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SHEPHARD'S APPROXIMATION THEOREM FOR CONVEX BODIES AND THE MILMAN THEOREM

CHRISTIAN BERG

In [4] G. C. Shephard proved an interesting approximation theorem concerning indecomposable convex polyhedra in the q -dimensional space \mathbb{R}^q (cf. p. 23 of the present paper).

The purpose of the present paper is to give a new proof of this theorem. First we find a correspondence between the set of homothety classes of convex bodies in \mathbb{R}^q and a compact convex set in the Banach space $C(\Omega_q)$ of continuous functions on the unit sphere Ω_q in \mathbb{R}^q such that the indecomposable classes correspond to the extreme points of this compact convex set. We next show that Shephard's approximation theorem is a consequence of Milman's theorem, valid for a compact convex set in a locally convex topological vector space [3, p. 9]. Our proof yields that Shephard's theorem is true not only for an indecomposable polyhedron but for any indecomposable convex body in \mathbb{R}^q .

Chapter 15 in the monograph [2] deals with the notion of decomposable and indecomposable polyhedra and the approximation theorem of G. C. Shephard.

Let \mathcal{C}_q denote the class of all convex bodies in \mathbb{R}^q consisting of more than one point. If $K, L \in \mathcal{C}_q$ and $\lambda > 0$, we have

$$K + L \in \mathcal{C}_q \quad \text{and} \quad \lambda K \in \mathcal{C}_q.$$

We consider \mathcal{C}_q as a metric space under the Hausdorff-distance

$$D(K, L) = \inf \{ \varepsilon > 0 \mid K \subseteq L + \varepsilon E_q, L \subseteq K + \varepsilon E_q \},$$

where E_q is the unit ball in \mathbb{R}^q . For each $K \in \mathcal{C}_q$ let $h(K)$ denote the supporting function of K . We consider $h(K)$ as an element of the Banach space $C(\Omega_q)$ of continuous real-valued functions defined on the unit sphere Ω_q in \mathbb{R}^q , equipped with the uniform norm. It is well known that the mapping $h: K \rightarrow h(K)$ of \mathcal{C}_q into $C(\Omega_q)$ is one-to-one and satisfies (cf. [1])

$$(1) \quad h(K + L) = h(K) + h(L), \quad h(\lambda K) = \lambda h(K) \quad \text{for } \lambda > 0,$$

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$$(2) \quad D(K, L) = \|h(K) - h(L)\|.$$

For $K \in \mathcal{C}_q$ one defines the Steiner-point $S(K)$ of K [2, p. 314] by

$$S(K) = \frac{q}{\|\omega_q\|} \int_{\Omega_q} \xi h(K)(\xi) d\omega_q(\xi)$$

and the mean width $B(K)$ of K [1, p. 50] by

$$B(K) = \frac{2}{\|\omega_q\|} \int_{\Omega_q} h(K)(\xi) d\omega_q(\xi),$$

where ω_q denotes the usual surface measure on Ω_q with total mass $\|\omega_q\|$. Note that $S(K) \in K$ and that $B(K) > 0$. The mappings $S: \mathcal{C}_q \rightarrow \mathbf{R}^q$ and $B: \mathcal{C}_q \rightarrow \mathbf{R}$ are both continuous and satisfy the linearity relations analogous to (1). Moreover S commutes with rigid motions, whereas B is invariant under rigid motions.

We shall consider the subset A of \mathcal{C}_q defined by

$$A = \{K \in \mathcal{C}_q \mid S(K) = o, B(K) = 1\},$$

where o denotes the origin of \mathbf{R}^q , and the corresponding set $h(A)$ of supporting functions. From the above remarks it is obvious that A is a closed subset of \mathcal{C}_q with the property that if $K, L \in A$ and $\lambda \in [0, 1]$, then $\lambda K + (1 - \lambda)L \in A$.

THEOREM 1. *The subset A of \mathcal{C}_q is compact, and h is a homeomorphism of A onto $h(A)$, which is a compact convex set in $C(\Omega_q)$.*

PROOF. For any convex body $K \in A$ and any point $a \in K$ the segment $[o, a]$ belongs to K since $o \in K$. Thus $B([o, a]) \leq B(K) = 1$. Since

$$B([o, a]) = \frac{2 \|\omega_{q-1}\| \|a\|}{\|\omega_q\| (q-1)},$$

$\|a\|$ and hence A is bounded. The selection theorem of Blaschke yields the compactness of A , and the proof is completed by (1) and (2).

