_q \quad \text{and} \quad \lambda K \in \mathscr{C}_q .$$

We consider $\mathscr{C}_q$ as a metric space under the Hausdorff-distance

$$D(K,L) = \inf\{\varepsilon > 0 \mid K \subseteq L+\varepsilon E_q,\, L \subseteq K+\varepsilon E_q\} ,$$

where E_q is the unit ball in R^q. For each $K \in \mathscr{C}_q$ let $h(K)$ denote the supporting function of K. We consider $h(K)$ as an element of the Banach space $C(\Omega_q)$ of continuous real-valued functions defined on the unit sphere Ω_q in R^q, equipped with the uniform norm. It is well known that the mapping $h\colon K \to h(K)$ of $\mathscr{C}_q$ into $C(\Omega_q)$ is one-to-one and satisfies (cf. [1])

$$(1) \qquad h(K+L) = h(K)+h(L), \qquad h(\lambda K) = \lambda h(K) \quad \text{for } \lambda > 0 ,$$

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(2) $$D(K,L) = \|h(K)-h(L)\| .$$

For $K \in \mathscr{C}_q$ one defines the Steiner-point $S(K)$ of K [2, p. 314] by

$$S(K) = \frac{q}{\|\omega_q\|} \int_{\Omega_p} \xi\, h(K)(\xi)\, d\omega_q(\xi)$$

and the mean width $B(K)$ of K [1, p. 50] by

$$B(K) = \frac{2}{\|\omega_q\|} \int_{\Omega_p} h(K)(\xi)\, d\omega_q(\xi) ,$$

where ω_q denotes the usual surface measure on Ω_q with total mass $\|\omega_q\|$. Note that $S(K) \in K$ and that $B(K) > 0$. The mappings $S: \mathscr{C}_q \to \mathsf{R}^q$ and $B: \mathscr{C}_q \to \mathsf{R}$ are both continuous and satisfy the linearity relations analogous to (1). Moreover S commutes with rigid motions, whereas B is invariant under rigid motions.

We shall consider the subset A of $\mathscr{C}_q$ defined by

$$A = \{K \in \mathscr{C}_q \mid S(K) = o, B(K) = 1\} ,$$

where o denotes the origin of R^q, and the corresponding set $h(A)$ of supporting functions. From the above remarks it is obvious that A is a closed subset of $\mathscr{C}_q$ with the property that if $K, L \in A$ and $\lambda \in [0,1]$, then $\lambda K + (1-\lambda)L \in A$.

THEOREM 1. *The subset A of $\mathscr{C}_q$ is compact, and h is a homeomorphism of A onto $h(A)$, which is a compact convex set in $C(\Omega_q)$.*

PROOF. For any convex body $K \in A$ and any point $a \in K$ the segment $[o,a]$ belongs to K since $o \in K$. Thus $B([o,a]) \leqq B(K) = 1$. Since

$$B([o,a]) = \frac{2\, \|\omega_{q-1}\|\, \|a\|}{\|\omega_q\|\,(q-1)} ,$$

$\|a\|$ and hence A is bounded. The selection theorem of Blaschke yields the compactness of A, and the proof is completed by (1) and (2).

A convex body $K \in \mathscr{C}_q$ is called *decomposable* if there exist convex bodies $L, M \in \mathscr{C}_q$, non-homothetic to K, such that $K = L + M$. If this is not the case, K is called *indecomposable*.

It is obvious that if $K, L \in \mathscr{C}_q$ are homothetic, then K is decomposable if and only if L is decomposable. Further, for any $K \in \mathscr{C}_q$ there exists a unique pair (λ, a), where $\lambda > 0$, $a \in \mathsf{R}^q$, such that $\lambda^{-1}(K-a) \in A$, namely $\lambda = B(K)$, $a = S(K)$.