A convex body $K \in \mathcal{C}_q$ is called *decomposable* if there exist convex bodies $L, M \in \mathcal{C}_q$, non-homothetic to K , such that $K = L + M$. If this is not the case, K is called *indecomposable*.

It is obvious that if $K, L \in \mathcal{C}_q$ are homothetic, then K is decomposable if and only if L is decomposable. Further, for any $K \in \mathcal{C}_q$ there exists a unique pair (λ, a) , where $\lambda > 0$, $a \in \mathbf{R}^q$, such that $\lambda^{-1}(K - a) \in A$, namely $\lambda = B(K)$, $a = S(K)$.

The supporting function of $\lambda^{-1}(K-a)$ is denoted $\eta(K)$ and is called *the normalized supporting function of K* . For $K \in \mathcal{C}_q$ and $\xi \in \Omega_q$ we have

$$\eta(K)(\xi) = B(K)^{-1}(h(K)(\xi) - S(K) \cdot \xi).$$

The normalized supporting function $\eta(K)$ lies in the compact convex set $h(A)$, and for $K, L \in \mathcal{C}_q$ we have $\eta(K) = \eta(L)$ if and only if K and L are homothetic. The mapping $\eta: \mathcal{C}_q \rightarrow h(A)$ is continuous.

THEOREM 2. *Let $K \in \mathcal{C}_q$. Then K is indecomposable if and only if the normalized supporting function $\eta(K)$ is an extreme point of the compact convex set $h(A)$.*

PROOF. It suffices to prove the theorem for a $K \in A$, that is, when $\eta(K) = h(K)$.

Suppose that K is decomposable. Then we have a decomposition $K = L + M$ with $L, M \in \mathcal{C}_q$, and $\eta(K)$ is different from $\eta(L)$ and $\eta(M)$. It follows that

$$\eta(K) = B(L)\eta(L) + B(M)\eta(M) \quad \text{and} \quad 1 = B(L) + B(M),$$

which show that $\eta(K)$ is not an extreme point of $h(A)$.

Conversely, if $\eta(K)$ is not extreme in $h(A)$, we can find $L, M \in A$ different from K and λ with $0 < \lambda < 1$ such that

$$\eta(K) = \lambda\eta(L) + (1-\lambda)\eta(M).$$

Since $K, L, M \in A$, we have

$$h(K) = \lambda h(L) + (1-\lambda)h(M) = h(\lambda L + (1-\lambda)M),$$

and consequently

$$K = \lambda L + (1-\lambda)M,$$

which shows that K is decomposable.

In the following let $\mathcal{K} \subseteq \mathcal{C}_q$ be a class of convex bodies stable under homothety, that is, if $K \in \mathcal{K}$, then $\lambda K + a \in \mathcal{K}$ for all $\lambda > 0$, $a \in \mathbb{R}^q$. Call a convex body $L \in \mathcal{C}_q$ approximable by such a class if there exist convex bodies $K_1 + \dots + K_n$, where $K_i \in \mathcal{K}$, arbitrarily near to L in the Hausdorff-distance.

LEMMA 1. *Let $\mathcal{K} \subseteq \mathcal{C}_q$ be a class of convex bodies stable under homothety, and let $L \in \mathcal{C}_q$. Then L is approximable by the class \mathcal{K} if and only if*

$$\eta(L) \in \text{cl conv } \eta(\mathcal{K}),$$

where $\text{cl conv } \eta(\mathcal{K})$ denotes the closed convex hull of the subset $\{\eta(K) \mid K \in \mathcal{K}\}$ of $h(A)$.

PROOF. Suppose that there exists a sequence $K_n \in \mathcal{C}_q$ such that $K_n \rightarrow L$ in \mathcal{C}_q , and such that

$$K_n = \sum_{i=1}^{i_n} K_n^i, \quad \text{where } K_n^i \in \mathcal{K}.$$

We then have $\eta(K_n) \rightarrow \eta(L)$ in $h(A)$ (uniformly over Ω_q) and

$$\eta(K_n) = \sum_{i=1}^{i_n} B(K_n)^{-1} B(K_n^i) \eta(K_n^i),$$

which is a convex combination of $\eta(K_n^i)$, $i=1, \dots, i_n$. This proves

$$\eta(L) \in \text{cl conv } \eta(\mathcal{K}).$$

Conversely, if this relation is satisfied, there exist convex bodies $K_n^i \in \mathcal{K}$ and numbers $\lambda_n^i > 0$, $i=1, \dots, i_n$, $n=1, 2, \dots$, such that

$$\sum_{i=1}^{i_n} \lambda_n^i = 1$$

and

$$\sum_{i=1}^{i_n} \lambda_n^i \eta(K_n^i) \rightarrow \eta(L) \quad \text{in } h(A).$$