The supporting function of $\lambda^{-1}(K-a)$ is denoted $\eta(K)$ and is called *the normalized supporting function of* K. For $K \in \mathscr{C}_q$ and $\xi \in \Omega_q$ we have

$$\eta(K)(\xi) = B(K)^{-1}\big(h(K)(\xi) - S(K)\cdot\xi\big) .$$

The normalized supporting function $\eta(K)$ lies in the compact convex set $h(A)$, and for $K,L \in \mathscr{C}_q$ we have $\eta(K)=\eta(L)$ if and only if K and L are homothetic. The mapping $\eta : \mathscr{C}_q \to h(A)$ is continuous.

Theorem 2. *Let $K \in \mathscr{C}_q$. Then K is indecomposable if and only if the normalized supporting function $\eta(K)$ is an extreme point of the compact convex set $h(A)$.*

Proof. It suffices to prove the theorem for a $K \in A$, that is, when $\eta(K)=h(K)$.

Suppose that K is decomposable. Then we have a decomposition $K=L+M$ with $L, M \in \mathscr{C}_q$, and $\eta(K)$ is different from $\eta(L)$ and $\eta(M)$. It follows that

$$\eta(K) = B(L)\eta(L) + B(M)\eta(M) \quad \text{and} \quad 1 = B(L) + B(M) ,$$

which show that $\eta(K)$ is not an extreme point of $h(A)$.

Conversely, if $\eta(K)$ is not extreme in $h(A)$, we can find $L, M \in A$ different from K and λ with $0<\lambda<1$ such that

$$\eta(K) = \lambda\eta(L) + (1-\lambda)\eta(M) .$$

Since $K, L, M \in A$, we have

$$h(K) = \lambda h(L) + (1-\lambda)h(M) = h(\lambda L + (1-\lambda)M) ,$$

and consequently

$$K = \lambda L + (1-\lambda)M ,$$

which shows that K is decomposable.

In the following let $\mathscr{K} \subseteq \mathscr{C}_q$ be a class of convex bodies stable under homothety, that is, if $K \in \mathscr{K}$, then $\lambda K + a \in \mathscr{K}$ for all $\lambda > 0$, $a \in \mathsf{R}^q$. Call a convex body $L \in \mathscr{C}_q$ approximable by such a class if there exist convex bodies $K_1 + \ldots + K_n$, where $K_i \in \mathscr{K}$, arbitrarily near to L in the Hausdorff-distance.

Lemma 1. *Let $\mathscr{K} \subseteq \mathscr{C}_q$ be a class of convex bodies stable under homothety, and let $L \in \mathscr{C}_q$. Then L is approximable by the class $\mathscr{K}$ if and only if*

$$\eta(L) \in \operatorname{cl} \operatorname{conv} \eta(\mathscr{K}) ,$$

where cl conv $\eta(\mathscr{K})$ *denotes the closed convex hull of the subset* $\{\eta(K) \mid K \in \mathscr{K}\}$ *of* $h(A)$.

PROOF. Suppose that there exists a sequence $K_n \in \mathscr{C}_q$ such that $K_n \to L$ in $\mathscr{C}_q$, and such that

$$K_n = \sum_{i=1}^{i_n} K^i{}_n, \quad \text{where} \quad K^i{}_n \in \mathscr{K} .$$

We then have $\eta(K_n) \to \eta(L)$ in $h(A)$ (uniformly over Ω_q) and

$$\eta(K_n) = \sum_{i=1}^{i_n} B(K_n)^{-1} B(K^i{}_n)\, \eta(K^i{}_n) ,$$

which is a convex combination of $\eta(K^i{}_n)$, $i=1,\ldots,i_n$. This proves

$$\eta(L) \in \text{cl conv}\, \eta(\mathscr{K}) .$$

Conversely, if this relation is satisfied, there exist convex bodies $K^i{}_n \in \mathscr{K}$ and numbers $\lambda^i{}_n > 0$, $i=1,\ldots,i_n$, $n=1,2,\ldots$, such that

$$\sum_{i=1}^{i_n} \lambda^i{}_n = 1$$

and

$$\sum_{i=1}^{i_n} \lambda^i{}_n\, \eta(K^i{}_n) \to \eta(L) \quad \text{in } h(A) .$$