Since \mathcal{K} is stable under homothety, the bodies K_n^i can be chosen such that $K_n^i \in A$, that is, $h(K_n^i) = \eta(K_n^i)$. Thus we get

$$h\left(\sum_{i=1}^{i_n} \lambda_n^i K_n^i\right) \rightarrow h\left(B(L)^{-1}(L - S(L))\right) \quad \text{in } h(A),$$

which by theorem 1 implies that

$$\sum_{i=1}^{i_n} \lambda_n^i K_n^i \rightarrow B(L)^{-1}(L - S(L)).$$

Consequently L is approximable by the class \mathcal{K} .

If we let $\mathcal{K} \subseteq \mathcal{C}_q$ be the class of all indecomposable convex bodies, lemma 1 combined with the Krein–Milman theorem yields the following result:

THEOREM 3. *Every convex body $L \in \mathcal{C}_q$ can be approximated arbitrarily well by sums of indecomposable convex bodies.*

As a consequence of lemma 1 combined with the Milman theorem, we get:

THEOREM 4. *Let $\mathcal{K} \subseteq \mathcal{C}_q$ be a class of convex bodies stable under homothety. If an indecomposable convex body $L \in \mathcal{C}_q$ is approximable by the class \mathcal{K} , then $L \in \text{cl } \mathcal{K}$.*

PROOF. By lemma 1 we have

$$\eta(L) \in \text{cl conv } \eta(\mathcal{K}),$$

and $\eta(L)$ must be an extreme point of $\text{cl conv } \eta(\mathcal{K})$, because it is extreme in $h(A)$. Thus by the Milman theorem we get $\eta(L) \in \text{cl } \eta(\mathcal{K})$, and the proof is easily completed by means of theorem 1.

It is straightforward to see that theorem 4 implies Shephard's approximation theorem:

Let $\mathcal{C} = \{K \in \mathcal{C}_q \mid S(K) = 0, \text{diam } K = 1\}$, and let $\mathcal{K}_0 \subseteq \mathcal{C}$ be a closed subset. If $P \in \mathcal{C}$ is an indecomposable polyhedron, and if P can be approximated arbitrarily well by convex bodies $K \in \mathcal{C}$ of the form $K = \sum_{i=1}^n \lambda_i K_i$, where $K_i \in \mathcal{K}_0$, $\lambda_i > 0$, then $P \in \mathcal{K}_0$.

We point out that theorem 3 does not tell anything new. It is well known that any convex body $K \in \mathcal{C}_2$, can be approximated by a convex polygon, and every convex polygon is a sum of segments and triangles, which are known to be indecomposable. For $q \geq 3$ the indecomposable convex bodies are even dense in \mathcal{C}_q , because every convex simplicial polyhedron (that is, a polyhedron the $(q-1)$ -dimensional facets of which are simplices) is indecomposable [4, lemma 23]. For $q \geq 3$ theorem 4 therefore has the consequence that if $\mathcal{K} \subseteq \mathcal{C}_q$ is a class stable under homothety, closed as a subset of \mathcal{C}_q and universally approximating, that is, every convex body $L \in \mathcal{C}_q$ can be approximated by \mathcal{K} , then $\mathcal{K} = \mathcal{C}_q$ (cf. [3, theorem 22]).

For $q=2$ the class $\mathcal{K} \subseteq \mathcal{C}_2$ consisting of all segments and triangles is closed in \mathcal{C}_2 , stable under homothety and universally approximating. By theorem 4 the class \mathcal{K} contains every indecomposable convex body so that the indecomposable plane convex bodies are precisely the segments and the triangles.

For $q \geq 3$ no exhaustive classification of the indecomposable convex bodies seems to be known. As an example of an indecomposable convex body in \mathbb{R}^3 , which is not a polyhedron, one could mention a cone.

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