Since $\mathscr{K}$ is stable under homothety, the bodies $K^i{}_n$ can be chosen such that $K^i{}_n \in A$, that is, $h(K^i{}_n) = \eta(K^i{}_n)$. Thus we get

$$h\left(\sum_{i=1}^{i_n} \lambda^i{}_n K^i{}_n\right) \to h\left(B(L)^{-1}(L - S(L))\right) \quad \text{in } h(A) ,$$

which by theorem 1 implies that

$$\sum_{i=1}^{i_n} \lambda^i{}_n K^i{}_n \to B(L)^{-1}(L - S(L)) .$$

Consequently L is approximable by the class $\mathscr{K}$.

If we let $\mathscr{K} \subseteq \mathscr{C}_q$ be the class of all indecomposable convex bodies, lemma 1 combined with the Krein–Milman theorem yields the following result:

THEOREM 3. *Every convex body* $L \in \mathscr{C}_q$ *can be approximated arbitrarily well by sums of indecomposable convex bodies.*

As a consequence of lemma 1 combined with the Milman theorem, we get:

THEOREM 4. *Let* $\mathscr{K} \subseteqq \mathscr{C}_q$ *be a class of convex bodies stable under homothety. If an indecomposable convex body* $L \in \mathscr{C}_q$ *is approximable by the class* $\mathscr{K}$, *then* $L \in \operatorname{cl} \mathscr{K}$.

PROOF. By lemma 1 we have

$$\eta(L) \in \operatorname{cl}\operatorname{conv} \eta(\mathscr{K}) ,$$

and $\eta(L)$ must be an extreme point of $\operatorname{cl}\operatorname{conv}\eta(\mathscr{K})$, because it is extreme in $h(A)$. Thus by the Milman theorem we get $\eta(L) \in \operatorname{cl} \eta(\mathscr{K})$, and the proof is easily completed by means of theorem 1.

It is straightforward to see that theorem 4 implies Shephard's approximation theorem:

Let $\mathscr{C} = \{K \in \mathscr{C}_q \mid S(K) = 0, \operatorname{diam} K = 1\}$, *and let* $\mathscr{K}_0 \subseteqq \mathscr{C}$ *be a closed subset. If* $P \in \mathscr{C}$ *is an indecomposable polyhedron, and if* P *can be approximated arbitrarily well by convex bodies* $K \in \mathscr{C}$ *of the form* $K = \sum_{i=1}^{n} \lambda_i K_i$, *where* $K_i \in \mathscr{K}_0$, $\lambda_i > 0$, *then* $P \in \mathscr{K}_0$.

We point out that theorem 3 does not tell anything new. It is well known that any convex body $K \in \mathscr{C}_2$, can be approximated by a convex polygon, and every convex polygon is a sum of segments and triangles, which are known to be indecomposable. For $q \geqq 3$ the indecomposable convex bodies are even dense in $\mathscr{C}_q$, because every convex simplicial polyhedron (that is, a polyhedron the $(q-1)$-dimensional facets of which are simplices) is indecomposable [4, lemma 23]. For $q \geqq 3$ theorem 4 therefore has the consequence that if $\mathscr{K} \subseteqq \mathscr{C}_q$ is a class stable under homothety, closed as a subset of $\mathscr{C}_q$ and universally approximating, that is, every convex body $L \in \mathscr{C}_q$ can be approximated by $\mathscr{K}$, then $\mathscr{K} = \mathscr{C}_q$ (cf. [3, theorem 22]).

For $q = 2$ the class $\mathscr{K} \subseteqq \mathscr{C}_2$ consisting of all segments and triangles is closed in $\mathscr{C}_2$, stable under homothety and universally approximating. By theorem 4 the class $\mathscr{K}$ contains every indecomposable convex body so that the indecomposable plane convex bodies are precisely the segments and the triangles.

For $q \geqq 3$ no exhaustive classification of the indecomposable convex bodies seems to be known. As an example of an indecomposable convex body in R^3, which is not a polyhedron, one could mention a cone.

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